Abstract. Generalizing the Ohio completeness property, we introduce the notion of $\kappa$-Ohio completeness. Although many results from a previous paper by the authors may easily be adapted for this new property, there are also some interesting differences. We provide several examples to illustrate this. We also have a consistency result; depending on the value of the cardinal $\kappa$, the countable union of open and $\omega_1$-Ohio complete subspaces may or may not be $\omega_1$-Ohio complete.

1. Introduction.

All spaces under consideration are Tychonoff. For all undefined notions we refer to [6]. A topological space $X$ is {Ohio complete} if for every compactification $\gamma X$ of $X$ there is a $G_\delta$-subset $S$ of $\gamma X$ such that $X \subseteq S$ and for every $y \in S \setminus X$, there is a $G_\delta$-subset of $\gamma X$ which contains $y$ and misses $X$. Ohio completeness was introduced by Arhangel’ski in [1] where it turned out to be a useful concept for the study of properties of remainders in compactifications.

Let $\kappa$ be an infinite cardinal number. It is quite natural to generalize Ohio completeness by saying that a space $X$ is $\kappa$-Ohio complete if for every compactification $\gamma X$ of $X$ there is a $G_\kappa$-subset $S$ of $\gamma X$ such that $X \subseteq S$ and for every $y \in S \setminus X$, there is a $G_\kappa$-subset of $\gamma X$ which contains $y$ and misses $X$. Here a subspace of a space $X$ is called a $G_\kappa$-subset if it is the intersection of at most $\kappa$-many open subsets of $X$.

Observe that any space is $\kappa$-Ohio complete, for some large enough $\kappa$. Also, if either the Čech-number or the compact covering number of a space does not exceed $\kappa$, then this space is $\kappa$-Ohio complete.

Ohio complete spaces were studied in [2] and [3]. In this paper we will focus our attention on unions of $\kappa$-Ohio complete subspaces. Since all the positive results proved in [3] can be easily generalized for $\kappa$-Ohio completeness, our main purpose will be to construct counterexamples for the $\kappa$-Ohio complete case. This will be done in the main section. We will construct a non $\kappa$-Ohio complete space which is the union of a locally countable family of closed and $\kappa$-Ohio complete

2000 Mathematics Subject Classification. Primary 54D35, 54G20; Secondary 54B05, 54B25.

Key Words and Phrases. $\kappa$-Ohio complete, sum theorems, compactification.
subspaces. Next we will show that, if $\kappa$ is a regular cardinal, the union of $\kappa$-many open and $\kappa$-Ohio complete subspaces need not be a $\kappa$-Ohio complete space.

The last section is devoted to positive results about open sums. We shall prove that the union of $\lambda$-many open and $\kappa$-Ohio complete subspaces is $kcoev(\kappa^+)$-Ohio complete. This result implies several interesting consistency results. In particular, if $\delta = \omega_1$, then the union of countably many open and $\omega_1$-Ohio complete subspaces is again $\omega_1$-Ohio complete. This statement may fail if $\delta > \omega_1$. Here the cardinal $\delta$ is the compact covering number of the space $\omega^\omega$ of irrationals (see [5] for more information).

Throughout the paper we will use the terminology introduced in [3]. Of course the terminology there was only introduced for Ohio completeness, but the $\kappa$-Ohio complete generalization is straightforward.

2. Examples.

In [3] it was proved that the union of a locally finite family of closed and Ohio complete subspaces is again Ohio complete and the same holds for $\kappa$-Ohio completeness. In contrast, the union of a locally countable family of closed and Ohio complete subspaces need not be Ohio complete by [3, Example 5.3]. The obvious generalization of this example shows that the union of a locally-$\kappa$ family of closed and Ohio complete subspaces need not be $\kappa$-Ohio complete. The following example is much better, since it shows that even a locally countable union of closed and Ohio complete subspaces may fail to be $\kappa$-Ohio complete.

**Example 2.1.** Fix an infinite cardinal number $\kappa$ and let $\tau = \kappa^+$. The space $\tau$ carries the discrete topology. Let $\gamma \tau = \tau \cup \{\infty\}$ be its one-point compactification. Let $\mathcal{A}$ be a family which is maximal with respect the following property:

1. $\mathcal{A} \subseteq [\tau]^\omega$,
2. $\forall A, B \in \mathcal{A} \quad (A \neq B \rightarrow |A \cap B| < \omega)$.

Recall that the well-known space $Y = \psi(\tau, \mathcal{A})$ is defined as follows (see for example [5, p.153]). The underlying set of $Y$ is $\tau \cup \mathcal{A}$, the points of $\tau$ are isolated and a basic neighborhood of $A \in \mathcal{A}$ has the form $\{A\} \cup (A \setminus F)$, where $F$ is finite. It is clear from the definition that $Y$ is covered by countable open sets.

We claim that the subspace $\mathcal{A}$ is not a $G_\kappa$-subset of $Y$. To prove this it suffices to show that every closed subset of $Y$ which misses $\mathcal{A}$ is finite. If this were not the case, then we could find a countably infinite closed subset $C$ of $Y$ which misses $\mathcal{A}$. Since the family $\mathcal{A}$ was chosen maximal, there must exist some $A \in \mathcal{A}$ such that $A \cap C$ is countably infinite. But then, every neighborhood of $A$ intersects $C$, a contradiction.
We let $Z = Y \times \gamma \tau$ and $X$ be the subspace of $Z$ given by:

$$X = (Y \times \tau) \cup (\mathcal{A} \times \{\infty\}).$$

We use [3, Lemma 5.1] (the proof can be easily adapted for $\kappa$-Ohio completeness) to show that $X$ is not $\kappa$-Ohio complete. Since $\mathcal{A}$ is not a $G_\kappa$-subset of $Y$, the space $X$ is not a $G_\kappa$-subset of $Z$. To conclude that $X$ is not $\kappa$-Ohio complete, observe that $Z \setminus X$ contains no non-empty $G_\kappa$-subset of $Z$.

It remains to verify that $X$ is the union of a locally countable family of closed and Ohio complete subspaces of $X$. Let $\pi$ be the projection of $X$ onto the first coordinate. We let the closed cover $\mathcal{C}$ of $X$ be given by:

$$\mathcal{C} = \{\pi^{-1}(y) : y \in Y\}.$$ 

Note that the fibers of $\pi$ are homeomorphic to either $\gamma \tau$ or $\tau$, which are both Ohio complete since they are locally compact. Furthermore, since $Y$ is locally countable in itself and $\pi$ is continuous, it follows that $\mathcal{C}$ is locally countable in $X$.

We now turn towards open sum theorems. In [3, Example 5.7] it was shown that the countable union of open and Ohio complete subspaces need not be Ohio complete. We can then ask whether the union of $\kappa$-many open and $\kappa$-Ohio complete subspaces is $\kappa$-Ohio complete. From the next theorem it will follow that, at least for $\kappa$ regular, this is not the case. Moreover it will follow that, under the assumption $\kappa < \omega$, the countable union of open and Ohio complete subspaces need not even be $\kappa$-Ohio complete. In the final section of this paper we will show that on the other hand the countable union of open and Ohio complete subspaces is always $\omega$-Ohio complete.

Fix an infinite regular cardinal $\kappa$ and consider the space $2^\kappa$. We call a set a $G_{<\kappa}$-subset if it is a $G_\lambda$-subset for some $\lambda < \kappa$. We denote by $(2^\kappa)_{<\kappa}$ its $G_{<\kappa}$-modification. So the topology on $(2^\kappa)_{<\kappa}$ is generated by the collection of all $G_{<\kappa}$-subsets of $2^\kappa$. Now consider the following subset of $(2^\kappa)$:

$$E_{<\kappa} = \{x \in 2^\kappa : |\{\alpha < \kappa : x_\alpha \neq 0\}| < \kappa\}.$$ 

Note that this set is dense in the space $(2^\kappa)_{<\kappa}$. Recall that the Baire number of a space with no isolated points, also called the Novák number, is the minimal cardinality of a family of closed nowhere dense subsets whose union is the whole space. In [8, Lemma 1.3(b)] it is proved that the Baire number of the space $(2^\kappa)_{<\kappa}$ is always greater than or equal to $\kappa^+$. This result implies the following.
Lemma 2.2. For regular $\kappa$, the set $E_{<\kappa}$ is not a $G_\kappa$-subset of $(2^\kappa)_{<\kappa}$.

Proof. Note that since $\kappa$ is regular, the set $E_{<\kappa}$ is equal to the union of sets of the form $2^\alpha \times \{0\}^{\kappa / \alpha}$, for $\alpha < \kappa$. These sets are all closed and nowhere dense in $(2^\kappa)_{<\kappa}$.

So if $E_{<\kappa}$ were a $G_\kappa$-subset, then its complement would be the union of $\kappa$-many closed sets which are all nowhere dense since $E_{<\kappa}$ is dense in $(2^\kappa)_{<\kappa}$. So by the previous observation, we would have that the Baire number of the space $(2^\kappa)_{<\kappa}$ were less than or equal to $\kappa$, contradicting [8, Lemma 1.3(b)].

The example constructed in the following theorem is very similar to [3, Example 5.7], see also [4, Example 2.4].

Theorem 2.3. Let $\kappa$ and $\lambda$ be infinite cardinal numbers, with $\kappa$ regular. There exists a space $X$ with the following properties:

1. If $X$ is $\lambda$-Ohio complete, then $E_{<\kappa}$ is a $G_\lambda$-set in $(2^\kappa)_{<\kappa}$,
2. $X$ is the union of $\kappa$-many open and $\kappa$-Ohio complete subspaces.

Proof. We set $E = E_{<\kappa}$. For every $e \in E$, we fix a collection $A(e)$ of one-to-one functions from $\kappa$ into $D = 2^\kappa \setminus E$, which is maximal with respect to the following conditions:

(i) $\forall f \in A(e) \forall \alpha < \kappa (f(\alpha) \upharpoonright \alpha = e \upharpoonright \alpha)$,
(ii) $\forall f, g \in A(e) (f \neq g \rightarrow |\text{ran}(f) \cap \text{ran}(g)| < \kappa)$.

Fix a discrete space $Y$ of cardinality $\lambda^+$ and let $\omega Y = Y \cup \{\infty\}$ be its one-point compactification. For every $\alpha \in \kappa$, with $E_\alpha$ we denote the subspace $2^\alpha \times \{0\}^{\kappa / \alpha}$. Put $A_\alpha = \bigcup_{e \in E_\alpha} A(e)$ and $A = \bigcup_{\alpha \in \kappa} A_\alpha$ and let $Z = A \cup (D \times \omega Y)$. If $f \in A$ and $\alpha < \kappa$, we let

$$U(f, \alpha) = \{f\} \cup \bigcup \{f(\beta) \times \omega Y : \alpha < \beta < \kappa\}.$$  

The collection $\mathcal{B}$, which serves as a base for a topology on $Z$, is given by

$$\{U(f, \alpha) : f \in A, \alpha < \kappa\} \cup \{R \times U : R \subseteq D, U \text{ is an open subset of } \omega Y\}.$$  

We leave it to the reader to verify that topologized in this way, the space $Z$ is Hausdorff and zero-dimensional and hence Tychonoff.

We let $X$ be the subspace of $Z$ given by $A \cup (D \times Y)$. In the following claim we will prove assertion (1). The proof is almost identical to the argument used in [4, Example 2.4].
CLAIM 1. If $X$ is $\lambda$-Ohio complete, then $E$ is a $G_\lambda$-set in $(2^\kappa)_{<\kappa}$.

PROOF OF CLAIM. Striving for a contradiction, assume that $X$ is $\lambda$-Ohio complete but $E$ is not a $G_\lambda$-subset of $(2^\kappa)_{<\kappa}$.

Since for every $d \in D$, the subspace $\{d\} \times \omega Y$ of $Z$ is homeomorphic to $\omega Y$, the set $Z \setminus X$ contains no non-empty $G_\lambda$-subsets of $Z$. Then, by [3, lemma 5.1] $X$ must be a $G_\lambda$-subset of $Z$. Hence $D = \bigcup_{\alpha < \lambda} G_\alpha$, where each $G_\alpha \times \{\infty\}$ is closed in $A \cup (D \times \{\infty\})$.

By assumption $D$ is not the union of $\lambda$-many closed subsets of $(2^\kappa)_{<\kappa}$, so for some $\alpha < \lambda$, $E \cap Cl_{<\kappa}(G_\alpha) \neq \emptyset$ (where the closure is taken in $(2^\kappa)_{<\kappa}$). So we may fix $e \in E$, such that for every $\beta < \kappa$, there is some $g \in G_\alpha$ such that $g \upharpoonright \beta = e \upharpoonright \beta$.

But this means that we may find an injective function $f : \kappa \to G_\alpha$ such that for every $\beta < \kappa$, $f(\beta) \upharpoonright \beta = e \upharpoonright \beta$.

Since the collection $A(e)$ was maximal, it follows that for some $f' \in A(e)$, we have that $|\text{ran}(f) \cap \text{ran}(f')| = \kappa$. But this means that $f'$ is in the closure of the set $G_\alpha \times \{\infty\}$ (closure in $A \cup (D \times \{\infty\})$), which is a contradiction. $\square$

We will now prove assertion (2), that is that $X$ is the union of $\kappa$-many open and $\kappa$-Ohio complete subspaces. For each $\alpha < \kappa$, we let $X_\alpha = A_\alpha \cup (D \times Y)$. It is not hard to verify that $X_\alpha$ is an open subspace of $X$ and of course $X = \bigcup\{X_\alpha : \alpha < \kappa\}$. It remains to prove that each $X_\alpha$ is $\kappa$-Ohio complete.

CLAIM 2. For each $\alpha < \kappa$, the space $X_\alpha$ is $\kappa$-Ohio complete.

PROOF OF CLAIM. Fix $\alpha < \kappa$. Note that both $A_\alpha$ and $D \times Y$ are discrete subspaces of $X_\alpha$. Since a discrete space is ($\kappa$-)Ohio complete, we have that $X_\alpha$ is the union of two ($\kappa$-)Ohio complete subspaces. The space $D \times Y$ is clearly an open subspace of $X_\alpha$, and the set $A_\alpha$ is a $G_\kappa$-subset of $X_\alpha$. Indeed, one verifies easily that $A_\alpha = \bigcap_{\beta \leq \alpha} \bigcup_{f \in A_\alpha} (U(f, \beta) \cap X_\alpha)$.

So it follows that $X_\alpha$ is the union of two $G_\kappa$-subsets which are both $\kappa$-Ohio complete. By Corollary 4.2 in [3] it follows that $X_\alpha$ is $\kappa$-Ohio complete. $\square$

This completes the proof of the theorem. $\square$

The previous theorem may be applied to obtain several interesting examples.

EXAMPLE 2.4. Assume $\lambda < \omega$. Then the union of countably many open and Ohio complete subspaces need not be $\lambda$-Ohio complete.

PROOF. Consider the space $X$ constructed in the previous theorem with $\kappa = \omega$. If $X$ were $\lambda$-Ohio complete, then $E_{<\lambda}$ would be a $G_\lambda$-subset of $(2^\kappa)_{<\omega}$. 

\kappa-Ohio completeness
However in this case $E_{\omega}$ is homeomorphic to the space of rationals in $(2^\omega)_{\omega}$, which is just the Cantor set. Of course, since $\lambda < \omega$, the set of rationals is not a $G_\lambda$-subset of $2^\omega$. \hfill \Box

**Example 2.5.** Assume $\kappa$ regular. Then the union of $\kappa$-many open and $\kappa$-Ohio complete subspaces need not be $\kappa$-Ohio complete.

**Proof.** It suffices to apply Theorem 2.3 with $\lambda = \kappa$, and then use Lemma 2.2. \hfill \Box

**3. Positive results.**

Having obtained several counterexamples, we now provide some positive results on open sum theorems for $\kappa$-Ohio completeness. In particular, the results of this section will allow us to prove that, under the assumption $\emptyset = \omega_1$, the countable open union of $\omega_1$-Ohio complete subspaces is $\omega_1$-Ohio complete.

Recall that the covering number of a space $X$, denoted by $kcov(X)$, is the minimal cardinality of a collection $\mathcal{K}$ of compact subsets of $X$ which covers $X$. In the next lemma we will refer to $kcov(\kappa^\beta)$. In this case the space $\kappa$ always carries the discrete topology. Of course $\kappa \leq kcov(\kappa^\beta) \leq \kappa^\beta$.

**Lemma 3.1.** Let $X$ be a space. Then the union of $\lambda$-many $G_{\kappa^\beta}$-subsets of $X$ is a $G_{kcov(\kappa^\beta)}$-subset of $X$.

**Proof.** Let $\mathcal{G}$ be a family of $G_{\kappa^\beta}$-subsets of $X$, with $|\mathcal{G}| = \lambda$. For every $G \in \mathcal{G}$, we fix a sequence $(G_\alpha)_{\alpha \in \kappa^\beta}$ of open subsets of $X$ such that $G = \bigcap_{\alpha \in \kappa^\beta} G_\alpha$. Since $|\mathcal{G}| = \lambda$, the space $\kappa^\beta$ is homeomorphic to $\kappa^\lambda$, and then we may write $\kappa^\beta = \bigcup_{\tau \in kcov(\kappa^\lambda)} K_{\tau}$, where each $K_{\tau}$ is a compact subset of $\kappa^\beta$.

For every $\tau \in kcov(\kappa^\lambda)$, let $f_\tau : \mathcal{G} \to [\kappa]^\omega$ be the function defined as $f_\tau(G) = p_G(K_{\tau})$, where $p_G$ denotes the projection of $\kappa^\beta$ onto the $G$-th factor, and we let $\mathcal{F} = \{ f_\tau : \tau \in kcov(\kappa^\lambda) \}$.

For every $f \in \mathcal{F}$, we let $W_f = \bigcup_{G \in \mathcal{G}} \bigcap_{\alpha \in f(G)} G_\alpha$ and $W = \bigcap_{f \in \mathcal{F}} W_f$. It is clear that $W_f$ is an open subset of $X$ containing $\bigcup \mathcal{G}$, and then $W$ is a $G_{kcov(\kappa^\beta)}$-subset of $X$ containing $\bigcup \mathcal{G}$. We shall now prove that actually $W = \bigcup \mathcal{G}$.

To this end, suppose that $x \notin \bigcup \mathcal{G}$. Then, for every $G \in \mathcal{G}$ we may fix an index $\alpha_G \in \kappa$ such that $x \notin G_{\alpha_G}$. Since the point $y = (\alpha_G)_{G \in \mathcal{G}} \in \kappa^\beta$, there exists some $\tau \in kcov(\kappa^\beta)$ such that $y \in K_{\tau}$. By construction, $x \notin W_f$, so that $x \notin W$. \hfill \Box

We say that a subspace $X$ of a space $Z$ is $\kappa$-Ohio embedded in $Z$, if there is a $G_{\kappa}$-subset $S$ of $Z$ such that $X \subseteq S$ and for every $y \in S \setminus X$, there is a $G_{\kappa}$-subset of $Z$ which contains $y$ and misses $X$. Such a $G_{\kappa}$-subset $S$ will be called a $\kappa$-good $G_{\kappa}$-subset with respect to $X$. For more information about $\kappa$-Ohio embedded
THEOREM 3.2. Let \( X \) be a space. Suppose that \( \mathcal{U} \) is a cover of \( X \) consisting of \( G_\kappa \)-subsets, with \( |\mathcal{U}| = \lambda \). If \( X \subseteq Z \), and every element of \( \mathcal{U} \) is \( \kappa \)-Ohio embedded in \( Z \), then \( X \) is kcov\((\kappa^\lambda)\)-Ohio embedded in \( Z \).

PROOF. Let \( \mathcal{U} = \{U_\alpha : \alpha \in \lambda\} \). For every \( \alpha \in \lambda \), we may fix a \( G_\kappa \)-subset \( S_\alpha \) of \( Z \) which is \( \kappa \)-good with respect to \( U_\alpha \). Note that since \( U_\alpha \) is a \( G_\kappa \)-subset of \( X \), we may assume without loss of generality that \( S_\alpha \cap X = U_\alpha \). Then, by Lemma 3.1, the set \( S = \bigcup_{\alpha \in \lambda} S_\alpha \) is a \( G_{\text{kcov}(\kappa^\lambda)} \)-subset of \( Z \). We claim that \( S \) is \( \text{kcov}(\kappa^\lambda) \)-good with respect to \( X \). First of all, note that \( X \subseteq S \), since \( U_\alpha \subseteq S_\alpha \), for \( \alpha \in \lambda \). So it remains to show that every point in \( S \setminus X \) can be separated from \( X \) by a \( G_{\text{kcov}(\kappa^\lambda)} \)-subset of \( Z \). Actually we will prove more: such a point can be separated from \( X \) by a \( G_\kappa \)-subset of \( Z \).

So, fix an arbitrary point \( z \in S \setminus X \). Then \( z \in S_\alpha \setminus U_\alpha \) for some \( \alpha \in \lambda \). Then, by construction, there is a \( G_\kappa \)-subset \( T \) of \( Z \) such that \( z \in T \) and \( T \cap U_\alpha = \emptyset \). But then, since \( S_\alpha \cap X = U_\alpha \), the set \( S_\alpha \cap T \) is a \( G_\kappa \)-subset of \( Z \) which contains \( z \) and misses \( X \).

Since \( \mathfrak{d} = \text{kcov}(\omega_1^\omega) \), the following corollary shows that the union of countably many open and \( \kappa \)-Ohio complete subspaces is \( \mathfrak{d} \)-Ohio complete. So Example 2.4 is best possible.

COROLLARY 3.3. Let \( X \) be a space. Let \( \mathcal{U} \) be a cover of \( X \) consisting of \( G_\kappa \)-subsets, with \( |\mathcal{U}| = \lambda \). Suppose that every element of \( \mathcal{U} \) is contained in a \( \kappa \)-Ohio complete subspace of \( X \). Then \( X \) is kcov\((\kappa^\lambda)\)-Ohio complete.

PROOF. Fix an arbitrary compactification \( \gamma X \) of \( X \). Since every element of \( \mathcal{U} \) is contained in a \( \kappa \)-Ohio complete subspace of \( X \), it follows from [3, Proposition 2.1] and [3, Corollary 2.5] that every element of \( \mathcal{U} \) is \( \kappa \)-Ohio embedded in \( \gamma X \). So the previous theorem, \( X \) is kcov\((\kappa^\lambda)\)-Ohio embedded in \( \gamma X \). Since \( \gamma X \) was an arbitrary compactification of \( X \), this shows that \( X \) is kcov\((\kappa^\lambda)\)-Ohio complete.

Our main interest is in open sum theorems. In particular we have the following.

COROLLARY 3.4. Assume \( \mathfrak{d} = \omega_1 \). Let \( \mathcal{U} \) be a countable open cover of \( X \) such that every element of \( \mathcal{U} \) is contained in an \( \omega_1 \)-Ohio complete subspace of \( X \), then \( X \) is \( \omega_1 \)-Ohio complete.

PROOF. It suffices to observe that \( \text{kcov}(\omega_1^\omega) = \mathfrak{d} \) (see for example [9, Proposition 3.6]), and then apply Corollary 3.3. 

\( \square \)
Let us denote the least cardinal $\kappa$ for which the union of $\lambda$-many open and $\kappa$-Ohio complete subspace fails to be $\kappa$-Ohio complete with the symbol $\mathcal{O}(\kappa)$. Then the following theorem holds.

**Theorem 3.5.** Let $\kappa$ be an infinite cardinal number.

1. If $\kappa$ is regular then $\mathcal{O}(\kappa) \leq \kappa$.
2. If $\lambda$ is a cardinal number such that $\text{kcov}(\kappa^\lambda) = \kappa$, then $\mathcal{O}(\kappa) > \lambda$.
3. Assuming the $\text{GCH}$, then $\mathcal{O}(\kappa) \geq \text{cf}(\kappa)$.
4. Assuming the $\text{GCH}$ and $\kappa$ regular then $\mathcal{O}(\kappa) = \kappa$.
5. If $\kappa < \mathfrak{d}$, then $\mathcal{O}(\kappa) = \omega$.
6. $\mathcal{O}(\omega_1) = \begin{cases} \omega_1, & \text{if } \mathfrak{d} = \omega_1, \\ \omega, & \text{if } \mathfrak{d} > \omega_1. \end{cases}$

**Proof.** Assertion (1) follows from Example 2.5. Corollary 3.3 implies (2). For assertion (3), observe that $\text{GCH}$ implies that, if $\lambda < \text{cf}(\kappa)$, then $\kappa^\lambda = \kappa$ (see [7, Theorem 5.15]) and then $\text{kcov}(\kappa^\lambda) = \kappa$. This implies $\mathcal{O}(\kappa) \geq \text{cf}(\kappa)$. Assertions (1) and (3) imply (4). Assertion (5) follows from Example 2.4. Finally, (6) follows from Corollary 3.4 and Example 2.4 ($\lambda = \omega_1$).

**Question 3.6.** If $\kappa$ is a singular cardinal, is it still true that $\mathcal{O}(\kappa) \leq \kappa$?

Finally, we consider locally countable and point-countable open sum theorems. By [3, Corollary 4.5], a point-finite and hence also locally finite union of open and $\omega_1$-Ohio complete subspaces is again $\omega_1$-Ohio complete. However as we have seen, if $\mathfrak{d} > \omega_1$, then even a countable union of open and $\omega_1$-Ohio complete subspaces may fail to be $\omega_1$-Ohio complete.

Now, if $\mathfrak{d} = \omega_1$, then the countable open sum theorem is true for $\omega_1$-Ohio completeness, but we do not know the answer to the following.

**Question 3.7.** Assume $\mathfrak{d} = \omega_1$. Does the point-countable or locally countable open sum theorem for $\omega_1$-Ohio completeness hold?

**Remark 3.8.** Let us point out that the notion of Ohio completeness could be generalized in even a more general way. Given infinite cardinal numbers $\kappa$ and $\lambda$, we say that a space $X$ is $(\kappa, \lambda)$-Ohio complete if for every compactification $\gamma X$ of $X$ there is a $G_\delta$-subset $S$ of $\gamma X$ such that $X \subseteq S$ and for every $y \in S \setminus X$, there is a $G_\delta$-subset of $\gamma X$ which contains $y$ and misses $X$.

So this notion is a further elaboration of the Ohio completeness property. Of course, many of the results in [3] may be rephrased in terms of this notion, with the two (possibly distinct) variables $\kappa$ and $\lambda$. The interested reader may verify that in certain results the first of these two variables plays a more important role.
than the second and in other results it is the other way around. In particular, in
the second example of this paper (Theorem 2.3) we added a one-point compacti-
fication of a space \( Y \) in the second coordinate, where the size of \( Y \) may be
arbitrarily large. So if \( \kappa \) is regular, then for any cardinal \( \lambda \), the union of \( \kappa \)-many
open and \( \kappa \)-Ohio complete spaces may fail to be \( (\kappa, \lambda) \)-Ohio complete.

References


Désirée BASILE
Faculty of Sciences
Department of Mathematics
Vrije Universiteit
De Boelelaan 1081A
1081 HV Amsterdam
the Netherlands
E-mail: basile@DMI.Unict. IT

Jan VAN MILL
Faculty of Sciences
Department of Mathematics
Vrije Universiteit
De Boelelaan 1081A
1081 HV Amsterdam
the Netherlands
E-mail: vanmill@few.vu.nl

Guit-Jan RIDDERBOS
Faculty of Electrical Engineering
Mathematics and Computer Science
TU Delft, Postbus 5031
2600 GA Delft
the Netherlands
E-mail: G.F.Ridderbos@tudelft.nl