TOPOLOGY PROCEEDINGS
Volume 33, 2009
Pages 153–161

http://topology.auburn.edu/tp/

Homogeneous spaces and transitive actions by $\aleph_0$-bounded groups

by

Jan van Mill

Electronically published on August 12, 2008

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 33 (2009)
Pages 153-161

E-Published on August 12, 2008

HOMOGENEOUS SPACES AND TRANSITIVE ACTIONS BY $\aleph_0$-BOUNDED GROUPS

JAN VAN MILL

ABSTRACT. We construct a homogeneous connected Polish space X on which no $\aleph_0$-bounded topological group acts transitively. In fact, X is homeomorphic to a convex subset of Hilbert space ℓ^2.

1. INTRODUCTION

Unless otherwise stated, all spaces under discussion are separable and metrizable.

In [4], an example of a homogeneous Polish space was constructed on which no $\aleph_0$-bounded topological group acts transitively. That space is not connected and this fact was used essentially in the verification of its properties. In this note we will construct a similar example that is connected. It has very strong connectivity properties since it is homeomorphic to a convex subset of Hilbert space ℓ^2. It is basically the same example as the one in [4] with the Cantor set replaced by the Hilbert cube. But the connectivity properties of the example make the verification of its properties more complicated.

2. PRELIMINARIES

We assume the reader is familiar with both the basic results in infinite-dimensional topology [3] and the construction in [4].

2000 *Mathematics Subject Classification.* Primary 22F05, 57S05, 22F30; Secondary 03E15.

Key words and phrases. Topological group, transitive action, Polish space.

Let $\mathbb{I} = [0,1]$, and $Q = \prod_{n=1}^{\infty}[-1,1]_n$, with admissible metric $\varrho(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$.

If $f\colon X \to Y$ is a function, then

$$\Gamma(f) = \{\big(x, f(x)\big) : x \in X\} \subseteq X \times Y$$

denotes its graph. The identity function on a set X will be denoted by 1_X.

The following result which is implicit in [1, Lemma 3.6] will be important in our construction.

Lemma 2.1. *Let A_1 and A_2 be subsets of a compact space X for which there exists a homeomorphism $h\colon X \setminus A_1 \to X \setminus A_2$. Assume that X has an open base $\mathcal{U}$ such that for every nonempty $U \in \mathcal{U}$ we have that $U \setminus A_1$ and $U \setminus A_2$ are both nonempty and connected. Moreover, let $p_i\colon \overline{\Gamma(h)} \to X$ be the projection maps, $i = 1, 2$. Then p_1 and p_2 are monotone surjections such that $p_1^{-1}(A_1) = p_2^{-1}(A_2)$.*

Proof. First observe that $X \setminus A_1$ and $X \setminus A_2$ are both dense in X. Moreover, $p_1{\restriction}\Gamma(h)\colon \Gamma(h) \to X \setminus A_1$ is a homeomorphism. Hence by [2, 3.5.6], $p_1\big(\overline{\Gamma(h)} \setminus \Gamma(h)\big) \subseteq A_1$. As a consequence, $p_1^{-1}(q) = \{\big(q, h(q)\big)\}$ for every $q \in X \setminus A_1$. Similarly, $p_2^{-1}(q) = \{(h^{-1}(q), q)\}$ for every $q \in X \setminus A_2$ and hence $p_1^{-1}(A_1) = p_2^{-1}(A_2)$. Since $X \setminus A_i$ is dense in X, each p_i is onto. By symmetry, it suffices to prove that p_1 is monotone. Assume that for some point $x \in A_1$, $p_1^{-1}(x)$ is not connected. Write $p_1^{-1}(x)$ as $F \cup G$, where F and G are disjoint nonempty closed sets. Choose disjoint open sets F' and G' in $\overline{\Gamma(h)}$ containing F and G, respectively. Since p_1 is a closed map, there is an element $U \in \mathcal{U}$ such that $x \in U$ and $p_1^{-1}(U) \subseteq F' \cup G'$. Observe that

$$U \setminus A_1 = \big(U \cap p_1(F' \cap \Gamma(h)\big) \cup \big(U \cap p_1(G' \cap \Gamma(h)\big).$$

This clearly contradicts the connectivity of $U \setminus A_1$. $\square$

Remark 2.2. It is clear that Lemma 2.1 can be generalized. We leave this to the reader.

Our example will be the complement of a σZ-set W in $Y = Q \times \mathbb{I}$. It is not difficult to prove that for every nonempty connected open subset U of Y the set $U \setminus W$ is connected (in fact, path-connected; any path in U between points in $U \setminus W$ may be deformed to a path in $U \setminus W$ via a deformation $\alpha\colon Y \times \mathbb{I} \to Y$ such that $\alpha(Y \times (0,1]) \subseteq Y \setminus W$). Hence we are in a position to apply Lemma 2.1.

A (not necessarily metrizable) topological group G is called $\aleph_0$-*bounded* provided that for every neighborhood U of the identity e there is a countable subset F of G such that $G = FU$. It was proved by Guran that a topological group G is $\aleph_0$-bounded if and only if it is topologically isomorphic to a subgroup of a product of separable metrizable groups. For a proof, see Uspenskiĭ [6].

A topologically complete (separable metrizable) space X is sometimes called *Polish*.

For a space X we let $\mathcal{H}(X)$ denote the homeomorphism group of X. A topology on $\mathcal{H}(X)$ is called *admissible* if it makes $\mathcal{H}(X)$ a topological group and makes the natural action of $\mathcal{H}(X)$ on X continuous. If X is compact, then the *compact-open* topology on $\mathcal{H}(X)$ is admissible and Polish.

3. The example

In this section we present the construction of our main example.

The example is basically the same as the one in [4] with the Cantor set replaced by the Hilbert cube. Its connectivity makes the verification of its properties more complicated. We ignore statements and proofs that can be copied almost verbatim from similar results in [4].

Consider the product $Y = Q \times \mathbb{I}$. Let $\pi_1 : Y \to Q$ and $\pi_2 : Y \to \mathbb{I}$ be the projection maps. If $x \in Y$, then x_1 abbreviates $\pi_1(x)$. Similarly for the second coordinate. On Y we use the admissible metric

$$d(x, y) = \max\{\varrho(x_1, y_1), |x_2 - y_2|\}.$$

Let $\mathcal{H}$ be the closed subgroup of $\mathcal{H}(Y)$ consisting of all those elements f with the following property: for all $q \in Q$ and $s, t \in \mathbb{I}$ such that $s < t$ we have that $\pi_1\big(f(q,s)\big) = \pi_1\big(f(q,t)\big)$ and $\pi_2\big(f(q,s)\big) < \pi_2\big(f(q,t)\big)$. This means that f permutes the collection $\big\{\{q\} \times \mathbb{I} : q \in Q\big\}$ and is 'order preserving' on each interval of the form $\{q\} \times \mathbb{I}$ for $q \in Q$.

Let Φ denote the collection of all pairs of functions $\langle \phi, \phi' \rangle$ having the following properties:

(1) $\operatorname{dom}(\phi) = \operatorname{dom}(\phi')$ is a countable dense subset of Q,
(2) $\operatorname{range}(\phi) \cup \operatorname{range}(\phi') \subseteq (0, 1)$,

(3) $\phi \ll \phi'$, i.e., $\phi(d) < \phi'(d)$ for every $d \in \operatorname{dom}(\phi) = \operatorname{dom}(\phi')$,
(4) if $x, y \in Y$, $x_1 = y_1$ and $x_2 < y_2$, then for every $\varepsilon > 0$ there exists $d \in \operatorname{dom}(\phi) = \operatorname{dom}(\phi')$ such that

$$\varrho(x_1, d) = \varrho(y_1, d) < \varepsilon, \quad |x_2 - \phi(d)| < \varepsilon, \quad |y_2 - \phi'(d)| < \varepsilon.$$

(Equivalently, $d\big(x, (d, \phi(d)\big) < \varepsilon$ and $d\big(y, (d, \phi'(d)\big) < \varepsilon$.)

Observe that $\operatorname{range}(\phi)$ and $\operatorname{range}(\phi')$ are countable dense subsets of $(0,1)$.

For $\langle \phi, \phi' \rangle \in \Phi$ such that $D = \operatorname{dom}(\phi) = \operatorname{dom}(\phi')$, put

$$U\langle \phi, \phi' \rangle = \bigcup_{x \in Q \setminus D} \{x\} \times (0,1) \cup \bigcup_{d \in D} \{d\} \times \big(\phi(d), \phi'(d)\big).$$

There is an element $\langle \phi, \phi' \rangle \in \Phi$, and for that element we put $Z = U\langle \phi, \phi' \rangle$. Observe that $W = Y \setminus Z$ is a σZ-set in Y. This implies by [1, Theorem 3.1] that Z is homeomorphic to a convex subset of ℓ^2.

Theorem 3.1. *Z is a homogenous, Polish space.*

An inspection of the proofs in [4, §3] shows that for this all we need to check is that Lemma 3.2 below holds. Indeed, the clopen set C in the Cantor set $\triangle$ in Lemma 3.1 in [4] corresponds in our setting to a connected open set C in Q. Standard homogeneity properties of Q (see [3]) yield that the proof of Lemma 3.1 in [4] can easily be adapted to get what we want. In the proof of Lemma 3.2 in [4], no specifics from $\triangle$ were used. A similar statement holds for an arbitrary space X, hence also for Q. The proofs of Proposition 3.3 and Lemma 3.4 in [4] can be copied almost verbatim with trivial changes only. There remains Lemma 3.5 in [4] of which we present the complete proof in the new setting in our next result.

Lemma 3.2. *If $x \in D$ and $y \in Q \setminus D$, then there is an element $h \in \mathcal{H}(Z)$ such that $h(\{x\} \times \big(\phi(x), \phi'(x))\big) = \{y\} \times (0,1)$.*

Proof. Let $t = \phi(x)$ and $t' = \phi'(x)$. Observe that $0 < t < t' < 1$. We let $f\colon \mathbb{I} \to \mathbb{I}$ be the continuous surjection that maps $[0,t]$ onto 0, $[t',1]$ onto 1, and $[t,t']$ linearly onto $[0,1]$, i.e.,

$$f(s) = \begin{cases} 0 & (0 \le s \le t), \\ \frac{s}{t'-t} - \frac{t}{t'-t} & (t \le s \le t'), \\ 1 & (t' \le s \le 1). \end{cases}$$

Then f is approximable by homeomorphisms of $\mathbb{I}$. Indeed, let $(t_n)_n$ be a sequence in $(0,t)$ such that $t_n \nearrow t$, and $(t'_n)_n$ a sequence in $(t',1)$ such that $t'_n \searrow t'$. For every n, let f_n be the unique homeomorphism of $\mathbb{I}$ mapping $[0,t_n]$ linearly onto $[0, {}^1\!/\!_n]$, $[t_n, t'_n]$ linearly onto $[{}^1\!/\!_n, 1 - {}^1\!/\!_n]$, and $[t'_n, 1]$ linearly onto $[1 - {}^1\!/\!_n, 1]$. Moreover, for every n, let $H(n)$ be the standard isotopy of $\mathbb{I}$ connecting f_n and f_{n+1}. That is, $H(n)_s$ maps $[0, (1-s)t_n + st_{n+1}]$ linearly onto $[0, (1-s)\,{}^1\!/\!_n + s\,{}^1\!/\!_{n+1}]$ for every $s \in \mathbb{I}$, etc. Hence $H(n)_0 = f_n$ and $H(n)_1 = f_{n+1}$. Observe that $f_n \to f$.

Let $\alpha\colon Q \to \mathbb{I}$ be a Urysohn function such that $\alpha^{-1}(0) = \{x\}$ and $\alpha(Q) = \mathbb{I}$. Recall that $Y = Q \times \mathbb{I}$. Define $\xi\colon Y \to Y$ as follows:

$$\xi(q,t) = \begin{cases} \big(q, H(n)_{-2^n\alpha(q)+2}(t)\big) & (2^{-n} \le \alpha(q) \le 2^{-n+1}), \\ \big(q, f(t)\big) & (\alpha(q) = 0 \Leftrightarrow q = x). \end{cases}$$

Then ξ is a continuous surjection, and maps the complement of $\{x\} \times ([0,t] \cup [t',1])$ homeomorphically onto the complement of $\{(x,0),(x,1)\}$. Define $\psi, \psi'\colon D \setminus \{x\} \to (0,1)$ so that for every $z \in D \setminus \{x\}$ we have that $\xi\big(z, \phi(z)\big) = \big(z, \psi(z)\big)$ and $\xi\big(z, \phi'(z)\big) = \big(z, \psi'(z)\big)$. Observe that $\langle \psi, \psi' \rangle \in \Phi$ and $\xi(Z) = U\langle \psi, \psi' \rangle$. By [4, Proposition 3.3] there is an element $h \in \mathcal{H}$ such that $h(U\langle \psi, \psi' \rangle) = U\langle \phi, \phi' \rangle = Z$. Then $q = h \circ \xi$ maps the complement of $\{x\} \times ([0,t] \cup [t',1])$ homeomorphically onto the complement of $\{h(x,0), h(x,1)\}$, hence $\{x\} \times (t,t')$ homeomorphically onto $\big\{\pi_1\big(h(x,0)\big)\big\} \times (0,1)$, and

$$q(Z) = h\big(\xi(Z)\big) = h(U\langle \psi, \psi' \rangle) = U\langle \phi, \phi' \rangle = Z,$$

as required. □

We will now analyze all the homeomorphisms of Z. For that it will be convenient to introduce some notation. Define $\varphi, \varphi'\colon Q \to \mathbb{I}$ as follows:

$$\varphi(q) = \begin{cases} 0 & (q \in Q \setminus D), \\ \phi(q) & (q \in D), \end{cases} \qquad \varphi'(q) = \begin{cases} 1 & (q \in Q \setminus D), \\ \phi'(q) & (q \in D). \end{cases}$$

Elements of the collection

$$\mathcal{T} = \big\{ \{q\} \times \big(\varphi(q), \varphi'(q)\big) : q \in Q \big\}$$

are thought of as 'vertical sections' of Z. Clearly, $Z = \bigcup \mathcal{T}$.

Theorem 3.3. *Each homeomorphism of Z permutes the collection $\mathcal{T}$.*

Proof. Let $h \in \mathcal{H}(Z)$, and let $\Gamma(h)$ denote the graph of h in $Y \times Y$. We adopt the notation in Lemma 2.1 with $A_1 = A_2 = \bigcup_{d \in D} \{d\} \times ([0, \phi(d)] \cup [\phi'(d), 1]) \cup (Q \times \{0, 1\})$.

Put $K = Q \times \{0, 1\}$. Then $L = p_1\big(p_2^{-1}(K)\big)$ is a compact subset of $Y \setminus Z$.

Take an arbitrary $q \in Q$, and consider the vertical section

$$S = \{q\} \times \big(\varphi(q), \varphi'(q)\big).$$

Claim 1. $\pi_1\big(h(S)\big)$ is a single point.

Proof. Assume that $\pi_1\big(h(S)\big)$ contains the points u and v. Pick $u', v' \in \big(\varphi(q), \varphi'(q)\big)$ such that

$$(\pi_1 \circ h)(q, u') = h(q, u')_1 = u, \quad (\pi_1 \circ h)(q, v') = h(q, v')_1 = v.$$

We may assume without loss of generality that $u' \leq v'$. Pick $\alpha, \beta \in \big(\varphi(q), \varphi'(q)\big)$ such that $\alpha < u' \leq v' < \beta$. Let $(d_n)_n$ be a sequence in D such that

(1) $\lim_{n \to \infty} d_n = q$,
(2) for every n, $\beta < \phi(d_n) < \phi'(d_n)$.

For every n, consider the segment

$$S_n = \{d_n\} \times [\alpha, \beta].$$

Observe that by (2), $S_n \subseteq Y \setminus Z$. We claim that there exists N such that $S_n \cap L = \emptyset$ for every $n \geq N$. If not, then we may assume without loss of generality that every S_n and L intersect, say in the point x_n. Since L is compact, we may assume without loss of generality that $x_n \to x$, where $x \in L$. Observe that $L \subseteq Y \setminus Z$ and hence that $x \notin Z$. But, clearly, $x \in \{q\} \times [\alpha, \beta] \subseteq Z$, which gives us the desired contradiction. So we may assume without loss of generality that

$$S_n \cap L = \emptyset \qquad (\forall\, n \in \mathbb{N}). \tag{3}$$

Now fix n for a moment. Then by (3), $p_1^{-1}(S_n) \cap p_2^{-1}(Q \times \{0, 1\}) = \emptyset$. This implies by Lemma 2.1 that $S'_n = p_2\big(p_1^{-1}(S_n)\big)$ is a continuum that is contained in $\bigcup_{d \in D} \{d\} \times [0, \phi(d)] \cup \bigcup_{d \in D} \{d\} \times [\phi'(d), 1]$. By the Sierpiński Theorem from [5] (see also [2, 6.1.27]), there is a unique $d \in D$ such that either $S'_n \subseteq \{d\} \times [0, \phi(d)]$ or $S'_n \subseteq \{d\} \times [\phi'(d), 1]$. From this we conclude that for all n,

$$\pi_1\big(p_2\big(p_1^{-1}(S_n)\big)\big) \text{ is a single point.} \tag{4}$$

Observe that $(d_n, u') \to (q, u')$ and that $p_1^{-1}(q, u')$ is a single point while every $p_1^{-1}(d_n, u')$ may very well be nontrivial. Since p_1 is a closed map, it easily follows that $p_1^{-1}(d_n, u') \to p_1^{-1}(q, u')$. (Simply use that p_1 is a closed map and $p_1^{-1}(q, u')$ is a singleton.) Hence

$$p_2\big(p_1^{-1}(d_n, u')\big) \to p_2\big(p_1^{-1}(q, u')\big) = h(q, u').$$

It follows similarly that

$$p_2\big(p_1^{-1}(d_n, v')\big) \to p_2\big(p_1^{-1}(q, v')\big) = h(q, v').$$

Hence by (4) we get $u = v$, as required. $\diamondsuit$

So we conclude that there is an element $q' \in Q$ such that

$$h(S) = \{q'\} \times \pi_2\big(h(S)\big) = \{q'\} \times (a, b) \subseteq Z.$$

Observe that both S and $h(S)$ are closed subsets of Z. Hence $h(S) = \{q'\} \times \big(\varphi(q'), \varphi'(q')\big)$, i.e., $h(S) \in \mathcal{T}$. Since h is a bijection, this consequently implies that h permutes $\mathcal{T}$. $\square$

4. Actions on Z

Since Z is a homogeneous topological space by Theorem 3.1, it is natural to ask whether there are actions of topological groups on Z that are more interesting than the natural action of the discrete group $\mathcal{H}(Z)$ on Z. In the following result we formulate a criterion that tells us that interesting actions on certain spaces do not exist. This criterion was extracted from [4, Theorem 4.2]. A collection $\mathcal{T}$ of subsets of a space X is called *invariant* provided that for all $f \in \mathcal{H}(X)$ and all $T \in \mathcal{T}$ we have that $f(T) \in \mathcal{T}$. Observe that this implies that for all $f \in \mathcal{H}(X)$ and all $T \in \mathcal{T}$ we also have that $f^{-1}(T) \in \mathcal{T}$. Hence every homeomorphism of X permutes the elements of $\mathcal{T}$. Examples are the collection of all components of X, or the collection of all subsets of X of a given dimension.

Theorem 4.1. *Let X be a topological space admitting a pairwise disjoint invariant collection $\mathcal{T}$ such that every nonempty open subset of X contains a nonempty member from $\mathcal{T}$. Assume that there exists $\varepsilon > 0$ such that $\bigcup\{T \in \mathcal{T} : \operatorname{diam} T < \varepsilon\}$ is meager in X. Then if G is an $\aleph_0$-bounded topological group that acts on X by a separately continuous action, there is an element $x \in X$ such that its orbit Gx is meager in X.*

Proof. It will be convenient to let $\mathcal{U}_e$ denote the collection of all open neighborhoods of the neutral element e in G. Striving for a contradiction, assume that for every $x \in X$ its orbit Gx is not meager in X.

Claim 1. Let $p \in X$ and $U \in \mathcal{U}_e$. Then Up is not meager in X. If moreover $Gp \cap T \neq \emptyset$ for some $T \in \mathcal{T}$, then $\bigcup\{T \in \mathcal{T} : Up \cap T \neq \emptyset\}$ is not meager in X as well.

Proof. Since G is $\aleph_0$-bounded, there is a countable set H in G such that $HU = G$. This means that $Gp = HUp = \bigcup_{h \in H} hUp$. Observe that for every $h \in H$ the function $x \mapsto hx$ is a homeomorphism of X. Hence if Up is meager then so is hUp for every $h \in H$, and hence so is Gp since H is countable. This proves that Up is not meager.

Now assume that $\bigcup\{T \in \mathcal{T} : Up \cap T \neq \emptyset\}$ is meager in X. Then since $\mathcal{T}$ is invariant, by the same argumentation as the one above, for every $h \in H$ we have that $\bigcup\{T \in \mathcal{T} : hUp \cap T \neq \emptyset\}$ is meager. Since H is countable, $\bigcup\{T \in \mathcal{T} : Gp \cap T \neq \emptyset\}$ is meager as well. There exists by assumption an element $T \in \mathcal{T}$ such that $T \cap Gp \neq \emptyset$, say $q \in T \cap Gp$. Since $\mathcal{T}$ is invariant, this means that Gq is contained in the meager set $\bigcup\{T \in \mathcal{T} : Gp \cap T \neq \emptyset\}$, and this is a contradiction. $\diamondsuit$

Take an arbitrary $p \in X$, and let V be a closed neighborhood of p in X of diameter less than ε. Let $\gamma_p \colon G \to X$ denotes the (continuous) function $g \mapsto gp$. Then $W = \gamma_p^{-1}(V)$ is a neighborhood of the neutral element of G. Choose an open neighborhood U of e such that $U^2 \subseteq W$.

We claim that if $q \in \overline{Up}$ then $Uq \subseteq V$. This is easy. Indeed, if $h \in U$ then

$$hq \in h\big(\overline{Up}\big) = \overline{hUp} \subseteq \overline{U^2p} \subseteq \overline{Wp} \subseteq \overline{V} = V,$$

as required.

Observe that $\overline{Up}$ is not nowhere dense by Claim 1, i.e., its interior E is nonempty. Pick $T \in \mathcal{T}$ such that $\emptyset \neq T \subseteq E$, and let $q_0 \in T$. Then by Claim 1, $\bigcup\{T \in \mathcal{T} : Uq_0 \cap T \neq \emptyset\}$ is not meager in X. There consequently exists by our assumptions an element $T' \in \mathcal{T}$ such that $\operatorname{diam} T' \geq \varepsilon$ and $T' \cap Uq_0 \neq \emptyset$. Let $q_1 \in T' \cap Uq_0$. There exists an element $g \in U$ such that $gq_0 = q_1$. Hence $gT \cap T' \neq \emptyset$, so $gT = T'$ since $\mathcal{T}$ is pairwise disjoint, i.e., $\operatorname{diam} gT \geq \varepsilon$.

On the other hand, $T \subseteq \overline{Up}$, and hence by the above, gT, being a subset of $UT \subseteq V$, has diameter less than ε. This is a contradiction. $\square$

Corollary 4.2. *No $\aleph_0$-bounded topological group G acts transitively on Z by a separately continuous action.*

Proof. By Theorem 3.3, the collection of all 'vertical sections' $\mathcal{T}$ is invariant. Observe that all but countably many elements of $\mathcal{T}$ have diameter 1 and that every element of $\mathcal{T}$ is meager in Z. It consequently follows that $\mathcal{T}$ satisfies the conditions in Theorem 4.1 with $\varepsilon = 1/2$. So we are done by the fact that Z is Polish. $\square$

Corollary 4.3. *$\mathcal{H}(Z)$ does not admit an $\aleph_0$-bounded admissible topology.*

The results in this note suggest the following interesting open problems.

Question 4.4. Does $\mathcal{H}(Z)$ admit an admissible topology which is not discrete?

Question 4.5. Is there an infinite homogeneous Polish space X having the property that the only admissible topology on $\mathcal{H}(X)$ is the discrete topology?

References

[1] R. D. Anderson, D. W. Curtis, and J. van Mill, *A fake topological Hilbert space*, Trans. Amer. Math. Soc. **272** (1982), 311–321.

[2] R. Engelking, *General topology*, Heldermann Verlag, Berlin, second ed., 1989.

[3] J. van Mill, *Infinite-dimensional topology: prerequisites and introduction*, North-Holland Publishing Co., Amsterdam, 1989.

[4] J. van Mill, *Homogeneous spaces and transitive actions by Polish groups*, Israel J. Math. **165** (2008), 133-159.

[5] W. Sierpiński, *Un théorème sur les continus*, Tôhoku Math. **13** (1918), 300–303.

[6] V. V. Uspenskiĭ, *Why compact groups are dyadic*, General topology and its relations to modern analysis and algebra, VI (Prague, 1986), Res. Exp. Math., vol. 16, Heldermann, Berlin, 1988, pp. 601–610.

Faculty of Sciences, Department of Mathematics, VU University Amsterdam, De Boelelaan 1081^a, 1081 HV Amsterdam, The Netherlands

E-mail address: vanmill@few.vu.nl