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A NEW BOUND ON THE CARDINALITY OF POWER HOMOGENEOUS COMPACTA

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Abstract. It was recently proved by R. de la Vega that if X is a homogeneous compactum then $|X| \leq 2^{t(X)}$. We extend his argument to show that the same inequality holds for power homogeneous compacta.

1. Introduction

The *tightness* of a space X, $t(X)$, is the least cardinal number κ with the property that if $A \subseteq X$ and $x \in \overline{A}$, then there is some set $B \in [A]^{\leq\kappa}$ such that $x \in \overline{B}$. A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism h of X such that $h(x) = y$. A space is called *power homogeneous* if X^μ is homogeneous for some cardinal number μ.

Recently R. de la Vega proved in [12] that the cardinality of any homogeneous compactum X, does not exceed $2^{t(X)}$, thus solving a longstanding problem in the field of cardinal functions. In this paper we extend his result to the class of compact power homogeneous spaces. We also show how the argument of de la Vega can be carried out using a classical closing-off argument without using elementary submodels. The possibility of this construction was noted independently by R. Buzyakova. In fact our argument boils down to a slightly modified version of the closing-off argument already used by Buzyakova in [5].

The proof of de la Vega depends on a result proved by Arhangel′skiĭ in [1] which states that in every compact space with $t(X) \leq \kappa$, there is some closed G_κ-subset in X which is contained in the closure of some member of $[X]^{\leq\kappa}$. De la Vega proved in [12] that if closed G_κ-subsets with this property cover the compact

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space X and $t(X) \leq \kappa$, then the weight of X is bounded by 2^κ. By Arhangel′skiĭ's result such a cover exists on homogeneous compacta of tightness at most κ. In section 2 we show that the same is true for power homogeneous compacta.

As in de la Vega's argument, we apply an inequality which is due to van Mill to obtain our main result.

Our results of section 2 are inspired by the notion of a κ-twister, studied by Arhangel′skiĭ in [2] and [3]. Although our primary interest is in power homogeneous spaces, we will present our results in the more general setting of arbitrary product spaces.

In the final section of this paper we show that some of our results generalize beyond the class of compact spaces.

2. The G_κ-density in homogeneous products

By $\psi(X)$, $\pi\chi(X)$ and $\chi(X)$ we denote the *pseudocharacter*, *π-character* and *character* respectively. By $\psi w(X)$, $\pi(X)$ and $w(X)$ we denote the *pseudo-weight*, *π-weight* and *weight*. By $nw(X)$ we denote the *network-weight*, by $L(X)$ we denote the *Lindelöf-degree*.

We say that the G_κ-density at a point x in X does not exceed κ if there exists a closed G_κ-subset H of X and a set $S \in [X]^{\leq\kappa}$ such that $x \in H \subseteq \overline{S}$. We say that the G_κ-density of X does not exceed κ, if the G_κ-density does not exceed κ at all points x in X.

All product spaces in this paper carry the usual product topology. Whenever $\{X_i : i \in I\}$ is a collection of topological spaces and $X = \prod\{X_i : i \in I\}$ is the product space, then for $A \subseteq I$ by X_A we denote the product $\prod\{X_i : i \in A\}$. By π_A we denote the natural projection of X onto X_A. If $i \in I$, then we write π_i instead of $\pi_{\{i\}}$. If $x \in X$ then by x_A we denote the point $\pi_A(x)$. If $Y \subseteq X$, then $Y_A = \pi_A[Y]$.

We say that $x \in X$ is a G_κ-point iff $\{x\}$ is a G_κ-subset of X, i.e. iff $\psi(x, X) \leq \kappa$. Recall that if X is compact, then x is a G_κ-point iff the character of x in X does not exceed κ.

All spaces under consideration are Hausdorff. The following proposition was proved in [1, Theorem 2.2.4].

Proposition 2.1. *If X is a compact space with $t(X) \leq \kappa$, then the G_κ-density does not exceed κ at some point $e \in X$.*

It is the aim of this section to show that if $X = \prod\{X_i : i \in I\}$ is a homogeneous compactum with $t(X_i) \leq \kappa$ for all $i \in I$, then the G_κ-density of X_i does not exceed κ for all $i \in I$. We start with some preliminary lemmas.

Theorem 2.2. *Let $X = \prod\{X_i : i \in I\}$ and suppose $h : X \to X$ is a homeomorphism. Suppose that $h(x) = z$ and for some $i \in I$, $\pi\chi(x_i, X_i) \le \kappa$. Then there is some set $A \in [I]^{\le\kappa}$ such that*

$$h^{-1}\pi_A^{-1}(z_A) \subseteq \pi_i^{-1}(x_i).$$

PROOF. Fix a local π-base $\mathcal{U}$ at x_i in X_i with $|\mathcal{U}| \le \kappa$. For every $U \in \mathcal{U}$, we pick $y_U \in \pi_i^{-1}[U]$ such that $(y_U)_B = x_B$, where $B = I \setminus \{i\}$. For every $U \in \mathcal{U}$, we may fix some basic open set G_U such that

$$h(y_U) \in G_U \subseteq h[\pi_i^{-1}[U]]$$

By A_U we denote the finite set of co-ordinates on which G_U is defined. Thus $G_U = \pi_{A_U}^{-1}\left[\pi_{A_U}[G_U]\right]$. We define $A = \bigcup\{A_U : U \in \mathcal{U}\}$. Then A is a subset of I with $|A| \le \kappa$. We will prove that $\pi_i h^{-1}[\pi_A^{-1}(z_A)] = \{x_i\}$.

Suppose to the contrary that $y \in \pi_i h^{-1}[\pi_A^{-1}(z_A)]$ and $y \ne x_i$. We fix a neighbourhood W of x_i such that $y \notin \overline{W}$. Let $\mathcal{V} = \{U \in \mathcal{U} : U \subseteq W\}$. Then $\mathcal{V}$ is a local π-base at x_i. So we have

$$x_i \in \mathrm{Cl}\,\{\pi_i(y_V) : V \in \mathcal{V}\}.$$

By construction we have

$$x \in \mathrm{Cl}\,\{y_V : V \in \mathcal{V}\}.$$

But then we also have that $z_A \in \mathrm{Cl}\,\{h(y_V)_A : V \in \mathcal{V}\}$.

Since $y \in \pi_i h^{-1}[\pi_A^{-1}(z_A)]$, there is some $w \in \pi_A^{-1}(z_A)$ such that $\pi_i h^{-1}(w) = y$. For $V \in \mathcal{V}$, We define the point $c_V \in X$ as follows,

$$(c_V)_j = \begin{cases} h(y_V)_j & \text{if } j \in A, \\ w_j & \text{if } j \notin A. \end{cases}$$

Note that $c_V \in G_V$. Since $w_A = z_A$ and $(c_V)_A = h(y_V)_A$ it follows that $w \in \mathrm{Cl}\,\{c_V : V \in \mathcal{V}\}$. But then $y = \pi_i h^{-1}(w) \in \mathrm{Cl}\,\{\pi_i h^{-1}(c_V) : V \in \mathcal{V}\}$. However, by construction we also have that $\pi_i h^{-1}(c_V) \in V$, because $c_V \in G_V$. It follows that $\mathrm{Cl}\,\{\pi_i h^{-1}(c_V) : V \in \mathcal{V}\} \subseteq \overline{W}$. Since $y \notin \overline{W}$, this is impossible. □

Corollary 2.3. *Let $X = \prod\{X_i : i \in I\}$ and suppose that $h : X \to X$ is a homeomorphism. Suppose further that for some $i \in I$, $\pi\chi(X_i) \le \kappa$. Let C be a subset of X with $|C| \le \kappa$. Then there is a set of co-ordinates $A \in [I]^{\le\kappa}$ such that for all $y \in C$,*

$$h^{-1}\pi_A^{-1}(h(y)_A) \subseteq \pi_i^{-1}(y_i).$$

□

The following corollary to Theorem 2.2 is due to Arhangel′skiĭ [2, Theorem 2.3].

Corollary 2.4. *Suppose $X = \prod\{X_i : i \in I\}$ is homogeneous. Suppose further that for all $i \in I$, X_i contains some G_κ-point. If for some $j \in I$, $\pi\chi(X_j) \leq \kappa$ then $\psi(X_j) \leq \kappa$.*

Proof. For all $i \in I$, let $e_i \in X_i$ be such that $\psi(e_i, X_i) \leq \kappa$. Let $z \in X_j$ be arbitrary. Choose $x \in X$ with $x_j = z$ and let $h : X \to X$ be some homeomorphism with $h(x) = e$, where of course e is the point of X whose i^{th} co-ordinate is e_i. By Theorem 2.2 there is some set $A \in [I]^{\leq\kappa}$ such that

$$\pi_j h^{-1} \pi_A^{-1}(e_A) = \{z\}.$$

Let $B = I \setminus \{j\}$ and $Y = \{y \in X : y_B = x_B\}$. Then $\pi_j \restriction Y : Y \to X_j$ is a homeomorphism. Since $G = h^{-1}\pi_A^{-1}(e_A)$ is a G_κ-subset of X, the set $G \cap Y$ is a G_κ-subset of Y. Since $G \cap Y = \{x\}$, it follows that $\{z\}$ is a G_κ-subset of X_j. $\square$

Corollary 2.5. *Suppose $X = \prod\{X_i : i \in I\}$ is a homogeneous compactum. Suppose further that for all $i \in I$, X_i contains some point e_i with $\chi(e_i, X_i) \leq \kappa$. If for some $j \in I$, $\pi\chi(X_j) \leq \kappa$ then $\chi(X_j) \leq \kappa$.* $\square$

For power homogeneous spaces we obtain the following corollary, which is related to van Mill [7, Remark 2.7], see also Bella [4].

Corollary 2.6. *If X is power homogeneous and contains some point e with $\psi(e, X) \leq \pi\chi(X)$, then $|X| \leq w(X)^{\pi\chi(X)}$. It follows that if in addition X is regular, then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

Proof. It is well known and easy to prove that $|X| \leq w(X)^{\psi(X)}$, so the first inequality follows from Corollary 2.4. The second inequality follows from the fact that $w(X) \leq \pi\chi(X)^{c(X)}$ for regular spaces which is due to Šapirovskiĭ [10, Theorem 3]. $\square$

We fix a product space $X = \prod\{X_i : i \in I\}$ and some collection $\{S_i : i \in I\}$ where $S_i \subseteq X_i$ for all $i \in I$. We set $S = \prod\{S_i : i \in I\} \subseteq X$. Whenever $A \in [I]^{\leq\kappa}$ we define a set $\mathcal{S}(A) \subseteq X$ as follows. For each $i \in I$, we pick some $s_{0,i} \in S_i$. If $B \subseteq I$, and $s' \in S_B$, we choose $s(B, s') \in S$ such that $s(B, s')_B = s'$ and for all $i \in I \setminus B$, $s(B, s')_i = s_{0,i}$. Now for $A \subseteq I$, we define

$$\mathcal{S}(A) = \{s(B, s') : B \in [A]^{<\omega} \text{ and } s' \in S_B\}.$$

We will need the following Lemma, its proof is straightforward.

Lemma 2.7. *If $A \in [I]^{\leq\kappa}$ and for all $i \in A$, $|S_i| \leq \kappa$ then $|\mathcal{S}(A)| \leq \kappa$ and $\mathcal{S}(A)_A$ is dense in S_A. Furthermore if $A \subseteq B$, then $\mathcal{S}(A) \subseteq \mathcal{S}(B)$. If (A_n) is an* *increasing sequence of infinite subsets of I and $A = \bigcup_n A_n$ then*

$$\mathcal{S}(A) = \bigcup_n \mathcal{S}(A_n).$$

$\square$

Theorem 2.8. *Let $X = \prod\{X_i : i \in I\}$ and $S = \prod\{S_i : i \in I\} \subseteq X$ where $|S_i| \leq \kappa$ for all $i \in I$. Suppose that for some $j \in I$, $\pi\chi(X_j) \leq \kappa$. If $h : X \to X$ is a homeomorphism and $B \in [I]^{\leq\kappa}$ then there is a set $A \in [I]^{\leq\kappa}$ such that $B \subseteq A$ and*

$$(*) \qquad \forall s \in \mathcal{S}(A) \ \big(h^{-1}\pi_A^{-1}(s_A) \subseteq \pi_j^{-1}(\pi_j h^{-1}(s))\big).$$

Proof. Set $A_0 = B$. By applying Corollary 2.3 to the set $h^{-1}[\mathcal{S}(A_n)]$, we get a set $A_{n+1} \in [I]^{\leq\kappa}$ such that

$$\forall s \in \mathcal{S}(A_n) \ \big(h^{-1}\pi_{A_{n+1}}^{-1}(s_{A_{n+1}}) \subseteq \pi_j^{-1}(\pi_j h^{-1}(s)).$$

We may assume that $A_n \subseteq A_{n+1}$. If we let $A = \bigcup_n A_n$, then $(*)$ follows from the previous Lemma. $\square$

Corollary 2.9. *Let $X = \prod\{X_i : i \in I\}$. Suppose that X is homogeneous and for all $i \in I$, the G_κ-density of some point of X_i does not exceed κ. If for some $j \in I$, $\pi\chi(X_j) \leq \kappa$ then the G_κ-density of all points of X_j does not exceed κ.*

Proof. For each $i \in I$, fix some point $e_i \in X_i$, a closed G_κ-subset $H_i \subseteq X_i$ and $S_i \in [X_i]^{\leq\kappa}$ such that $e_i \in H_i \subseteq \overline{S}_i$. By H we denote the set $\prod\{H_i : i \in I\}$.

Let $z \in X_j$ be arbitrary and pick $x \in X$ with $x_j = z$. Since X is homogeneous, there is a homeomorphism h of X with $h(x) = e$, where of course e is the point of X whose i^{th} co-ordinate is e_i. By the previous Theorem and Lemma 2.7 we get a set $A \in [I]^{\leq\kappa}$ and a subset $\mathcal{S}$ of X with $|\mathcal{S}| \leq \kappa$, such that

(1) $e_A \in H_A \subseteq \overline{\mathcal{S}_A}$,
(2) $\forall s \in \mathcal{S} \ \big(h^{-1}\pi_A^{-1}(s_A) \subseteq \pi_j^{-1}(\pi_j h^{-1}(s))\big)$.

For every $s \in \mathcal{S}$, the set $\pi_j h^{-1}\pi_A^{-1}(s_A)$ consists of the single point $\pi_j h^{-1}(s)$. We let $T = \{\pi_i h^{-1}(s) : s \in \mathcal{S}\}$. Since $H_A \subseteq \overline{\mathcal{S}_A}$ and π_A is open, we have

$$\pi_A^{-1}[H_A] \subseteq \overline{\pi_A^{-1}[\mathcal{S}_A]}.$$

It follows that

$$\pi_j h^{-1}[\pi_A^{-1}[H_A]] \subseteq \mathrm{Cl}\, \pi_j h^{-1}\pi_A^{-1}[\mathcal{S}_A] = \overline{T}.$$

Let $B = I \setminus \{j\}$ and $Y = \{y \in X : y_B = x_B\}$. Then $\pi_j{\restriction}Y : Y \to X_j$ is a homeomorphism. Now let $K = Y \cap h^{-1}[\pi_A^{-1}[H_A]]$. Then K is a closed G_κ-subset of Y which contains x. We let $F = \pi_j[K]$. Then F is a closed G_κ-subset of X_j and we have

$$z = x_j \in F = \pi_j[K] \subseteq \pi_j h^{-1}[\pi_A^{-1}[H_A]] \subseteq \overline{T}.$$

Then $z \in F \subseteq \overline{T}$ is the desired statement. □

Corollary 2.10. *Suppose $X = \prod\{X_i : i \in I\}$ is compact and homogeneous and suppose that $t(X_i) \leq \kappa$ for each $i \in I$. Then for all $i \in I$, the G_κ-density of X_i does not exceed κ.*

Proof. This follows from the previous corollary, Proposition 2.1 and the inequality $\pi\chi(X_i) \leq t(X_i)$ for the compact spaces X_i, which was proved by Šapirovskiĭ in [9]. □

For power homogeneous compacta we obtain the following.

Corollary 2.11. *Suppose X is compact and power homogeneous with $t(X) \leq \kappa$. Then the G_κ-density of X does not exceed κ.* □

From this corollary, de la Vega's argument in [12] and van Mill's inequality $|X| \leq w(X)^{\pi\chi(X)}$ for power homogeneous compacta (cf. [7]) it follows that the cardinality of power homogeneous compacta does not exceed $2^{t(X)}$.

3. A closing-off argument

In this section we provide an elementary version of the proof of de la Vega presented in [12]. With the results of the previous section we obtain our main result.

Lemma 3.1. *Suppose H is a closed G_κ-subset of X and $S \in [X]^{\leq\kappa}$ is such that $H \subseteq \overline{S}$. Then there is a collection $\mathcal{B}$ of open subsets of X with $|\mathcal{B}| \leq 2^\kappa$, such that $\mathcal{B}$ is a local ψ-base for every point $x \in H$.*

Proof. Let $H = \bigcap_{\alpha<\kappa} G_\alpha$, where $G_\alpha \subseteq X$ is open. The collection $\mathcal{B}$ is given by

$$\mathcal{B} = \{G_\alpha : \alpha < \kappa\} \cup \{X \setminus \overline{C} : C \subseteq S\}.$$

Then $|\mathcal{B}| \leq \kappa + 2^\kappa = 2^\kappa$ and it is easily verified that $\mathcal{B}$ is as desired. □

Lemma 3.2. *Suppose X is a compact space with $t(X) \leq \kappa$. Then for any closed subspace Y of X with $d(Y) \leq 2^\kappa$, we have $w(Y) \leq 2^\kappa$.*

Proof. Let D be a dense subset of Y with $|D| \leq 2^\kappa$. Since $t(X) \leq \kappa$ it is easily verified that the collection $\mathcal{F}$ given by

$$\mathcal{F} = \{\overline{C} : C \in [D]^{\leq\kappa}\}$$

forms a network in Y. Since Y is compact we have $w(Y) = nw(Y)$ ([1, Theorem 2.1.9]). It follows that $w(Y) \leq |\mathcal{F}| \leq 2^\kappa$. $\square$

The following Lemma is well-known, we include its simple proof.

Lemma 3.3. *Let X be a compact space with $w(X) \leq 2^\kappa$. Then the cardinality of the family of all closed G_κ-subsets of X does not exceed 2^κ.*

Proof. Let $\mathcal{B}$ be a base of X, which is closed under finite unions, such that $|\mathcal{B}| \leq 2^\kappa$. If H is a closed G_κ-subset of X, then by compactness there is a subfamily $\mathcal{U}$ of $\mathcal{B}$ with $|\mathcal{U}| \leq \kappa$, such that $H = \bigcap \mathcal{U}$. Thus we may define an injection from the family of closed G_κ-subsets of X into $\mathcal{B}^\kappa$. $\square$

Lemma 3.4. *Suppose X is a compact space with $t(X) \leq \kappa$. Suppose further that the G_κ-density of X does not exceed κ. If Y is a closed subspace of X with $d(Y) \leq 2^\kappa$ then there is a collection $\mathcal{B}$ of open subsets of X with $|\mathcal{B}| \leq 2^\kappa$ such that $\mathcal{B}$ is a local ψ-base for all points $y \in Y$.*

Proof. Since the G_κ-density of X does not exceed κ, we may find a cover $\mathcal{G}$ of Y which consists of closed G_κ-subsets of X such that every member of $\mathcal{G}$ is contained in the closure of some member of $[X]^{\leq\kappa}$. By Lemma 3.2 and 3.3 we may assume that $|\mathcal{G}| \leq 2^\kappa$. By Lemma 3.1 we find for every $G \in \mathcal{G}$ a pseudo-base $\mathcal{B}_G$ for all points of G such that $|\mathcal{B}_G| \leq 2^\kappa$. Then $\mathcal{B} = \bigcup\{\mathcal{B}_G : G \in \mathcal{G}\}$ is a pseudo-base in X for all points of Y and clearly $|\mathcal{B}| \leq 2^\kappa$. $\square$

Theorem 3.5. *Suppose X is a compact space with $t(X) \leq \kappa$. Suppose further that the G_κ-density of X does not exceed κ. Then $w(X) \leq 2^\kappa$.*

Proof. By transfinite recursion we construct an increasing sequence $\{Y_\alpha : \alpha < \kappa^+\}$ of closed subspaces of X and an increasing sequence $\{\mathcal{P}_\alpha : \alpha < \kappa^+\}$ of families of open subsets of X such that the following conditions are satisfied for all $\alpha < \kappa^+$.

(1) $\mathcal{P}_\alpha$ is a local ψ-base for all points of Y_α,
(2) $|\mathcal{P}_\alpha| \leq 2^\kappa$,
(3) $d(Y_\alpha) \leq 2^\kappa$,
(4) Whenever $\mathcal{U}$ is a finite subfamily of $\mathcal{P}_\alpha$ which covers Y_α such that $X \setminus \bigcup \mathcal{U}$ is non-empty, then $Y_{\alpha+1} \setminus \bigcup \mathcal{U}$ is non-empty.

We put $Y_0 = \emptyset$ and $\mathcal{P}_0 = \emptyset$. Take any $\beta < \kappa^+$ and suppose that Y_α and $\mathcal{P}_\alpha$ have been defined for all $\alpha < \beta$. Then we proceed as follows.

Case 1: β is a limit ordinal. Put $Y_\beta = \mathrm{Cl}_X \bigcup\{Y_\alpha : \alpha < \beta\}$. Then by (3), the density of Y_β does not exceed 2^κ. It follows from Lemma 3.4 that there is a collection $\mathcal{B}$ of open subsets of X with $|\mathcal{B}| \leq 2^\kappa$ such that $\mathcal{B}$ is a local ψ-base for all points $y \in Y_\beta$. Now let $\mathcal{P}_\beta = \mathcal{B} \cup \bigcup\{\mathcal{P}_\alpha : \alpha < \beta\}$.

Case 2: $\beta = \alpha + 1$ for some $\alpha < \kappa^+$. Let $\mathcal{E}_\alpha$ be the collection of all finite subfamilies $\mathcal{U}$ of $\mathcal{P}_\alpha$ such that $\mathcal{U}$ covers Y_α and $X \setminus \bigcup\mathcal{U}$ is non-empty. Since $|\mathcal{P}_\alpha| \leq 2^\kappa$, the cardinality of the collection $\mathcal{E}_\alpha$ does not exceed 2^κ. For each $\mathcal{U} \in \mathcal{E}_\alpha$, pick $c(\mathcal{U}) \in X \setminus \bigcup\mathcal{U}$, and put

$$Y_\beta = \mathrm{Cl}_X\{c(\mathcal{U}) : \mathcal{U} \in \mathcal{E}_\alpha\} \cup Y_\alpha.$$

The density of Y_β does not exceed 2^κ. Again by Lemma 3.4 we may find a collection $\mathcal{B}$ with $|\mathcal{B}| \leq 2^\kappa$ such that $\mathcal{B}$ is a local ψ-base for all points $y \in Y_\beta$. We put $\mathcal{P}_\beta = \mathcal{B} \cup \mathcal{P}_\alpha$.

This completes the recursion. We put $Y = \bigcup\{Y_\alpha : \alpha < \kappa^+\}$. Then clearly $d(Y) \leq \kappa^+ \cdot 2^\kappa = 2^\kappa$. Furthermore, since the sequence $\{Y_\alpha : \alpha < \kappa^+\}$ is increasing and $t(X) \leq \kappa$, it follows that Y is closed in X. We will show that $X = Y$.

Assume to contrary that $x \in X \setminus Y$. Put $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha < \kappa^+\}$. Clearly, $\mathcal{P}$ is a pseudobase for all points $y \in Y$. Therefore, there exists a cover $\mathcal{U}$ of Y which consists of members of $\mathcal{P}$ such that $x \notin \bigcup\mathcal{U}$. By compactness of Y we may assume that $\mathcal{U}$ is finite. It follows that $\mathcal{U} \subseteq \mathcal{P}_\alpha$ for some $\alpha < \kappa^+$. But then by condition (4), it follows that $\mathcal{U}$ does not cover $Y_{\alpha+1}$, which contradicts the fact that $\mathcal{U}$ covers Y. It follows that $X = Y$.

Since $d(Y) \leq 2^\kappa$, it follows from Lemma 3.2 that $w(X) = w(Y) \leq 2^\kappa$. □

Corollary 3.6. *Suppose $X = \prod\{X_i : i \in I\}$ is a homogeneous compactum and for all $i \in I$, $t(X_i) \leq \kappa$. Then for all $i \in I$, $w(X_i) \leq 2^\kappa$.*

Proof. This follows from Corollary 2.10 and Theorem 3.5. □

Corollary 3.7. *Suppose X is a power homogeneous compactum, then $|X| \leq 2^{t(X)}$.*

Proof. This follows from Corollary 2.11, Theorem 3.5 and the inequalities $|X| \leq w(X)^{\pi\chi(X)}$ for power homogeneous compacta ([7, Theorem 2.5]) and $\pi\chi(X) \leq t(X)$ for compacta ([9, Theorem 1]). □

An obvious corollary tells us that under GCH, the tightness and character coincide in power homogeneous compacta.

Corollary 3.8 (GCH)**.** *If X is a power homogeneous compactum then $\chi(X) = t(X)$.*

Proof. Since $|X| \leq 2^{t(X)}$, it follows from the Čech-Pospišil Theorem and GCH that for some $e \in X$, $\chi(e, X) \leq t(X)$. Since $\pi\chi(X) \leq t(X)$, it follows from Corollary 2.5 that $\chi(X) \leq t(X)$. □

Corollary 3.9 ($2^\omega < 2^{\omega_1}$). *A power homogeneous compactum has countable tightness if and only if it is first countable.* □

4. Generalizations

In this section we will indicate how the results obtained so far can be generalized. Recall that the *pointwise compactness type* of a space X (notation pct(X)) is the smallest infinite cardinal number κ such that X can be covered by a family of compact subsets F of X such that $\chi(F, X) \leq \kappa$. A space is said to be of point-countable type if $\mathrm{pct}(X) \leq \omega$.

By $\mathcal{G}_\kappa(X)$ we denote the collection of all closed G_κ-subsets of a space X. By $\mathcal{C}_\kappa(X)$ we denote the collection of all compact members of $\mathcal{G}_\kappa(X)$. Note that if $\mathrm{pct}(X) \leq \kappa$, then $X = \bigcup \mathcal{C}_\kappa(X)$.

A family $\mathcal{B}$ of non-empty subsets of a space X is called a local π-*network* of X at a point x if every open neighbourhood of x contains some member of $\mathcal{B}$. For a family $\mathcal{E}$ of non-empty subsets of X we define the $\pi_{\mathcal{E}}$-*character* of X at x by

$$\pi\chi_{\mathcal{E}}(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{E} \text{ is a local } \pi\text{-network of } X \text{ at } x\}.$$

This cardinal function is only defined if $\mathcal{E}$ contains some local π-network of X at x. Examples of possible families are τX, the family of all open subsets and $\mathcal{G}_\kappa(X)$, the family of all closed G_κ-subsets of X. If $\mathcal{E} = \mathcal{G}_\kappa(X)$ we write $\pi\kappa\chi(x, X)$ instead of $\pi\chi_{\mathcal{E}}(x, X)$. By $\pi\kappa\chi(X)$ we denote the supremum of $\pi\kappa\chi(x, X)$ for all $x \in X$. Note that $\pi\kappa\chi(X) \leq \pi\chi(X)$.

It is straightforward to see that Proposition 2.1 allows for improvements. In particular, if a space X contains some compact G_κ-subset and $t(X) \leq \kappa$, then the G_κ-density does not exceed κ at some point $e \in X$. It follows that if $t(X)\mathrm{pct}(X) \leq \kappa$, then the G_κ-density does not exceed κ at some point $e \in X$.

Our most fundamental result is Theorem 2.2. The construction carried out, uses the fact that basic open subsets of a product space depend on only finitely many co-ordinates. We applied this observation to open subsets that are related to a local π-base of size κ. We may apply similar observations to G_κ-subsets of a product space. Starting with a π-network consisting of not more than κ G_κ-subsets of X_i, we may still carry out the construction of Theorem 2.2 and subsequent results, since 'basic' G_κ-subsets of product spaces depend on not more than κ co-ordinates.

Since $\pi\chi(X) \leq t(X)$ for compact spaces ([9, Theorem 1]), it is easily verified that $\pi\kappa\chi(X) \leq \kappa$ where $\kappa = t(X)\mathrm{pct}(X)$. These observations yield the following generalizations of Corollary 2.10 and Corollary 2.11.

Proposition 4.1. *Suppose $X = \prod\{X_i : i \in I\}$ is homogeneous with $t(X_i)\mathrm{pct}(X_i) \leq \kappa$ for all $i \in I$. Then for all $i \in I$, the G_κ-density of X_i does not exceed κ.* □

Proposition 4.2. *Suppose X is power homogeneous with $t(X)\mathrm{pct}(X) \leq \kappa$. Then the G_κ-density of X does not exceed κ.* □

We will now show how the closing-off argument of section 3 may be generalized. On inspection of the proof of Theorem 3.5 one immediately realizes that Lemma 3.4 is the main ingredient of the proof. In turn, this Lemma depends on the previous Lemmas of section 3. These Lemmas may be generalized as follows.

We showed that in compact spaces with $t(X) \leq \kappa$, for any closed subset Y, if the density of Y does not exceed 2^κ, then its weight does not exceed 2^κ. This follows from the fact that $nw(Y) = w(Y)$ for all compact spaces. If a space Y is covered by compact spaces of character (in Y) not exceeding κ then a similar proof may be carried out. In particular we use the transitivity of the character to show that if $t(X)\mathrm{pct}(X) \leq \kappa$ for some regular space X, then if Y is a closed subspace of X with $d(Y) \leq 2^\kappa$, then $w(Y) \leq 2^\kappa$.

Furthermore, it is easily shown that if $w(X) \leq 2^\kappa$ and $L(X) \leq \kappa$, then the cardinality of the family of closed G_κ-subsets of X does not exceed 2^κ.

Having generalized the necessary results, we obtain the following lemma.

Lemma 4.3. *Suppose X is a regular space with $t(X)\mathrm{pct}(X)L(X) \leq \kappa$. Suppose further that the G_κ-density of X does not exceed κ. If Y is a closed subspace of X with $d(Y) \leq 2^\kappa$ then there is a collection $\mathcal{B}$ of open subsets of X with $|\mathcal{B}| \leq 2^\kappa$ such that $\mathcal{B}$ is a local ψ-base for all points $y \in Y$.* □

The previous lemma provides the necessary input for the closing-off argument. Carrying out the argument with the generalized results (Proposition 4.1 and Lemma 4.3), we obtain the following corollary.

Corollary 4.4. *If $X = \prod\{X_i : i \in I\}$ is homogeneous and $t(X_i)\mathrm{pct}(X_i) \leq \kappa$ for all $i \in I$, then whenever X_j is regular,*

$$w(X_j) \leq 2^{\kappa \cdot L(X_j)}.$$

Corollary 4.5. *If X is a power homogeneous regular space then*

$$w(X) \leq 2^{t(X)\mathrm{pct}(X)L(X)}.$$

The remainder of this section is dedicated to showing how we may improve van Mill's inequality of [7]. We only provide a partial generalization, which will lead to a generalization of Corollary 3.7 for homogeneous spaces (Corollary 4.9). We first recall some techniques introduced by van Douwen in [6].

Recall that if $\mathcal{B}$ is an invariant collection of subsets of X and $\phi \in \mathcal{B}^\kappa$, then

$$w(x,\phi) = \big\{A \subseteq \kappa : x \in \mathrm{Cl} \bigcup\{\phi(a) : a \in A\}\big\}.$$

Subsequently we define

$$\begin{aligned} W(x,\kappa,\mathcal{B}) &= \{w(x,\phi) : \phi \in \mathcal{B}^\kappa\}, \\ W(\kappa,\mathcal{B}) &= \bigcup\{W(x,\kappa,\mathcal{B}) : x \in X\}. \end{aligned}$$

Whenever $H \subseteq X$, we say that H *h-generates* X if for all $y \in X$, there is some homeomorphism h of X and some $x \in H$ such that $h(x) = y$. If $h(x) = y$ for some homeomorphism, then $W(x,\kappa,\mathcal{B}) = W(y,\kappa,\mathcal{B})$. It follows that if H h-generates X then

$$W(\kappa,\mathcal{B}) = \bigcup\{W(x,\kappa,\mathcal{B}) : x \in H\}.$$

Proposition 4.6. *Suppose $\mathcal{B}$ is an invariant family such that $|\mathcal{B}| \leq 2^\kappa$, $\pi\chi_{\mathcal{B}}(X) \leq \kappa$ and $|W(\kappa,\mathcal{B})| \leq 2^\kappa$. If $M \in [X]^{\leq\kappa}$ then $|\overline{M}| \leq 2^\kappa$.*

Proof. We may fix $\mathcal{U} \in [\mathcal{B}]^{\leq\kappa}$ such that $\mathcal{U}$ is a local π-network for all elements of $\overline{M}$ (cf. [7, Lemma 2.2]). Now fix $\phi \in \mathcal{B}^\kappa$ such that $\mathrm{ran}(\phi) \supseteq \mathcal{U}$. Then $\{w(x,\phi) : x \in \overline{M}\}$ is a subset of $W(\kappa,\mathcal{B})$ and therefore its cardinality does not exceed 2^κ. Since $\mathrm{ran}(\phi)$ is a local π-network for all members of $\overline{M}$, it follows that the assignment $x \mapsto w(x,\phi)$ is one-to-one on $\overline{M}$ (cf. [7, Lemma 2.2]). This implies that $|\overline{M}| \leq |\{w(x,\phi) : x \in \overline{M}\}| \leq 2^\kappa$. $\square$

Theorem 4.7. *Let $t(X) \leq \kappa$ and suppose $X = \bigcup \mathcal{C}_\kappa(X)$. Suppose further that $w(X) \leq 2^\kappa$ and some $H \in [X]^{\leq 2^\kappa}$, h-generates X. Then $|X| \leq 2^\kappa$.*

Proof. Let $\mathcal{B} = \mathcal{C}_\kappa(X)$. It is easily shown that $|\mathcal{B}| \leq w(X)^\kappa \leq 2^\kappa$ (compare with Lemma 3.3). Since $t(X) \leq \kappa$ and $X = \bigcup \mathcal{C}_\kappa(X)$, it follows that $\pi\chi_{\mathcal{B}}(X) \leq \kappa$, because $\pi\chi(Y) \leq t(Y) \leq t(X)$ for compact subsets Y of X.

Note that $|W(x,\kappa,\mathcal{B})| \leq |\mathcal{B}|^\kappa \leq 2^\kappa$. Since H h-generates X, it follows that $|W(\kappa,\mathcal{B})| \leq 2^\kappa \cdot |H| = 2^\kappa$. It follows from the previous proposition that if $M \in [X]^{\leq\kappa}$, then $\overline{M} \leq 2^\kappa$.

Since $w(X) \leq 2^\kappa$, we have $d(X) \leq 2^\kappa$. So fix $D \in [X]^{\leq 2^\kappa}$ such that $\overline{D} = X$. Since $t(X) \leq \kappa$, we have

$$X = \bigcup\{\overline{M} : M \in [D]^{\leq\kappa}\}.$$

It follows that $|X| \leq 2^\kappa \cdot |[D]^{\leq\kappa}| = 2^\kappa$. $\square$

We define the *homogeneity-index*, hind(X), as the the smallest cardinal number κ such that some subset of X of size κ, h-generates X. Recall that for infinite cardinals μ, $\log\mu = \min\{\lambda \leq \mu : 2^\lambda \geq \mu\}$.

Corollary 4.8. *Suppose $X = \prod\{X_i : i \in I\}$ is homogeneous and $t(X_i)\mathrm{pct}(X_i) \leq \kappa$ for all $i \in I$. Then whenever X_j is regular,*

$$|X_j| \leq 2^{\kappa \cdot L(X_j) \log \mathrm{hind}(X_j)}.$$

Proof. Let $\mu = \kappa \cdot L(X_j) \log \mathrm{hind}(X_j)$. Since $\mathrm{pct}(X_j) \leq \mu$, we have $X_j = \bigcup \mathcal{C}_\mu(X_j)$. It remains to apply Corollary 4.4 and Theorem 4.7. □

Corollary 4.9. *Suppose X is a power homogeneous regular space. Then*

$$|X| \leq 2^{t(X)\mathrm{pct}(X)L(X) \log \mathrm{hind}(X)}.$$

For the size of homogeneous regular spaces we obtain the bound $2^{t(X)\mathrm{pct}(X)L(X)}$. This bound was already discovered by de la Vega (see [11, Theorem 4.13]) and it generalizes his result for compact homogeneous spaces in [12]. We do not know whether this generalization is true for regular power homogeneous spaces.

Question 4.10. *Is the size of arbitrary regular power homogeneous spaces bounded by $2^{t(X)\mathrm{pct}(X)L(X)}$?*

Remark (added September 2006). The third author has recently shown in [8] that Question 4.10 has a positive answer.

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