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Ohio completeness and products

Désirée Basile, Jan van Mill *

Faculteit Exacte Wetenschappen, Afdeling Wiskunde, Vrije Universiteit, De Boelelaan 1081A, 1081 HV Amsterdam, The Netherlands

Abstract

In [A.V. Arhangel'skiĭ, Remainders in compactifications and generalized metrizability properties, Topology Appl. 150 (2005) 79–90], Arhangel'skiĭ introduced the notion of Ohio completeness and proved it to be a useful concept in his study of remainders of compactifications and generalized metrizability properties. We will investigate the behavior of Ohio completeness with respect to closed subspaces and products. We will prove among other things that if an uncountable product is Ohio complete, then all but countably many factors are compact. As a consequence, $\mathbb{R}^\kappa$ is not Ohio complete, for every uncountable cardinal number κ.

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Keywords: Ohio complete; Product

1. Introduction

All spaces under discussion are Tychonoff.

A space X is *Ohio complete*, Arhangel'skiĭ [1, p. 82], if *every compactification* γX of X has the following property: there exists a G_δ-subset Y of γX such that $X \subseteq Y$ and for every $y \in Y \setminus X$ there exists a G_δ-subset S of γX with $y \in S$ and $S \cap X = \emptyset$. We say that every point $y \in Y \setminus X$ can be *separated* from X by a G_δ-subset of γX. It was proved in [1] that if X is Čech-complete, or Lindelöf, or a p-space, or has a G_δ-diagonal, then X is Ohio complete.

Arhangel'skiĭ introduced the notion of Ohio completeness to study generalized metrizability properties of remainders of compactifications. Special attention was paid to topological groups.

It is obvious that Ohio completeness and realcompactness are strongly related notions. But they are not the same. Simply observe that every locally compact space is Ohio complete but that there are locally compact spaces that are not realcompact. Consider, for example, the familiar ordinal space $W(\omega_1)$.

The aim of this paper is to study some basic properties of Ohio complete spaces. We will show, for example, that a C^*-embedded closed subspace of an Ohio complete space is Ohio complete. As a consequence, a closed subspace of a *normal* Ohio complete space is again Ohio complete. We do not know whether every closed subspace of an Ohio complete space is again Ohio complete. This is, as we believe, a tricky and interesting open problem. We prove that if there is an Ohio complete space X having a closed subspace which is not Ohio complete, then there is a compact space Z such that $X \times Z$ is not Ohio complete. So if such an example exists, then Ohio completeness behaves very

* Corresponding author.
E-mail addresses: basile@dmi.unict.it (D. Basile), vanmill@few.vu.nl (J. van Mill).

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badly with respect to products, even when one of the factors is compact. This motivated us to study products of Ohio complete spaces. Our main result is that if an uncountable product of spaces is a closed subspace of some Ohio complete space, then all but countably many of its factors are compact. As a consequence, no $\mathbb{R}^\kappa$ for uncountable κ is a closed subspace of an Ohio complete space.

It is well known that a space X is realcompact if and only if every point in $\beta X \setminus X$ can be separated from X by a G_δ-subset of βX [2, Theorem 3.11.10]. Hence $\beta\mathbb{R}^{\omega_1}$ is a 'good' compactification of $\mathbb{R}^{\omega_1}$ from the standpoint of Ohio completeness. But by the above, $\mathbb{R}^{\omega_1}$ also has a 'bad' compactification. This shows that '*every compactification*' in the definition of Ohio completeness cannot be weakened to '*some compactification*'.

2. Preliminaries

A space is *crowded* if it has no isolated points.

Let X and Y be disjoint spaces, $M \subseteq X$ closed, and $f : M \to Y$ continuous. Then $X \cup_f Y$ is the space we get from the topological sum $X \oplus Y$ by identifying each set of the form $\{y\} \cup f^{-1}(y)$, where $y \in f(M)$, to a single point. So $X \cup_f Y$ is endowed with the quotient topology with respect to the equivalence relation on $X \oplus Y$ of which

$$\big\{\{y\} \cup f^{-1}(y):\ y \in f(M)\big\}$$

is its collection of nontrivial equivalence classes. The space $X \cup_f Y$ is called the *adjunction space* determined by X, Y and f. See [2, p. 93] for more details.

For a space X, we let $\mathcal{C}(Y)$ denote the collection of all compactifications of X with its standard partial order given by $aX \leqslant bX$ if there exists a continuous function $f : bX \to aX$ which restricts to the identity on X. For a space X we let βX denote its Čech–Stone compactification.

We use standard conventions with respect to ordinals and cardinals. A *cardinal* is an initial ordinal, and an *ordinal* is the set of smaller ordinals. Ordinals are endowed with the discrete topology. Sometimes we need the order topology on a given ordinal. We use the standard notation $W(\alpha)$ for the topological space with underlying set the ordinal α endowed with its order topology. Observe that for ω we do not need to distinguish between $W(\omega)$ and ω since in both cases the topology is the discrete topology.

A space is *zero-dimensional* if it has a base for its topology consisting of *clopen* (= both closed and open) sets.

The following theorem is basically due to Mrowka [7]. For the convenience of the reader we include its proof.

Theorem 2.1. *Let X be a space and $\kappa \geqslant \omega$. Then the following statements are equivalent*:

(1) *X admits a closed embedding in ω^κ,*
(2) *X has a zero-dimensional compactification γX such that*
   (a) *$\mathrm{w}(\gamma X) \leqslant \kappa$,*
   (b) *there is a family $\mathcal{S}$ consisting of closed G_δ-subsets of γX such that $|\mathcal{S}| \leqslant \kappa$ and $\bigcup \mathcal{S} = \gamma X \setminus X$.*

Proof. For (1) $\Rightarrow$ (2), assume that X is a closed subspace of ω^κ. Let γX be the closure of X in $W(\omega+1)^\kappa$.

For (2) $\Rightarrow$ (1), for every $S \in \mathcal{S}$ let $f_S : \gamma X \to W(\omega+1)$ be a continuous function such that $f_S^{-1}(\{\omega\}) = S$. Define $f : \gamma X \to W(\omega+1)^{\mathcal{S}}$ in the obvious way by $f(x)_S = f_S(x)$ $(S \in \mathcal{S})$. It is easy to prove that $f(X) \subseteq \omega^{\mathcal{S}}$ and $f(\gamma X \setminus X) \subseteq W(\omega+1)^{\mathcal{S}} \setminus \omega^{\mathcal{S}}$. It means that $Y = f(X)$ is a closed subspace of $\omega^{\mathcal{S}}$. It is clear that $f \restriction X : X \to f(X)$ is a perfect map. Since $\mathrm{w}(\gamma X) \leqslant \kappa$ and γX is zero-dimensional, we may assume that γX is a subspace of 2^κ. Now define $g : X \to 2^\kappa \times \omega^{\mathcal{S}}$ by $g(x) = \langle x, f(x)\rangle$. Then g is a topological embedding. We claim that $g(X)$ is closed in $2^\kappa \times \omega^{\mathcal{S}} \approx \omega^\kappa$. Indeed, assume that $\langle p, q\rangle \in \overline{g(X)}$. Then $q \in f(X)$ since $f(X)$ is closed. Hence $f^{-1}(q)$ is compact. Assume that $p \notin f^{-1}(q)$. There are disjoint open subsets $U, V \subseteq \omega^\kappa$ such that $f^{-1}(q) \subseteq U$ and $p \in V$. Since $f : X \to f(X)$ is perfect, there is a neighborhood W of q in $\omega^{\mathcal{S}}$ such that $f^{-1}(W \cap f(X)) \subseteq U$. Consider the neighborhood $V \times W$ of $\langle p, q\rangle$. It contains for some $x \in X$ the point $\langle x, f(x)\rangle$. But then, obviously, $x \in V \cap U$, which is a contradiction. So we conclude that $p \in f^{-1}(q)$, i.e., $\langle p, q\rangle \in g(X)$. $\square$

A space X is called a P-space if all of its G_δ's are open. The one-point Lindelöfication of a discrete set of cardinality ω_1 is an example of a Lindelöf P-space of weight ω_1 which is not discrete. Nice examples with many interesting additional properties were constructed by Kunen [6]. We need such an example which has no isolated points along with a 'nice' compactification. Such a space is described in the proof of our next result.

Theorem 2.2. *There is a Lindelöf P-space X with a zero-dimensional compactification γX having the following properties*:

(a) *X is crowded, $|X| = \omega_1$, and $\mathrm{w}(\gamma X) = \omega_1$,*
(b) *there is a family $\mathcal{S}$ consisting of closed G_δ-subsets of γX such that $|\mathcal{S}| = \omega_1$ and $\bigcup \mathcal{S} = \gamma X \setminus X$.*

Proof. Let $Y = W(\omega_1 + 1)$, and let S denote the set of isolated points of Y. That is, S is the set of successor ordinals in ω_1. Then $T = S \cup \{\omega_1\}$ is a Lindelöf P-space of weight ω_1 and Y is a 'nice' compactification of T. But all points of T but one are isolated. So our aim is to modify Y and T.

Consider the space $\mathcal{K}(Y)$ of all nonempty closed subspaces of Y endowed with the Vietoris topology. That is, basic neighborhoods of elements of $\mathcal{K}(Y)$ have the form

$$\langle \mathcal{U} \rangle = \Bigl\{L \in \mathcal{K}(Y)\colon\ \Bigl(L \subseteq \bigcup \mathcal{U}\Bigr)\ \&\ (\forall U \in \mathcal{U},\ L \cap U \neq \emptyset)\Bigr\},$$

where $\mathcal{U}$ is any family consisting of finitely many open subsets of Y. Observe that $\mathcal{K}(Y)$ is compact by [2, Problem 3.12.27]. In addition, $\mathcal{K}(Y)$ is zero-dimensional by [2, Problem 6.3.22(e)]. Now define

$$\mathcal{Z} = \bigl\{K \in \mathcal{K}(Y)\colon\ \omega_1 \in K\bigr\}.$$

Then clearly $\mathcal{Z}$ is a closed and hence compact subspace of $\mathcal{K}(Y)$. Observe that the set T^* consisting of all elements $K \in \mathcal{Z}$ having the property that $K \cap \omega_1$ is a finite (and possibly empty) subset of S, is dense in $\mathcal{Z}$.

Claim 1. *T^* is a crowded Lindelöf P-space.*

Proof. Let $F \in T^*$. For a basic neighborhood $\langle \mathcal{U} \rangle$ of F, pick an element $U \in \mathcal{U}$ containing ω_1. Pick an arbitrary element $\alpha \in U \cap S$ such that $\alpha \notin F$. Then $F \cup \{\alpha\} \in T^* \cap (\langle \mathcal{U} \rangle \setminus \{F\})$, hence T^* is crowded.

For every $n \geqslant 1$ define $f_n : Y^n \to \mathcal{K}(Y)$ by $f_n(y_1, \dots, y_n) = \{y_1, \dots, y_n\}$. As is well known, each f_n is continuous. It is not difficult to prove directly that T^n is Lindelöf (alternatively, apply Noble [9]). As a consequence, $\bigcup_{n \geqslant 1} f_n(T^n)$ is Lindelöf. We will verify that T^* is a closed subspace of $\bigcup_{n \geqslant 1} f_n(T^n)$, and is therefore Lindelöf as well. In fact, let $K \in \bigcup_{n \geqslant 1} f_n(T^n) \setminus T^*$. Hence $\omega_1 \notin K$, and so there exists an open subset $U \subseteq Y$ such that $K \subseteq U$ and $\omega_1 \notin U$. It follows that $K \in \langle \{U\} \rangle \cap \bigcup_{n \geqslant 1} f_n(T^n) \subset \bigcup_{n \geqslant 1} f_n(T^n) \setminus T^*$.

We will next prove that T^* is a P-space. To this end, take an arbitrary element $F \in T^*$, and for every $n < \omega$, let $\langle \mathcal{U}_n \rangle$ be a basic open neighborhood of F. We will prove that $\bigcap_{n<\omega} \langle \mathcal{U}_n \rangle$ contains a basic neighborhood of F. Indeed, for every $n < \omega$, pick $U_n \in \mathcal{U}_n$ such that $\omega_1 \in U_n$. Let U be an open neighborhood of ω_1 in Y such that $U \subseteq \bigcap_{n<\omega} U_n$. Put $\mathcal{V} = \{\{\alpha\}\colon\ \alpha \in F \cap \omega_1\} \cup \{U\}$. Then clearly

$$F \in \langle \mathcal{V} \rangle \subseteq \bigcap_{n<\omega} \langle \mathcal{U}_n \rangle,$$

as required. □

Now let $X = T^*$, and $\gamma X = \mathcal{Z}$. Then $|X| = \omega_1$. It follows by [2, p. 245] that $\mathrm{w}(\gamma X) \leqslant \omega_1$, and γX is zero-dimensional since $\mathcal{K}(Y)$ is. Observe that $\mathrm{w}(\gamma X) \geqslant \omega_1$ since T^* contains a copy of T.

Claim 2. $\mathcal{Z} \setminus T^* = \{K \in \mathcal{Z}\colon\ K$ *contains some limit ordinal* $\gamma < \omega_1\}$.

Proof. Assume that $K \in \mathcal{Z}$ is disjoint from the set of all limit ordinals in ω_1. Then $K \cap \omega_1$ is finite since K is closed, and is contained in S. But then $K \in T^*$. □

For every limit ordinal $\gamma < \omega_1$, put

$$\mathcal{A}_\gamma = \bigl\{K \in \mathcal{K}(Y)\colon\ \gamma \in K\bigr\}.$$

Pick an arbitrary limit ordinal $\gamma < \omega_1$, and let $(\gamma_n)_n$ be a sequence of ordinal numbers in γ such that $\sup\{\gamma_n\colon n < \omega\} = \gamma$. For every $n < \omega$, put $\mathcal{G}_n = \{[\gamma_n + 1, \gamma], Y\}$. Observe that $\mathcal{G}_n$ consists of clopen sets. An easy verification shows that

$$\mathcal{A}_\gamma = \bigcap_{n<\omega} \langle \mathcal{G}_n \rangle,$$

hence $\mathcal{A}_\gamma$ is a closed G_δ-subset of $\mathcal{K}(Y)$. So we are done since there are only ω_1 limit ordinals in ω_1. □

We end this section by a simple result on topological sums of Ohio complete spaces.

Theorem 2.3. *The topological sum of Ohio complete spaces is Ohio complete.*

Proof. Let $X = \bigoplus_{\alpha \in I} X_\alpha$ be the topological sum of the Ohio complete spaces X_α, $\alpha \in I$, and let γX be a compactification of X. For every $\alpha \in I$, let $\widehat{X}_\alpha$ be an open subset of γX such that $\widehat{X}_\alpha \cap X = X_\alpha$. Then X_α is dense in $\widehat{X}_\alpha$, and $\widehat{X}_\alpha \cap \widehat{X}_\beta = \emptyset$ if $\alpha \neq \beta$. Since X_α is Ohio complete, and $\overline{X}_\alpha$ is a compactification of X_α, there exists a G_δ-subset Z_α of $\overline{X}_\alpha$ that contains X_α and such that every point in $Z_\alpha \setminus X_\alpha$ can be separated from X_α by a G_δ-subset of $\overline{X}_\alpha$.

Consider now $S_\alpha = Z_\alpha \cap \widehat{X}_\alpha$ for some $\alpha \in I$, and observe that it contains X_α and is a G_δ-subset of both $\overline{X}_\alpha$ and γX. Pick an arbitrary point $x \in S_\alpha \setminus X_\alpha$. There exists a G_δ-subset T of $\overline{X}_\alpha$ containing x but missing X_α. Hence $T \cap S_\alpha$ is a G_δ-subset of γX containing x but missing X. We consequently conclude that every point in $S_\alpha \setminus X_\alpha$ can be separated from X by a G_δ-subset of γX.

For $\alpha \in I$, $T_\alpha = \overline{X}_\alpha \setminus S_\alpha$ is an F_σ-subset of γX, and hence can be written as $\bigcup_{n\in\omega} F_n^\alpha$, where each F_n^α is closed in γX. Consider now the closed subset $G_n = \overline{\bigcup_{\alpha\in I} F_n^\alpha}$ of γX. Clearly, $K \cup \bigcup_{n\in\omega} G_n$, where $K = \gamma X \setminus \bigcup_{\alpha\in I} \widehat{X}_\alpha$, is an F_σ-subset of γX that contains $\bigcup_{\alpha\in I} T_\alpha$ and misses X. So its complement P is a G_δ-subset of γX that contains X. Now, pick an arbitrary point $x \in P \setminus X$. There exists $\alpha \in I$ such that $x \in S_\alpha \setminus X_\alpha$. Hence by the above, x can be separated from X by a G_δ-subset of γX. □

3. Ohio completeness: closed subspaces

In this section we will present some results about the behavior of Ohio completeness with respect to closed subspaces.

Theorem 3.1. *Let Y be a closed C^*-embedded subspace of an Ohio complete space X. Then Y is Ohio complete.*

Proof. Let γY be any compactification of Y. Observe that since Y is C^*-embedded in X, the closure $\overline{Y}$ of Y in βX is βY [3, p. 89]. Let $f : \beta Y \to \gamma Y$ be the Stone extension of the inclusion $Y \hookrightarrow \gamma Y$. Consider the adjunction space Z determined by βX, γY and f. It is clear that Z is a compactification of X, say $Z = bX$, having the property that the closure of Y in bX is exactly γY. Now it is easy to prove that Y is Ohio complete. Let T be a G_δ-subset of bX that contains X, such that every point of $T \setminus X$ can be separated from X by a G_δ-subset of bX. Then $S = Z \cap \gamma Y$ is clearly a G_δ-subset of γY such that every point of $S \setminus Y$ can be separated from Y by a G_δ-subset of γY. □

As a corollary we have:

Corollary 3.2. *Let Y be a closed subspace of an Ohio complete normal space X. Then Y is Ohio complete.*

The simple proof of Theorem 3.1 was based on the fact that Y is C^*-embedded in X. It is well known of course that any nonnormal space X contains a closed subspace Y that is not C^*-embedded. For Y one can simply take the union of two disjoint closed subsets of X that cannot be separated by disjoint open subsets in X. If X is, for example, the square of the Sorgenfrey line, then one can take Y to be discrete and hence Ohio complete. So these considerations lead us to the question whether Theorem 3.1 is optimal, i.e., whether a closed subspace of an Ohio complete space is again Ohio complete. This may seem to be a simple problem, but despite our efforts, we were unable to answer it. In fact, we believe that it is a tricky and interesting question because of its implications for the behavior of Ohio completeness with respect to products, see Theorem 3.4 below.

Question 3.3. Is a closed subspace of an Ohio complete space again Ohio complete?

Theorem 3.4. *If there is an Ohio complete space with a closed subspace that is not Ohio complete, then the product of two Ohio complete spaces need not be Ohio complete, even if one factor is compact.*

Proof. Let X be an Ohio complete space containing a closed subspace Y which is not Ohio complete. Consider the product $Z = X \times \beta Y$ and its subspace $\triangle(Y) = \{(y, y)\colon\ y \in Y\}$. Then $\triangle(Y)$ is a closed and C^*-embedded copy of Y in Z. Hence Z is not Ohio complete by Theorem 3.1. $\square$

4. A realcompact space which is not Ohio complete

Throughout this section, let X be the space with compactification γX constructed in Theorem 2.2.

Example 4.1. There is a realcompact space of cardinality and weight ω_1 which is not Ohio complete.

Proof. Since X has weight ω_1 and no nonempty open subset of X is countable (since X is a crowded P-space), a trivial transfinite induction shows that we can split X into two dense subsets, say A and B. We claim that A is realcompact, but not Ohio complete.

Since A is a P-space of weight ω_1, it is paracompact by Kunen [5, Lemma 1.3] (simply prove that any open cover of A has a disjoint clopen refinement). Since every closed and discrete subspace of A has cardinality at most ω_1, it follows that A is realcompact (Katětov [4]; see also Shirota [10], Gillman and Jerison [3, p. 229]).

We now claim that A is not Ohio complete. Indeed, consider the compactification γX of A. Let S be a G_δ-subset of γX containing A. Then $U = S \cap X$ is a G_δ-subset of X containing A. Hence U is open, X being a P-space. Since B is dense in X, we may pick $p \in S \cap B$. Let T be any G_δ-subset of γX that contains p. Then $T \cap X$ is a neighborhood of p in X and hence intersects A. So this shows that p cannot be separated from A by a G_δ-subset of γX. $\square$

Remark 4.2. Under the Continuum Hypothesis (abbreviated: CH), the space in Example 4.1 can be chosen to be a topological group. Indeed, let $X = (2^{\omega_1})_\delta$, i.e., 2^{ω_1} with the G_δ-topology. Then X is a topological group of weight ω_1 under CH. Let K be a dense subgroup of X of cardinality ω_1. A moments reflection shows that for the subset A in the proof of Example 4.1 we may take a subgroup of K.

As is clear from Example 4.1 and the remarks made at the end of Section 1, if a space is not Ohio complete then it may have many 'good' compactifications. It is convenient to introduce a notation for the collection of all 'good' compactifications of a given space X. Indeed, let $\mathcal{O}(X)$ denote the collection of all compactifications γX of X for which there exists a G_δ-subset S of γX such that $X \subseteq S$ and for every $p \in S \setminus X$ can be separated from X by a G_δ-subset of γX.

The following result shows that in general there are many 'good' compactifications provided there is at least one.

Proposition 4.3. *Let X be a space and $\gamma X \in \mathcal{O}(X)$. Then $\{\delta X\colon\ \delta X \in \mathcal{C}(X)$ and $\delta X \geqslant \gamma X\} \subseteq \mathcal{O}(X)$.*

Proof. Let δX be a compactification of X such that $\delta X \geqslant \gamma X$. By definition, there exists a continuous mapping $f : \delta X \to \gamma X$ which restricts to the identity on X. Since $\gamma X \in \mathcal{O}(X)$, there exists a G_δ-subset Z of γX such that $X \subseteq Z$ and every $z \in Z \setminus X$ can be separated from X by a G_δ-subset γX. The set $f^{-1}(Z)$ is a G_δ-subset of δX which clearly contains X. Now, pick a point $z \in f^{-1}(Z) \setminus X$. Then $f(z) \in Z \setminus f(X) = Z \setminus X$. So there exists a G_δ-subset S of γX with $f(z) \in S$ and $S \cap X = \emptyset$. Pulling back, we obtain the G_δ-subset $f^{-1}(S)$ of δX such that $z \in f^{-1}(S)$ and $f^{-1}(S) \cap X = \emptyset$. Hence $\delta X \in \mathcal{O}(X)$. $\square$

Proposition 4.4. *Let X be a closed subspace of an Ohio complete space Y, and let γX be the closure of X in βY. Then $\{\delta X\colon\ \delta X \in \mathcal{C}(X)$ and $\delta X \leqslant \gamma X\} \subseteq \mathcal{O}(X)$.*

Proof. Similar to the proof of Theorem 3.1. $\square$

These simple results show that if X is a closed subspace of an Ohio complete space Y, then all compactifications of Y that are somehow related to compactifications of X are 'good'. Since there are in general many other compactifications, this unfortunately does not answer Question 3.3.

5. Creating 'bad' compactifications

Let κ be an infinite cardinal. We say that a subset A is a $G_{<\kappa}$-subset of X if there is a family $\mathcal{U}$ of open subsets of X such that $|\mathcal{U}| < \kappa$, and $\bigcap \mathcal{U} = A$.

Theorem 5.1. *Let X be a space. For a given point $x \in X$ put $\kappa = \chi(x, X)$. Then at least one of the following statements is true*:

(i) $\kappa = \omega$.
(ii) *There is a family $\mathcal{L}$ of fewer than κ compact subsets of $\beta X \setminus X$ such that $\bigcup \mathcal{L}$ does not have compact closure in $\beta X \setminus X$.*
(iii) *The point x is contained in a closed $G_{<\kappa}$-subset of X which is Lindelöf.*
(iv) *For every compactification aX of X there is a compactification bX of X such that $bX \leqslant aX$ and $bX \notin \mathcal{O}(X)$.*

Proof. Assume that (i), (ii) and (iii) are false, and let aX be an arbitrary compactification of X. We will prove that (iv) holds. Observe that $\neg$(ii) holds if we replace βX by aX.

We will first show that κ is regular. Observe that $\chi(x, aX) = \chi(x, X)$ since X is dense in aX. Let $\mathcal{V}$ be a neighborhood base of x in aX consisting of closed G_δ-subsets of aX. If κ is singular, then we can split $\mathcal{V}$ into subfamilies $\{\mathcal{V}_i\colon i \in I\}$ such that $|\mathcal{V}_i| < \kappa$ for every $i \in I$, while moreover $|I| < \kappa$. If $i \in I$, then $\bigcap \mathcal{V}_i$ is a $G_{<\kappa}$-subset of aX containing x, hence by its compactness it cannot be contained in X by $\neg$(iii); pick an arbitrary point $y_i \in \bigcap \mathcal{V}_i \setminus X$. Then x is in the closure of the set $\{y_i\colon i \in I\}$, which is impossible by $\neg$(ii). Hence by $\neg$(i), κ is an uncountable regular cardinal.

Claim 1. *Each $G_{<\kappa}$-subset U of aX which contains x contains a compact subset K having the following properties*:

(1) *$K \subseteq U \setminus X$,*
(2) *there is no G_δ-subset S of aX such that $K \subseteq S \subseteq aX \setminus X$.*

Proof. Let P be a closed $G_{<\kappa}$-subset of x in aX such that $P \subseteq U$. By $\neg$(iii) there is an open collection $\mathcal{V}$ in U such that $P \cap X \subseteq \bigcup \mathcal{V}$ while moreover $P \cap X \nsubseteq \bigcup \mathcal{V}'$ for every countable $\mathcal{V}' \subseteq \mathcal{V}$. Put $K = P \setminus \bigcup \mathcal{V}$. Then K is a compact subset of $U \setminus X$, and we claim that it is as required. To prove this, assume that there exists a G_δ-subset S of aX such that $K \subseteq S \subseteq aX \setminus X$. Observe that

$$aX \setminus S \subseteq (aX \setminus P) \cup \bigcup \mathcal{V}.$$

Since $aX \setminus S$ is σ-compact, it is Lindelöf. There consequently is a countable subcollection $\mathcal{V}'$ of $\mathcal{V}$ such that

$$aX \setminus S \subseteq (aX \setminus P) \cup \bigcup \mathcal{V}'.$$

But then $P \cap X \subseteq \bigcup \mathcal{V}'$, which is a contradiction. □

Let $\{V_\alpha\colon \alpha < \kappa\}$ be a neighborhood base of x in aX. For $\alpha < \kappa$ we will construct a continuous function $f_\alpha : aX \to \mathbb{I}$, and a compact subset K_α of aX such that

(3) $f_\alpha(x) = 0$ and $K_\alpha \subseteq (\bigcap_{\beta<\alpha} f_\beta^{-1}(0) \cap \bigcap_{\beta \leqslant \alpha} V_\beta) \setminus X$,
(4) there is no G_δ-subset S of aX such that $K_\alpha \subseteq S \subseteq aX \setminus X$,
(5) $f_\alpha^{-1}([0, 1)) \subseteq V_\alpha \cap \bigcap_{\beta \leqslant \alpha}(aX \setminus K_\beta)$.

Assume that for some $\alpha < \kappa$ we constructed f_β and K_β for all $\beta < \alpha$ (the ordinal α could be 0).

Observe that $U = \bigcap_{\beta<\alpha} f_\beta^{-1}(0) \cap \bigcap_{\beta\leqslant\alpha} V_\beta$ is a $G_{<\kappa}$-subset of aX containing x. Pick K_α for U as in Claim 1. By ¬(ii), there is an open neighborhood V of x in aX such that $V \subseteq V_\alpha \cap \bigcap_{\beta\leqslant\alpha}(aX \setminus K_\beta)$. There consequently is a continuous function $f_\alpha : aX \to \mathbb{I}$ such that $f_\alpha(x) = 0$ and $f_\alpha^{-1}([0,1)) \subseteq V$. This completes the transfinite construction.

Now define $f : aX \to \mathbb{I}^\kappa$ by

$$f(p)_\alpha = f_\alpha(p) \quad (\alpha < \kappa).$$

Then f is clearly a continuous function. For every $\alpha < \kappa$ let the point $p(\alpha) \in \mathbb{I}^\kappa$ be defined by

$$p(\alpha)_\xi = \begin{cases} 0 & (\xi < \alpha), \\ 1 & (\alpha \leqslant \xi < \kappa). \end{cases}$$

Let $\underline{0}$ be the point of $\mathbb{I}^\kappa$ having all coordinates equal to 0. It is easy to see that the subset

$$P = \{\underline{0}\} \cup \{p(\alpha) \colon \alpha < \kappa\}$$

of $\mathbb{I}^\kappa$ is closed in $\mathbb{I}^\kappa$ and hence is compact. Observe that P is a homeomorphic copy of the ordinal space $W(\kappa+1)$.

Claim 3. *$f^{-1}(\underline{0}) = \{x\}$, and for every $\alpha < \kappa$, $f(K_\alpha) = \{p(\alpha)\}$.*

Proof. This is clear from (3) and (5) and the fact that $\{V_\alpha \colon \alpha < \kappa\}$ is a neighborhood base of x in aX. □

Now put $K = \overline{\bigcup_{\alpha<\kappa} K_\alpha}$.

Claim 4. *$K \cap X = \{x\}$.*

Proof. It is clear from (5) that $x \in K \cap X$. Assume that there exists $y \in (K \cap X) \setminus \{x\}$. Pick $\alpha < \kappa$ such that $y \notin \overline{V}_\alpha$. By (5), $K_\beta \subseteq V_\alpha$ for every $\beta \geqslant \alpha$. Since

$$y \in \overline{\bigcup_{\beta<\kappa} K_\beta} = \overline{\bigcup_{\beta<\alpha} K_\beta} \cup \overline{\bigcup_{\alpha\leqslant\beta<\kappa} K_\beta} \subseteq \overline{\bigcup_{\beta<\alpha} K_\beta} \cup \overline{V}_\alpha,$$

this means that $y \in \overline{\bigcup_{\beta<\alpha} K_\beta}$. But this contradicts ¬(ii). This is a contradiction. □

Let $g = f \restriction K$. Then $g : K \to P$ is a continuous surjection by Claim 2. Consider the adjunction space Z determined by aX, P and g. Let $\pi : aX \to Z$ be the natural quotient map. Observe that π replaces K by a copy of P. It will be convenient to identify P and that copy of itself. Also observe that by Claim 2 we have that $\pi^{-1}(\pi(y)) = \{y\}$ for every $y \in X$. Hence Z is a compactification of X, say $Z = bX$. We claim that $bX \notin \mathcal{O}(X)$. To this end, assume that S is an arbitrary G_δ-subset of bX which contains X. Since κ has uncountable cofinality, being regular, and $bX \setminus S$ is an F_σ-subset of bX that is contained in $bX \setminus X$, there exists $\alpha < \kappa$ such that $p(\alpha) \in S$. Striving for a contradiction, assume that there is a G_δ-subset T of bX containing $p(\alpha)$ and missing X. Then $\pi^{-1}(T)$ is a G_δ-subset of aX which misses X but contains K_α by Claim 2. This however contradicts (4). □

Corollary 5.2. *Let X be a closed subspace of an Ohio complete space containing a point x. Then for $\kappa = \chi(x, X)$ at least one of the following statements is true*:

(i) *$\kappa = \omega$.*
(ii) *There is a family $\mathcal{L}$ of fewer than κ compact subsets of $\beta X \setminus X$ such that $\bigcup \mathcal{L}$ does not have compact closure in $\beta X \setminus X$.*
(iii) *The point x is contained in a closed $G_{<\kappa}$-subset of X which is Lindelöf.*

Proof. From Proposition 4.4 it follows that (iv) of Theorem 5.1 does not hold for X. Hence one of (i), (ii) and (iii) must hold. □

As we observed in the introduction, it was proved in [1] that if X is Čech-complete, or Lindelöf, or a p-space, or has a G_δ-diagonal, then X is Ohio complete. It is not difficult to verify that these classes of spaces satisfy the conclusion of Corollary 5.2.

This leads us to a characterization of the Ohio complete P-spaces of weight ω_1.

Theorem 5.3. *Let X be a P-space of weight at most ω_1. Then the following conditions are equivalent*:

(1) *X is Ohio complete.*
(2) *X is a closed subspace of an Ohio complete space.*
(3) *X admits a clopen partition each element of which is Lindelöf.*

Proof. For (2) $\Rightarrow$ (3), pick an arbitrary $x \in X$, and let $\kappa = \chi(x, X)$. Then if $\kappa = \omega$, it follows that x is isolated in X since X is a P-space. So assume that $\kappa = \omega_1$. Since X is a P-space, the union of countably many compact subsets of $\beta X \setminus X$ clearly has compact closure in $\beta X \setminus X$. Hence by Corollary 5.2 it follows that x has a Lindelöf neighborhood, and hence a Lindelöf clopen neighborhood. So we conclude that X has a clopen cover by Lindelöf subspaces. But this cover can be refined by a clopen partition since X is a P-space of weight ω_1.

For (3) $\Rightarrow$ (1), simply use the fact that every Lindelöf space is Ohio complete, and apply Theorem 2.3. □

We do not know whether it is possible to characterize the Ohio complete spaces among arbitrary P-spaces in a similar way.

6. Products that are not Ohio complete

The aim of this section is to show among other things that $\mathbb{R}^{\omega_1}$ is not a closed subspace of an Ohio complete space.

Let A and B be the spaces in the proof of Example 4.1. Hence $X = A \cup B$ is the space in Theorem 2.2. It is clear that $\chi(x, A) = \omega_1$ for every $x \in A$. For if $x \in A$ is such that $\chi(x, A) = \omega$, then x is isolated in A and hence in the space X and that is impossible since X is crowded. We claim that no nonempty open subset of A is Lindelöf. To prove this, let $U \subseteq A$ be nonempty and open, and let $\widehat{U}$ be an open subset of X such that $\widehat{U} \cap A = U$. Since B is dense in X, we may pick an element $p \in \widehat{U} \cap B$. Since p is a P-point of X of character ω_1, it is clear that we can split $\widehat{U} \setminus \{p\}$ in a pairwise disjoint clopen family $\{V_\alpha\colon \alpha < \omega_1\}$. If only countably many of the V_α's are nonempty, then $\{p\}$ is a G_δ-subset of $\widehat{U}$, hence p is an isolated point of X. This again contradicts the fact that X is crowded. As a consequence, A being dense, $\{V_\alpha \cap A\colon \alpha < \omega_1\}$ is a clopen cover of U without countable subcover. Hence by Theorem 5.3 we get:

Proposition 6.1. *A cannot be embedded as a closed subspace of an Ohio complete space.*

Theorem 6.2. *A admits a closed embedding in ω^{ω_1}.*

Proof. By Theorem 2.1 we may assume that X is a closed subset of ω^{ω_1}. Take an arbitrary $x \in X$. We claim that $X \setminus \{x\}$ admits a closed embedding in ω^{ω_1}. Since X is a crowded P-space, and has weight ω_1, the space $X \setminus \{x\}$ is the disjoint union of a family $\mathcal{A}$ consisting of clopen subspaces of X. This family has clearly size ω_1, hence we may enumerate it faithfully as $\{A_\alpha\colon \alpha < \omega_1\}$. By a result of Mycielski [8], ω^{ω_1} has a closed discrete subspace D of cardinality ω_1 (the proof in [2, 3.1.H(a)] outlined for $\mathfrak{c}$ can easily be adapted to work for ω_1 as well). Enumerate D faithfully as $\{d_\alpha\colon \alpha < \omega_1\}$. Define $f : X \setminus \{x\} \to \omega^{\omega_1} \times \omega^{\omega_1}$ by

$$f(p) = \langle p, d_\alpha \rangle \quad \Longleftrightarrow \quad p \in A_\alpha.$$

Then f is a closed embedding.

So we conclude that for every $x \in B$ we have that $X \setminus \{x\}$ admits a closed embedding in ω^{ω_1}. Since $|X| = \omega_1$, standard methods now prove that $A = \bigcap_{x \in B} X \setminus \{x\}$ admits a closed embedding in ω^{ω_1} as well. □

Corollary 6.3. *Let κ be an infinite cardinal. If $X = \prod_{\alpha<\kappa} X_\alpha$ is a closed subspace of an Ohio complete space, then all but countably many of the X_α's are countably compact.*

Proof. If not, then X contains a closed subspace homeomorphic to ω^{ω_1}. So then we contradict the combination of Theorems 6.1 and 6.2. □

A natural question is if 'countably compact' in Corollary 6.3 can be improved to 'compact'. In the following we will prove that this can indeed be done.

Theorem 6.4. *Every countably compact closed subspace of an Ohio complete space is Čech complete.*

Proof. Let Y be an Ohio complete space containing a closed, countably compact subspace X. Consider βY, and pick a G_δ-subset S of βY containing Y such that every point of $S \setminus Y$ can be separated from Y by a G_δ-subset of βY. We claim that $S \cap \overline{X} = X$, proving that X is a G_δ-subset of $\overline{X}$. Here $\overline{X}$ denotes the closure of X in βY. Striving for a contradiction, put $T = S \cap \overline{X}$, and assume that there is a point $y \in T \setminus X$. There is a closed G_δ-subset A of $\overline{X}$ such that $y \in A$ and $A \cap X = \emptyset$. There is a continuous function $f : \overline{X} \to \mathbb{I}$ such that $f^{-1}(0) = A$. Now for every n, pick a point $x_n \in X$ such that $f(x_n) < \frac{1}{n}$. Then all limit points of the sequence $\{x_n \colon n \in \mathbb{N}\}$ belong to A, contradicting the countable compactness of X. □

Observe that in this proof we only need that Y has at least one 'good' compactification, and not that all compactifications are 'good'. So we actually have proved a stronger result than stated.

The following result is probably well known. Since we could not find it in the literature, we include its simple proof.

Lemma 6.5. *Suppose that for $\alpha < \omega_1$, X_α is not compact. Then $X = \prod_{\alpha<\omega_1} X_\alpha$ is not Čech complete.*

Proof. Pick an arbitrary point $x \in X$, and assume that x is contained in a G_δ-subset K of X. There is a countable set $A \subseteq \omega_1$ such that

$$B = \bigl\{y \in X \colon (\forall \alpha \in A)(x_\alpha = y_\alpha)\bigr\}$$

is contained in K. Since B is closed in X and not compact, it follows that K is not compact. This clearly implies that X is not a Čech-complete space. □

These results lead us to our main result.

Theorem 6.6. *If $X = \prod_{\alpha<\omega_1} X_\alpha$ is a closed subspace of an Ohio complete space, then all but countably many of the X_α's are compact.*

Proof. Suppose that this is not true. We may consequently assume without loss of generality that X_α is noncompact for every $\alpha < \omega_1$. Split ω_1 into a disjoint family of sets $\{E_\alpha \colon \alpha < \omega_1\}$ such that $|E_\alpha| = \omega_1$ for every $\alpha < \omega_1$. For every $\alpha < \omega_1$, put $X(\alpha) = \prod_{\beta \in E_\alpha} X_\beta$.

Fix $\beta < \omega_1$ for a moment. We claim that $X(\beta)$ is not countably compact. For if it were countably compact, then it would be Čech complete by Theorem 6.4, which contradicts Lemma 6.5.

Hence $X = \prod_{\beta<\omega_1} X(\beta)$ is a closed subspace of an Ohio complete space, and none of its factors is countably compact. But this contradicts Corollary 6.3. □

As a consequence:

Corollary 6.7. *$\mathbb{R}^{\omega_1}$ cannot be embedded as a closed subspace of an Ohio complete space.*

Corollary 6.8. *If κ is uncountable, then $\mathbb{R}^\kappa$ is not Ohio complete.*

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