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Amsterdam Properties of $C_p(X)$ Imply Discreteness of X

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Abstract. We prove, among other things, that if $C_p(X)$ is subcompact in the sense of de Groot, then the space X is discrete. This generalizes a series of previous results on completeness properties of function spaces.

Introduction

Historically, completeness properties were designed to represent some facets of compactness in non-compact spaces, so all of them are primarily generalizations of compactness. The real line gives a clear idea of the fact that local compactness also has strong completeness properties. Other generalizations, which nowadays are classical, are Čech-completeness and the Baire property.

However, there are plenty of important spaces (pseudocompact ones or products of the real lines, for example) which are not necesarily Čech-complete but still have some intuitively clear completeness properties. To capture the quintessence of completeness in products, Oxtoby [Ox] introduced the notion of pseudocompleteness; its importance can be seen from the facts that it is preserved by arbitrary products and that a metrizable space is pseudocomplete if and only if it has a dense Čech-complete subspace. Choquet [Ch] used strategies of topological games to define two classes of complete spaces (called nowadays *Choquet spaces* and *strong Choquet spaces*); these classes are productive and have many other nice categorical properties; in particular, all Choquet spaces are Baire.

The school of de Groot, on the other hand, tried to express in more general terms the fact that, apart from completeness, any Čech-complete space is an absolute G_δ; this, evidently, required properties stronger than pseudocompleteness. After proving to be very useful, they were baptized *Amsterdam properties* (the reader can find all definitions and technicalities in Section 1).

Next, all well-known classes of spaces were to be checked for some (or all) completeness properties. The turn of spaces $C_p(X)$ came in 1980 when Lutzer and McCoy [LM] characterized the Baire property in $C_p(X)$ and proved that Čech-completeness of $C_p(X)$ takes place if and only if X is countable and discrete. It turned out that pseudocompleteness of $C_p(X)$ does not imply discreteness of X but does imply that

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each countable subset of X is closed and discrete. They also gave equivalences of pseudocompleteness of $C_p(X)$ for wide classes of spaces.

However, if $C_p(X)$ is homeomorphic to some power of the reals, then X is discrete [Tk1]. It is folklore (and easy to see) that if $C_p(X)$ is complete as a uniform space or is a continuous linear image of a power of the real line, then X is also discrete.

Additional results on discreteness of X have been proved given some completeness property of $C_p(X)$. For example, if $C_p(X)$ is an F_σ-subset of $\mathbb{R}^X$ or G_δ or even $G_{\delta\sigma}$ in $\mathbb{R}^X$, then X is discrete (see [LM, DGLM, vM]). The space X also must be discrete if $C_p(X)$ is a closed continuous image of $\mathbb{R}^X$ (see [Tk3]) or if it is pseudocomplete and realcompact (see [Tk2]).

Following the mentioned line of research in this paper, we prove that if $C_p(X)$ is subcompact, then X is discrete; since subcompactness is the weakest of the Amsterdam properties, every one of those in $C_p(X)$ implies discreteness of X. This result shows that it is time to see what happens if $C_p(X)$ has a dense complete subspace. It is not even clear whether X has to be discrete if $C_p(X)$ contains a dense copy of a power of the real line. This was formulated as an open question in [Tk3]. We only succeeded in proving that, under Martin's axiom, if $C_p(X)$ contains some dense copy of $\mathbb{R}^\kappa$ for $\kappa < \mathfrak{c}$, then X is discrete.

1 Notation and Terminology

All spaces are assumed to be Tychonoff. If X is a space, then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \setminus \{\varnothing\}$. The set $\mathbb{R}$ is the real line with its usual topology and $\mathbb{I} = [0,1] \subset \mathbb{R}$. Given Tychonoff spaces X and Y, the symbol $C_p(X,Y)$ stands for the set of all continuous functions from X to Y endowed with the pointwise convergence topology; if $Y = \mathbb{R}$, then we write $C_p(X)$ instead of $C_p(X,Y)$. See [Ar] for a systematic presentation.

A space Y is called *pseudocomplete* if it has a sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases such that for any family $\{B_n : n \in \omega\}$ with $B_n \in \mathcal{B}_n$ and $\overline{B}_{n+1} \subset B_n$ for each $n \in \omega$, we have $\bigcap_{n\in\omega} B_n \neq \varnothing$. Two sets $A, B \subset Y$ are said to be *completely separated* if there exists a continuous function $f\colon Y \to \mathbb{R}$ such that $f(a) \leq 0$ for any $a \in A$ while $f(b) \geq 1$ for each $b \in B$; we consider that the empty set is completely separated from any subset of Y.

A family $\mathcal{U} \subset \tau^*(Y)$ is called *a regular filterbase* if, for any $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $\overline{W} \subset U \cap V$. A space Y is *subcompact* if it has a base $\mathcal{B} \subset \tau^*(Y)$ such that every regular filterbase $\mathcal{U} \subset \mathcal{B}$ has non-empty intersection. The space Y is *base-compact* if it has a base $\mathcal{B}$ such that $\bigcap\{\overline{U} : U \in \mathcal{U}\} \neq \varnothing$ for any family $\mathcal{U} \subset \mathcal{B}$ with the finite intersection property. If $\bigcap\{\overline{U} : U \in \mathcal{U}\} \neq \varnothing$ for any $\mathcal{U} \subset \mathcal{B}$ such that $\{\overline{U} : U \in \mathcal{U}\}$ has the finite intersection property, then Y is called *regularly co-compact*. Regular co-compactness, base compactness and subcompactness are also called *Amsterdam properties.*[1]

The rest of our notation is standard and follows [En, AL].

[1]There is another property, called co-compactness, that is usually included in the list of Amsterdam properties. Every regularly co-compact space is co-compact, but the Sorgenfrey line shows that the converse is false. The Sorgenfrey line also shows that co-compactness does not imply base-compactness.

2 Amsterdam Properties in Function Spaces

It turns out that even subcompactness of $C_p(X)$ implies that X is discrete. We will also establish that it is consistent that if $C_p(X)$ contains a dense copy of $\mathbb{R}^{\omega_1}$, then it is homeomorphic to $\mathbb{R}^{\omega_1}$.

Theorem 2.1 *Suppose that there is a subcompact subspace $C \subset C_p(X)$ with the following properties:*

(i) *if $f, g \in C$, then $f \cdot g \in C$ and $\max\{f, g\} \in C$;*
(ii) *if A and B are completely separated subspaces of X, then there exists $f \in C$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.*

Then the space X is discrete.

Proof Let $\mathcal{B}$ be a base which witnesses subcompactness of C. Given a function $g \in C$, a finite set $F \subset X$, and $\epsilon > 0$, let $O(g, F, \epsilon) = \{h \in C : |h(x) - g(x)| < \epsilon$ for any $x \in F\}$; the sets $O(g, F, \epsilon)$ form a local base at g in the space C. Denote by C^+ the set of all non-negative functions from C. Our plan is to prove first that any two disjoint countable sets are completely separated and then establish the same for all pairs of disjoint sets of higher cardinalities. The reasonings are similar but, unfortunately, there are some technical obstacles which do not allow us to give the same proof for all cardinalities at once.

Lemma 2.2 *With hypotheses and notation as in Theorem* 2.1*, suppose that Q is a countable subset which is completely separated from any finite subset of $X \setminus Q$. Then Q is completely separated from any countable subset of $X \setminus Q$.*

Proof Observe that the set Q may be finite in which case it is trivially completely separated from any finite subset of $X \setminus Q$.

Note that to say that Q is completely separated from each finite subset of $X \setminus Q$ is equivalent to saying that Q is a closed subset of X, because X is completely regular. Our formulation of Lemma 2.2 is chosen to emphasize the analogy between Lemma 2.2 and the uncountable cases considered below.

So take any countably infinite set $P \subset X \setminus Q$ and let $\{p_n : n \in \omega\}$ be a faithful enumeration of P. Let us also fix some enumeration $\{q_n : n \in \omega\}$ (with repetitions if Q is finite) of the set Q. There is a function $f_0 \in C$ such that $f_0(p_0) = 1$ and $f_0(Q) \subset \{0\}$; passing from f_0 to $(f_0)^2$ if necessary, we can assume that $f_0 \in C^+$. Pick any $U_0 \in \mathcal{B}$ with $f_0 \in U_0$; there exist a finite set $F_0 \subset X$ and $\epsilon_0 \in (0, 1)$ such that $\{p_0, q_0\} \subset F_0$ and $\overline{O(f_0, F_0, \epsilon_0)} \subset U_0$.

Choose $g_0 \in C$ for which $g_0(F_0 \setminus Q) \subset \{1\}$ and $g_0(Q) \subset \{0\}$; again, there is no loss of generality to assume that $g_0 \in C^+$. Fix a set $V_0 \in \mathcal{B}$ with $g_0 \in V_0$; there exists a finite set $H_0 \subset X$ such that $F_0 \subset H_0$ and $\overline{O(g_0, H_0, \eta_0)} \subset V_0$ for some $\eta_0 \in (0, 1)$.

Suppose that $n \in \omega$ and we have chosen elements $U_0, V_0, \dots, U_n, V_n$ of the base $\mathcal{B}$ together with finite subsets $F_0, H_0, \dots, F_n, H_n$ of the space X as well as functions $f_0, g_0, \dots, f_n, g_n \in C^+$ and real positive numbers $\epsilon_0, \eta_0, \dots, \epsilon_n, \eta_n$ with the following properties:

(i) $\overline{O(f_k, F_k, \epsilon_k)} \subset U_k$ and $\overline{O(g_k, H_k, \epsilon_k)} \subset V_k$ for every $k \le n$;
(ii) $H_k \subset F_{k+1}$, $U_{k+1} \subset O(f_k, F_k, \epsilon_k)$, and $V_{k+1} \subset O(g_k, H_k, \eta_k)$ whenever $k < n$;

(iii) $\{p_0, q_0, \ldots, p_k, q_k\} \subset F_k \subset H_k$ and $\epsilon_k, \eta_k \in (0, 2^{-k})$ for every $k \leq n$;
(iv) $f_{k+1}((H_k \setminus F_k) \setminus Q) \subset \{1\}$ for every $k < n$;
(v) $f_k(Q) \subset \{0\}$, $g_k(Q) \subset \{0\}$ and $g_k((F_k \setminus H_{k-1}) \setminus Q) \subset \{1\}$ (where $H_{-1} = \varnothing$) for every $k \leq n$;
(vi) $f_{k+1}|F_k = f_k$ and $g_{k+1}|H_k = g_k$ for every $k < n$.

Apply property (i) of the set C (see Theorem 2.1) to find a function $\varphi_0 \in C^+$ such that $\varphi_0((H_n \setminus F_n) \setminus Q) \subset \{1\}$ and $\varphi_0(Q) \subset \{0\}$. There exists a function $\varphi_1 \in C^+$ such that $\varphi_1(F_n) \subset \{0\}$ and $\varphi_1((H_n \setminus F_n) \setminus Q) \subset \{1\}$. The function $\varphi_0 \cdot \varphi_1$ is equal to zero on $F_n \cup Q$ and equals 1 on $(H_n \setminus F_n) \setminus Q$. We will also need a function $\varphi \in C^+$ such that $\varphi(F_n) \subset \{1\}$ and $\varphi((H_n \setminus F_n) \setminus Q) \subset \{0\}$.

It is clear that the function $f_{n+1} = \max\{f_n \cdot \varphi, \varphi_0 \cdot \varphi_1\} \in C$ is non-negative and $f_{n+1}|F_n = f_n$ while $f_{n+1}((H_n \setminus F_n) \setminus Q) \subset \{1\}$ and $f_{n+1}(Q) \subset \{0\}$. Take $U_{n+1} \in \mathcal{B}$ such that $f_{n+1} \in U_{n+1} \subset O(f_n, F_n, \epsilon_n)$. There exist a finite set $F_{n+1} \subset X$ and a number $\epsilon_{n+1} \in (0, 2^{-n-1})$ such that $H_n \cup \{p_0, q_0, \ldots, p_{n+1}, q_{n+1}\} \subset F_{n+1}$ and $\overline{O(f_{n+1}, F_{n+1}, \epsilon_{n+1})} \subset U_{n+1}$.

Analogously, we can construct a function $g_{n+1} \in C^+$ such that $g_{n+1}|H_n = g_n$ while $g_{n+1}((F_{n+1} \setminus H_n) \setminus Q) \subset \{1\}$ and $g_{n+1}(Q) \subset \{0\}$. Take $V_{n+1} \in \mathcal{B}$ such that $g_{n+1} \in V_{n+1} \subset O(g_n, H_n, \eta_n)$. There exists a finite set $H_{n+1} \subset X$ and a number $\eta_{n+1} \in (0, 2^{-n-1})$ such that $F_{n+1} \subset H_{n+1}$ and $\overline{O(g_{n+1}, H_{n+1}, \eta_{n+1})} \subset V_{n+1}$.

It is straightforward that after step $n + 1$ we still have all properties (i)–(vi) for the relevant sets and functions, so our inductive procedure gives us sequences

$$\{f_n, g_n : n \in \omega\} \subset C, \quad \{U_n, V_n : n \in \omega\} \subset \mathcal{B}$$

as well as families $\{F_n, H_n : n \in \omega\}$ and $\{\epsilon_n, \eta_n : n \in \omega\}$ for which the properties (i)–(vi) hold for all $n \in \omega$.

It follows from the properties (i) and (ii) that the families $\mathcal{U} = \{U_n : n \in \omega\}$ and $\mathcal{V} = \{V_n : n \in \omega\}$ are regular filterbases so we can pick $f \in \bigcap \mathcal{U}$ and $g \in \bigcap \mathcal{V}$. The properties (i)–(vi) imply that $P \subset f^{-1}(1) \cup g^{-1}(1)$ and $Q \subset f^{-1}(0) \cap g^{-1}(0)$, so P and Q are completely separated. ■

Lemma 2.3 *With hypotheses and notation as in Theorem* 2.1*, any two countable disjoint subsets of X are completely separated.*

Proof Let P be a finite subset of X; since it is completely separated from any finite $Q \subset X \setminus P$, we can apply Lemma 2.2 to see that it is completely separated from any countable subset of $X \setminus P$. This means that any countable subset of X is completely separated from any finite set from its complement. Applying Lemma 2.2 once more, we conclude that any two disjoint countable subsets of X are completely separated. ■

For any set $A \subset X$ and $f \in C$, let $G(A, f) = \{g \in C : g|A = f|A\}$; if $A = \varnothing$, then we consider that $G(A, f) = C$ for any $f \in C$.

Lemma 2.4 *With hypotheses and notation as in Theorem* 2.1*, for any $f \in C$, if $\mathcal{V} \subset \mathcal{B}$ and $f \in \bigcap \mathcal{V}$, then there exists a regular filterbase $\mathcal{V}' \subset \mathcal{B}$ such that $|\mathcal{V}'| \leq |\mathcal{V}| \cdot \omega$ while $\mathcal{V} \subset \mathcal{V}'$ and $f \in \bigcap \mathcal{V}'$.*

Proof The proof can be easily derived from the fact that for any $U, V \in \mathcal{V}$, there is $W \in \mathcal{B}$ for which $f \in W \subset \overline{W} \subset U \cap V$. ■

Lemma 2.5 *With hypotheses and notation as in Theorem* 2.1*, suppose that* $\mathcal{U} \subset \mathcal{B}$ *and* $f \in \bigcap \mathcal{U}$*. Then for any set* $A \subset X$*, we can find a set* $A' \subset X$ *with* $A \subset A'$ *and a regular filterbase* $\mathcal{U}' \subset \mathcal{B}$ *with* $\mathcal{U} \subset \mathcal{U}'$ *such that* $\max\{|\mathcal{U}'|, |A'|\} \le |A| \cdot |\mathcal{U}| \cdot \omega$ *and* $\bigcap \mathcal{U}' = G(A', f)$*.*

Proof Let $\mu = |A| \cdot |\mathcal{U}| \cdot \omega$; since the sets $O(f, F, \epsilon)$ form a local base at f, for any $B \in \mathcal{B}$ with $f \in B$ there is a finite $F(B) \subset X$ and $n(B) \in \mathbb{N}$ for which $O(f, F(B), \frac{1}{n(B)}) \subset B$.

For every finite $F \subset A$, the set $O(f, F, \frac{1}{n})$ is an open neighbourhood of f in the space C for each $n \in \mathbb{N}$. Therefore there exists a set $W(F, n) \in \mathcal{B}$ such that $f \in W(F, n) \subset O(f, F, \frac{1}{n})$.

Let $A_0 = A$; the family $\mathcal{U}_0 = \mathcal{U} \cup \{W(\{y\}, k) : y \in A_0,\ k \in \mathbb{N}\} \subset \mathcal{B}$ contains f in its intersection and $|\mathcal{U}_0| \le \mu$. Proceeding inductively, assume that we have a set $A_n \subset X$ and a family $\mathcal{U}_n \subset \mathcal{B}$ such that $|A_n| \cdot |\mathcal{U}_n| \le \mu$ and $f \in \bigcap \mathcal{U}_n$. By Lemma 2.4 there exists a regular filterbase $\mathcal{F} \subset \mathcal{B}$ such that $f \in \bigcap \mathcal{F}$ while $|\mathcal{F}| \le \mu$ and $\mathcal{U}_n \subset \mathcal{F}$.

Let $A_{n+1} = \bigcup\{F(B) : B \in \mathcal{F}\}$ and $\mathcal{U}_{n+1} = \mathcal{F} \cup \{W(\{y\}, k) : y \in A_{n+1},\ k \in \mathbb{N}\}$. It is easy to see that $|A_{n+1}| \cdot |\mathcal{U}_{n+1}| \le \mu$, so our inductive construction gives us increasing sequences $\{A_n : n \in \omega\}$ and $\{\mathcal{U}_n : n \in \omega\}$. If we let $A' = \bigcup\{A_n : n \in \omega\}$ and $\mathcal{U}' = \bigcup\{\mathcal{U}_n : n \in \omega\}$, then A' and $\mathcal{U}'$ are as promised. ■

Lemma 2.6 *With hypotheses and notation as in Theorem* 2.1*, suppose that* κ *is an uncountable cardinal, any two disjoint subsets of* X *of cardinality* $< \kappa$ *are completely separated, and a set* $P \subset X$ *of cardinality* $\le \kappa$ *is completely separated from any* $Q \subset X \setminus P$ *with* $|Q| < \kappa$*. Then* P *is completely separated from any set* $R \subset X \setminus P$ *with* $|R| = \kappa$*.*

Proof Fix $R \subset X \setminus P$ of cardinality κ and take a faithful enumeration $\{z_\alpha : \alpha < \kappa\}$ of the set R. We can also choose an increasing family $\mathcal{P} = \{P_\alpha : \alpha < \kappa\}$ of subsets of P such that $|P_\alpha| \le |\alpha| \cdot \omega$ for any $\alpha < \kappa$ and $\bigcup \mathcal{P} = P$. Observe first that

(i) if $A \subset X \setminus P$ and $|A| < \kappa$, then for any $f \in C^+$ there exists $g \in C^+$ such that $g|A = f|A$ and $g(P) \subset \{0\}$.

Indeed, take any $\varphi \in C$ such that $\varphi(P) \subset \{0\}$ and $\varphi(A) \subset \{1\}$; then the function $g = \varphi^2 \cdot f$ is as promised in (i). Our following step is to prove that

(ii) for any disjoint sets $A, B \subset X \setminus P$ with $|A| < \kappa$ and $|B| < \kappa$, if $f \in C^+$, then there is $g \in C^+$ such that $g|(A \cup P) = f|(A \cup P)$ and $g(B) \subset \{1\}$.

By property (ii) of the set C (see Theorem 2.1), there exists a function $\varphi_0 \in C$ such that $\varphi_0(B) \subset \{1\}$ and $\varphi_0(P) \subset \{0\}$. By the same reason we can find a function $\varphi_1 \in C$ such that $\varphi_1(B) \subset \{1\}$ and $\varphi_1(A) \subset \{0\}$. If $\varphi = (\varphi_0 \cdot \varphi_1)^2$, then $\varphi \in C^+$ while $\varphi(B) \subset \{1\}$ and $\varphi(A \cup P) \subset \{0\}$. Analogously, there exists a function $\delta \in C^+$ such that $\delta(B \cup P) \subset \{0\}$ and $\delta(A) \subset \{1\}$. Fix a function $\mu \in C$ with $\mu(P) \subset \{1\}$ and $\mu(A \cup B) \subset \{0\}$. Then the function $\nu = \max\{\mu^2, \delta\}$ belongs to C^+ while $\nu(A \cup P) \subset \{1\}$ and $\nu(B) \subset \{0\}$.

The function $h = f \cdot \nu \in C^+$ coincides with f on the set $A \cup P$ and $h(B) \subset \{0\}$, so $g = \max\{h, \varphi\}$ is equal to 1 on B and coincides with f on $A \cup P$, *i.e.,* (ii) is proved.

Define f_0 and g_0 to be the functions which are identically zero on X; it follows from property (ii) of the set C (see Theorem 2.1) that $\{f_0, g_0\} \subset C^+$. Let $E_0 = F_0 = \varnothing$; we will also need empty families $\mathcal{U}_0, \mathcal{V}_0 \subset \mathcal{B}$.

Proceeding by transfinite induction, assume that $\beta < \kappa$ and we have constructed a set $\{f_\alpha, g_\alpha : \alpha < \beta\} \subset C^+$, a β-sequence $\{\mathcal{U}_\alpha, \mathcal{V}_\alpha : \alpha < \beta\}$ of regular filterbases contained in $\mathcal{B}$, and a family $\{E_\alpha, F_\alpha : \alpha < \beta\}$ of subsets of X with the following properties:

(iii) $|E_\alpha \cup F_\alpha| \leq |\alpha| \cdot \omega$ and $|\mathcal{U}_\alpha \cup \mathcal{V}_\alpha| \leq |\alpha| \cdot \omega$ for any $\alpha < \beta$;
(iv) $E_\gamma \subset E_\alpha$, $\mathcal{U}_\gamma \subset \mathcal{U}_\alpha$ and $F_\gamma \subset F_\alpha$, $\mathcal{V}_\gamma \subset \mathcal{V}_\alpha$ if $\gamma < \alpha < \beta$;
(v) $f_\alpha|E_\gamma = f_\gamma$, $g_\alpha|F_\gamma = g_\gamma$ and $F_\gamma \subset E_\alpha \subset F_\alpha$ whenever $\gamma < \alpha < \beta$;
(vi) $\{z_\gamma\} \cup P_\gamma \subset E_\alpha$ whenever $\gamma < \alpha < \beta$;
(vii) $E_\alpha \setminus P \subset f_\alpha^{-1}(1) \cup g_\alpha^{-1}(1)$ and $P \subset f_\alpha^{-1}(0) \cap g_\alpha^{-1}(0)$ for all $\alpha < \beta$;
(viii) $\bigcap \mathcal{U}_\alpha = G(E_\alpha, f_\alpha)$ and $\bigcap \mathcal{V}_\alpha = G(F_\alpha, g_\alpha)$ for any $\alpha < \beta$.

If β is a limit ordinal, then let $E_\beta = \bigcup_{\alpha<\beta} E_\alpha$ and $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$; observe that it follows from (v) that $E_\beta = F_\beta$. If $\mathcal{U}_\beta = \bigcup_{\alpha<\beta} \mathcal{U}_\alpha$ and $\mathcal{V}_\beta = \bigcup_{\alpha<\beta} \mathcal{V}_\alpha$, then it follows from (iv) that both $\mathcal{U}_\beta$ and $\mathcal{V}_\beta$ are regular filterbases, so there exist $f \in \bigcap \mathcal{U}_\beta$ and $g \in \bigcap \mathcal{V}_\beta$. An immediate consequence of (v) and (viii) is that we have $\bigcap \mathcal{U}_\beta = G(E_\beta, f)$ and $\bigcap \mathcal{V}_\beta = G(F_\beta, g)$. Apply property (i) of the set C (see Theorem 2.1) to find functions $f_\beta, g_\beta \in C^+$ such that $f_\beta|(E_\beta \setminus P) = f|(E_\beta \setminus P)$, $g_\beta|(F_\beta \setminus P) = g|(F_\beta \setminus P)$ while $f_\beta(P) \subset \{0\}$ and $g_\beta(P) \subset \{0\}$. It follows from the properties (v) and (vii) that $f(E_\beta \cap P) \subset \{0\}$ and $g(F_\beta \cap P) \subset \{0\}$; therefore $f_\beta|E_\beta = f|E_\beta$ and $g_\beta|F_\beta = g|F_\beta$ which shows that the properties (vii) and (viii) hold for $\alpha = \beta$. It is evident that the conditions (iii)–(vi) are also satisfied for all $\alpha \leq \beta$.

Now, assume that $\beta = \lambda + 1$ and let $\gamma < \kappa$ be the minimal ordinal such that $z_\gamma \notin E_\lambda$. If $F'_\lambda = (F_\lambda \setminus E_\lambda) \setminus P$, then the set $F'_\lambda \cup \{z_\gamma\} \subset X \setminus (P \cup E_\lambda)$ has cardinality less than κ so we can apply property (ii) of the set C (see Theorem 2.1) to find a function $f_\beta \in C^+$ such that $f_\beta|(P \cup E_\lambda) = f_\lambda|(P \cup E_\lambda)$ and $f_\beta(F'_\lambda \cup \{z_\gamma\}) \subset \{1\}$. Apply Lemma 2.5 to find a set $E_\beta \subset X$ and a regular filterbase $\mathcal{U}_\beta \subset \mathcal{B}$ such that $\mathcal{U}_\lambda \subset \mathcal{U}_\beta$, $F_\lambda \cup \{z_\gamma\} \cup P_\lambda \subset E_\beta$, $\bigcap \mathcal{U}_\beta = G(E_\beta, f_\beta)$ and, besides, $|\mathcal{U}_\beta| \cdot |E_\beta| \leq |\beta| \cdot \omega$.

Apply property (ii) of the set C (see Theorem 2.1) to find $g_\beta \in C^+$ such that $g_\beta|(F_\lambda \cup P) = g_\lambda|(F_\lambda \cup P)$ while $g_\beta((E_\beta \setminus F_\lambda) \setminus P) \subset \{1\}$. By Lemma 2.5 there is a set $F_\beta \subset X$ and a regular filterbase $\mathcal{V}_\beta \subset \mathcal{B}$ such that $E_\beta \subset F_\beta$ and $\mathcal{V}_\lambda \subset \mathcal{V}_\beta$ while $\bigcap \mathcal{V}_\beta = G(F_\beta, g_\beta)$ and $|F_\beta| \cdot |\mathcal{V}_\beta| \leq |\beta| \cdot \omega$. It is easy to see that the properties (iii)–(viii) are satisfied for the relevant families for all $\alpha \leq \beta$. Therefore our inductive procedure can be continued to obtain a set $\{f_\alpha, g_\alpha : \alpha < \kappa\} \subset C^+$, a κ-sequence $\{\mathcal{U}_\alpha, \mathcal{V}_\alpha : \alpha < \kappa\}$ of regular filterbases contained in $\mathcal{B}$ and a family $\{E_\alpha, F_\alpha : \alpha < \kappa\}$ of subsets of X for which the properties (iii)–(viii) hold for all $\beta < \kappa$.

It is easy to see that both families $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ and $\mathcal{V} = \bigcup_{\alpha<\kappa} \mathcal{V}_\alpha$ are regular filterbases so there are functions $f \in \bigcap \mathcal{U}$ and $g \in \bigcap \mathcal{V}$. It follows from (v), (vi) and (viii) that $f(P) \subset \{0\}$ and $g(P) \subset \{0\}$.

Let $E = \bigcup_{\alpha<\kappa} E_\alpha$; the properties (v) and (vii) show that $E \setminus P \subset f^{-1}(1) \cup g^{-1}(1)$. Property (vi) guarantees that $R \subset E \setminus P$ so P and R are completely separated. ■

To finally establish that X is discrete, it suffices to prove that any two disjoint subsets $A, B \subset X$ are completely separated (actually, it suffices to show that any point is completely separated from its complement). We already saw that this is true if A and

B are countable. Suppose that κ is a cardinal and we have proved that any disjoint $A, B \subset X$ with $|A| < \kappa$, $|B| < \kappa$ are completely separated. If $A \subset X$ and $|A| < \kappa$, then A is completely separated from any $B \subset X \setminus A$ with $|B| < \kappa$, so we can apply Lemma 2.6 to see that A is completely separated from any $B \subset X \setminus A$ with $|B| \leq \kappa$. Thus, every set A of cardinality $\leq \kappa$ is completely separated from any disjoint set B of cardinality $< \kappa$. This shows that we can apply Lemma 2.6 again to conclude that A is completely separated from any disjoint set of cardinality $\leq \kappa$. In other words, any two disjoint sets of cardinality at most κ are completely separated, so our inductive proof can go on to establish that any two disjoint subsets of X are completely separated and hence X is discrete. ■

Corollary 2.7 *If $C_p(X)$ is subcompact, then X is discrete.*

Corollary 2.8 *If $C_p(X, [0, 1])$ is subcompact, then X is discrete.*

Proof It is easy to see that the set $C = C_p(X, [0, 1])$ satisfies the conditions (i) and (ii) of Theorem 2.1. ■

Corollary 2.9 *If either $C_p(X)$ or $C_p(X, [0, 1])$ is regularly co-compact or base-compact, then X is discrete.*

Proof This is because subcompactness is the weakest of our list of Amsterdam properties, so Corollaries 2.7 and 2.8 do the rest. ■

Proposition 2.10 *Under MA+¬CH, if $\kappa < \mathfrak{c}$ is a cardinal, then no space can contain two disjoint dense copies of $\mathbb{R}^\kappa$.*

Proof We can identify $\mathbb{R}^\kappa$ with the subspace $(0, 1)^\kappa$ of the compact space $\mathbb{I}^\kappa$. If $\pi_\alpha \colon \mathbb{I}^\kappa \to \mathbb{I}$ is the projection of $\mathbb{I}^\kappa$ onto its α-th factor for any $\alpha < \kappa$, then the set $\mathbb{I}^\kappa \setminus (0, 1)^\kappa = \bigcup\{\pi_\alpha^{-1}(\{0, 1\}) : \alpha < \kappa\}$ is the union of κ-many compact subsets of $\mathbb{I}^\kappa$. It is standard to show that this implies that $\mathbb{R}^\kappa$ is a G_κ-subset of any space which contains it as a dense subspace.

Now assume that C and D are disjoint dense homeomorphic copies of $\mathbb{R}^\kappa$ in a space Y. By our observation there exist families $\mathcal{K}$ and $\mathcal{L}$ of compact subsets of βY such that $|\mathcal{K}| \leq \kappa$ and $|\mathcal{L}| \leq \kappa$ while $\beta Y \setminus C = \bigcup \mathcal{K}$ and $\beta Y \setminus D = \bigcup \mathcal{L}$.

It is evident that $\mathcal{K}' = \mathcal{K} \cup \mathcal{L}$ is a family of at most κ-many nowhere dense subsets of βY such that $\bigcup \mathcal{K}' = \beta Y$. However, $c(\beta Y) \leq c(D) = \omega$ (recall that the space D is dense in βY and homeomorphic to $\mathbb{R}^\kappa$), so Martin's axiom is applicable to βY to conclude that it cannot be represented as a union of $< \mathfrak{c}$-many nowhere dense sets, a contradiction. ■

Theorem 2.11 *Under Martin's axiom and the negation of CH, if $C_p(X)$ has a dense subspace homeomorphic to $\mathbb{R}^\kappa$ for some cardinal $\kappa < \mathfrak{c}$, then X is discrete.*

Proof If $C_p(X)$ has a dense copy of $\mathbb{R}^\kappa$ and X is not discrete, then take a discontinuous function φ on the set X. Then $\varphi + C_p(X)$ is a dense disjoint copy of $C_p(X)$ in $\mathbb{R}^X$ so we also have two dense disjoint copies of $\mathbb{R}^\kappa$ in $\mathbb{R}^X$ which contradicts Proposition 2.10. ■

3 Open Questions

We give below the list of questions we could not solve while working on this paper. They might be simple or difficult, but they will all require new methods for their solution.

Question 1: Is it consistent with ZFC that there exists a Tychonoff space X such that $X = X_0 \cup X_1$ where every X_i is dense in X, homeomorphic to $\mathbb{R}^{\omega_1}$, and $X_0 \cap X_1 = \varnothing$?

Question 2: Is it true in ZFC that if $C_p(X)$ contains a dense copy of $\mathbb{R}^{\omega_1}$, then X is discrete?

Question 3: Suppose that $C_p(X)$ has a dense regularly co-compact subspace. Must X be discrete?

Question 4: Suppose that $C_p(X)$ has a dense base-compact subspace. Must X be discrete?

Question 5: Suppose that $C_p(X)$ has a dense subcompact subspace. Must X be discrete?

Question 6: Suppose that X is a zero-dimensional space such that $C_p(X, \{0, 1\})$ is subcompact. Must X be discrete?

Question 7: Assume that $C_p(X)$ is a countable union of its closed subcompact subspaces. Must X be discrete?

Question 8: Assume that $C_p(X)$ is a countable union of its closed base-compact subspaces. Must X be discrete?

Question 9: Suppose that $C_p(X, [0, 1])$ has a dense base-compact subspace. Must X be discrete?

Question 10: Suppose that $C_p(X, [0, 1])$ is a countable union of its closed base-compact subspaces. Must X be discrete?

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