Covering compacta by discrete subspaces

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Abstract

For any space $X$, denote by $\text{dis}(X)$ the smallest (infinite) cardinal $\kappa$ such that $\kappa$ many discrete subspaces are needed to cover $X$. It is easy to see that if $X$ is any crowded (i.e. dense-in-itself) compactum then $\text{dis}(X) \geq m$, where $m$ denotes the additivity of the meager ideal on the reals. It is a natural, and apparently quite difficult, question whether in this inequality $m$ could be replaced by $\mathfrak{c}$. Here we show that this can be done if $X$ is also hereditarily normal.

Moreover, we prove the following mapping theorem that involves the cardinal function $\text{dis}(X)$. If $f : X \to Y$ is a continuous surjection of a countably compact $T_2$ space $X$ onto a perfect $T_3$ space $Y$ then $|\{y \in Y : f^{-1}y \text{ is countable}\}| \leq \text{dis}(X)$.

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In this note we use standard topological terminology and notation, as e.g. in [1] or [3]. Our aim is to study the following new cardinal functions.

Definition 1. For any space $X$ we let $\text{dis}(X)$ ($\text{ls}(X)$, respectively $\text{rs}(X)$) denote the smallest infinite cardinal $\kappa$ such that $X$ can be covered by $\kappa$ many discrete (left separated, respectively right separated) subspaces.

Since discrete spaces are both left and right separated, we clearly have $\text{ls}(X) \leq \text{dis}(X)$ and $\text{rs}(X) \leq \text{dis}(X)$. Any crowded (i.e. dense-in-itself) compactum $X$ has a crowded closed subspace $Y$ that maps irreducibly onto the interval $[0, 1]$. Then $Y$ is separable, moreover, as any right separated subspace of $Y$ is nowhere dense in $Y$, we have

$$\text{dis}(X) \geq \text{rs}(X) \geq \text{rs}(Y) \geq N(Y) = N([0, 1]) = m.$$ 

Here $N(X)$ is the Novák-number of a space $X$, i.e. the smallest number of nowhere-dense sets needed to cover $X$; hence $m$ is also known as the additivity of the meager ideal on the reals.
It is less obvious to see but, as it has been shown in [2], the inequality \( \text{ls}(X) \geq m \) is also valid for any crowded compactum \( X \).

Now, it is a natural question to raise if these inequalities are sharp, in particular the following attractive problem can be formulated and will be examined below. (Of course, here \( c \) denotes the cardinality of the continuum.)

**Problem 2.** Is there a crowded compactum \( X \) with \( \text{dis}(X) < c \)?

As we have seen above, such a space \( X \) has a separable closed subspace \( Y \subset X \). Since the weight of a separable compactum is at most \( c \), any discrete subspace of \( Y \) has size \( \leq c \). Consequently, we must have \( |Y| = c \) and so by the Čech–Pospíšil theorem some point of \( Y \) must have character \( < c \). Our next result yields a much stronger statement in this vein.

**Theorem 3.** Every compactum \( X \) with \( \text{rs}(X) > 1 \) contains a point \( p \) such that

\[
\chi(p, X) < \text{rs}(X).
\]

**Proof.** As we have seen above, if \( \text{rs}(X) < m \) then \( X \) is scattered, hence \( X \) contains (a dense set of) isolated points, i.e. points of character 1. So assume now that \( \kappa = \text{rs}(X) \geq m > \omega \). Assume also, indirectly, that \( \chi(x, X) \geq \kappa \) holds for all \( x \in X \).

By definition, there is a sequence \( \langle S_\alpha : \alpha \in \kappa \rangle \) such that each \( S_\alpha \) is a right separated subspace of \( X \) and \( X = \bigcup \{S_\alpha : \alpha \in \kappa \} \). By transfinite recursion on \( \alpha \in \kappa \) we shall then define a decreasing sequence of non-empty closed sets \( H_\alpha \subset X \) such that

\[
\psi(H_\alpha, X) = \chi(H_\alpha, X) \leq |
\alpha| + \omega,
\]

moreover \( H_\alpha \cap S_\alpha = \emptyset \) holds for each \( \alpha \in \kappa \). Since, by the compactness of \( X \), we have \( \bigcap \{H_\alpha : \alpha \in \kappa \} \neq \emptyset \), this clearly leads to a contradiction.

So let \( \alpha \in \kappa \) and assume that \( H_\beta \) has been suitably defined for each \( \beta < \alpha \). Let us set then

\[
\widetilde{H}_\alpha = \bigcap \{H_\beta : \beta < \alpha \}.
\]

Clearly we have

\[
\psi(\widetilde{H}_\alpha, X) = \chi(\widetilde{H}_\alpha, X) \leq |
\alpha| + \omega,
\]

hence if \( \widetilde{H}_\alpha \cap S_\alpha = \emptyset \) holds then we may set \( H_\alpha = \widetilde{H}_\alpha \). So assume now that \( \widetilde{H}_\alpha \cap S_\alpha \neq \emptyset \) and therefore it has an isolated point, say \( x_\alpha \). But \( x_\alpha \) cannot be an isolated point of \( \widetilde{H}_\alpha \) since otherwise we would have \( \chi(x_\alpha, X) \leq |\alpha| + \omega < \kappa \). Thus if \( U \) is any open neighbourhood of \( x_\alpha \) such that \( U \cap (\widetilde{H}_\alpha \cap S_\alpha) = \{x_\alpha \} \), then \( U \cap (\widetilde{H}_\alpha \setminus \{x_\alpha \}) \neq \emptyset \) and \( \psi(U \cap (\widetilde{H}_\alpha \setminus \{x_\alpha \}), X) \leq |\alpha| + \omega \). We may then finish by defining \( H_\alpha \) as any non-empty closed subset of \( U \cap (\widetilde{H}_\alpha \setminus \{x_\alpha \}) \) that also satisfies \( \psi(H_\alpha, X) \leq |\alpha| + \omega \). The existence of such a closed subset is obvious from the regularity of the space \( X \). \( \square \)

It immediately follows from Theorem 3 that if \( X \) is a compactum satisfying \( \text{dis}(X) = \omega_1 \) then the points of first countability are dense (even \( G_\delta \)-dense) in \( X \). This leads us to the following weaker version of our main Problem 2:

**Problem 4.** Is it provable that \( \text{dis}(X) = c \) for each first countable crowded compactum \( X \)?

(Note that, by Archangelskii’s theorem, we have \( \text{dis}(X) \leq |X| = c \) in this case.) We are sorry to admit that we could not answer this problem, however we do have a partial positive solution to Problem 2 for hereditarily normal spaces.

**Theorem 5.** If \( X \) is a hereditarily normal crowded compactum then either \( \text{ls}(X) \geq c \) or \( \text{rs}(X) \geq c \) holds, hence surely \( \text{dis}(X) \geq c \).

**Proof.** Assume, indirectly, that we have both \( \text{ls}(X) < c \) and \( \text{rs}(X) < c \). As we have noted above, we may also assume that \( X \) is separable and consequently \( \mathcal{G}(X) \), the number of regular open subsets of \( X \), is equal to \( c \). Of course, in this case we also have \( |X| = c \).
Let us assume now that $\kappa$ is a regular cardinal. Then first using $ls(X) < \kappa$ and then $rs(X) < \kappa$ it is easy to find a subset of $X$ of size $\kappa$ that is both left and right separated. But then, see e.g. [3, 2.12], there is also a discrete subset $D \subset X$ with $|D| = \kappa$. If, on the other hand, $\kappa$ is singular then a similar argument yields us for each $\kappa < \kappa$ a discrete subset of $X$ of size $\kappa$, in particular a discrete $D \subset X$ with $|D| = cf(\kappa)$. Thus we have established that $X$ contains a discrete subspace of size $cf(\kappa)$ whether or not $\kappa$ is regular.

But then the hereditary normality of $X$ implies $\theta(X) \geq 2^{cf(\kappa)} = 2^{\omega \cdot cf(\kappa)} = \omega \cdot cf(\kappa) > \kappa$, a contradiction. (The finishing argument is widely known as Jones’ lemma, see e.g. [1, 2.1.10].) $\square$

Of course, Theorem 5 would be much more esthetic if one could prove that say $ls(X) \geq \kappa$ is always valid for a crowded $T_5$ space $X$ (or the same with $ls(X)$ replaced with $rs(X)$).

Now we turn to our last theorem that establishes a rather surprising connection between certain continuous maps and the cardinal function $\text{dis}(X)$. This result also sheds some light on potential counterexamples to Problem 4. A space is perfect if all closed sets are $G_\delta$.

**Theorem 6.** Let $f : X \to Y$ be a continuous surjection from a countably compact $T_2$ space $X$ onto a perfect $T_3$ space $Y$. Then we have

$$\left| \{y \in Y : f^{-1}(y) \text{ is countable}\} \right| \leq \text{dis}(X).$$

Moreover, if $\text{dis}(X) = \omega$ then we even have $|Y| \leq \omega$.

**Proof.** Let us start by noting that $Y$ is also countably compact, being the continuous image of $X$. This implies that $Y$ is first countable because any $G_\delta$ point in a countably compact $T_3$ space has countable character. This in turn implies that $f$ is a closed map. Indeed, if $F \subset X$ is closed then $f[F]$ is a countably compact subset of $Y$ and as such it is closed in $Y$ because $Y$ is first countable and $T_2$.

Next we show that for any set $A \subset X$ we have

$$|A \setminus f^{-1}(f[A'])| \leq \omega,$$

where, as usual, $A'$ denotes the derived set of all limit points of the set $A$. To see this, we first note that by our above remark $f[A']$ is closed and hence a $G_\delta$ set in $Y$. Thus we may write

$$f[A'] = \bigcap \{G_n : n < \omega\},$$

where each set $G_n$ is open in $Y$. Consequently, we have

$$f^{-1}(f[A']) = \bigcap \{f^{-1}(G_n) : n < \omega\}.$$

Now, for each $n < \omega$ we have $A' \subset f^{-1}(G_n)$, hence the countable compactness of $X$ implies that $A \setminus f^{-1}(G_n)$ is finite, consequently

$$A \setminus f^{-1}(f[A']) = \bigcup \{A \setminus f^{-1}(G_n) : n < \omega\}$$

is indeed countable.

Let us assume now that $Y$ is uncountable. We claim that in this case for every discrete subspace $D$ of $X$ there is a closed set $F \subset X \setminus D$ such that $f[F]$ is uncountable as well.

To see this, we distinguish two cases. First, if $f[D]$ is countable then $Y \setminus f[D]$ is an uncountable $F_\sigma$-set, hence clearly there is an uncountable closed set $Z$ in $Y$ that is disjoint from $f[D]$. Obviously, then $F = f^{-1}(Z)$ is as required. (In this case we have not used that $D$ is discrete.)

If, on the other hand, $f[D]$ is uncountable then by the previous observation we have $|D \setminus f^{-1}(f[D'])| \leq \omega$, hence $|f[D]| = |f[D']| > \omega$. But ‘$D$ is discrete’ just means that $D \cap D' = \emptyset$, hence in this case we may simply set $F = D'$. Now assume that we have $\text{dis}(X) = \omega$, hence

$$X = \bigcup \{D_n : n < \omega\}$$

where each $D_n$ is a discrete subspace of $X$. Our aim is to show that in this case $Y$ is countable. Indeed, if this were false then, using the above claim, we could define by a straightforward recursion a decreasing sequence
\( (F_n; n < \omega) \) of closed subsets of \( X \) such that \( F_n \cap D_n = \emptyset \) and \( f[F_n] \) is uncountable for each \( n < \omega \). But then we have \( \bigcap (F_n; n < \omega) \neq \emptyset \) as \( X \) is countably compact, contradicting that \( X \) is covered by the \( D_n \)'s. We have thus established the second part of the theorem, namely that \( \text{dis}(X) = \omega \) implies \( |Y| \leq \omega \).

So now we turn to the first (main) part. Let us start by noting that every countable closed subset of \( X \) is in fact compact. However, it is well known that any countable compact \( T_2 \) space is homeomorphic to a (countable successor) ordinal (taken, of course, with its natural order topology). With this in mind, we introduce here the following piece of notation: If \( Z \) is any topological space that is homeomorphic to some ordinal then \( \alpha(Z) \) will denote the smallest such ordinal. We are going to make use of the following easy to prove fact: If \( \alpha(Z) \) is a successor ordinal then there is a point \( z \in Z \) such that

\[
\alpha(Z \setminus \{z\}) < \alpha(Z).
\]

Let us now denote by \( I(\xi) \) the following statement: For every continuous surjection \( f \) from a countably compact \( T_2 \) space \( X \) onto a perfect \( T_3 \) space \( Y \) we have

\[
|\{ y \in Y : \alpha(f^{-1}(y)) \leq \xi \}| \leq \text{dis}(X).
\]

Since in this part of the proof we may assume that \( \text{dis}(X) > \omega \), it will clearly suffice to prove that \( I(\xi) \) holds for all countable ordinals \( \xi \). Indeed, this is so because if \( f^{-1}(y) \) is countable then \( \alpha(f^{-1}(y)) \) exists and is a countable ordinal.

Of course, the proof will proceed by transfinite induction. Since \( \alpha(f^{-1}(y)) \) is always a successor, the limit steps of the induction are trivial. So assume now that \( \xi = \eta + 1 \) and \( I(\eta) \) is valid. We want to show that then so is \( I(\xi) \).

Assume, indirectly, that this is not the case, hence

\[
|\{ y \in Y : \alpha(f^{-1}(y)) = \xi \}| > \text{dis}(X).
\]

Let \( Z \) be the set that appears on the left-hand side of this inequality. For each \( y \in Z \) we have \( \alpha(f^{-1}(y)) = \xi = \eta + 1 \) and hence we may pick by the above remark a point \( x_y \in f^{-1}(y) \) such that

\[
\alpha(f^{-1}(y) \setminus \{x_y\}) \leq \eta.
\]

Since \( |Z| > \text{dis}(X) \) we may clearly find a subset \( Z_0 \subseteq Z \) with \( |Z_0| > \text{dis}(X) \) such that \( D = \{x_y : y \in Z_0\} \) is a discrete subspace of \( X \). Recall that we have \( |D \setminus f^{-1}(f[D'])| \leq \omega \), hence if we set \( Z_1 = f[D] \cap f[D'] \) then \( Z_1 \subseteq Z_0 \) and \( |Z_0 \setminus Z_1| \leq \omega \), consequently we have \( |Z_1| = |Z_0| > \text{dis}(X) \).

Let us now consider the restriction of \( g \) to the closed subspace \( D' \) of \( X \). Then \( g \) is a continuous surjection from \( D' \) onto \( f[D'] \), hence the inductive hypothesis \( I(\eta) \) may be applied to it to conclude that

\[
|\{ y \in f[D'] : \alpha(g^{-1}(y)) \leq \eta \}| \leq \text{dis}(D') \leq \text{dis}(X).
\]

On the other hand, we have \( f[D'] \supseteq Z_1 \) and for each \( y \in Z_1 \),

\[
g^{-1}(y) \subset f^{-1}(y) \setminus \{x_y\}
\]

holds because \( D \cap D' = \emptyset \). But then, by the choice of \( x_y \), for all \( y \in Z_1 \) we have \( \alpha(g^{-1}(y)) \leq \eta \). This is a contradiction since \( |Z_1| > \text{dis}(X) \).

This contradiction completes the proof of the transfinite induction and with it the proof of our theorem. \( \square \)

Assume now that \( X \) is a crowded first countable compactum with \( \text{dis}(X) < \aleph_0 \), i.e. a counterexample to Problem 4. Then any uncountable closed subspace of \( X \) is of cardinality \( \aleph_0 \) and non-scattered. Hence it is an immediate corollary of Theorem 6 that if e.g. \( f \) is any continuous surjection of \( X \) onto the interval \([0, 1]\) (and such maps always exist) then for almost all (more precisely: for all but dis\((X)\) many) points \( r \in [0, 1] \) we have \( |f^{-1}(r)| = \aleph_0 \). In some, admittedly non-precise, sense this means that a counterexample to Problem 4 must be “complicated”.

References