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Characterizing Complete Erdős Space

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Abstract. The space now known as *complete Erdős space* $\mathfrak{E}_c$ was introduced by Paul Erdős in 1940 as the closed subspace of the Hilbert space ℓ^2 consisting of all vectors such that every coordinate is in the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. In a solution to a problem posed by Lex G. Oversteegen we present simple and useful topological characterizations of $\mathfrak{E}_c$. As an application we determine the class of factors of $\mathfrak{E}_c$. In another application we determine precisely which of the spaces that can be constructed in the Banach spaces ℓ^p according to the 'Erdős method' are homeomorphic to $\mathfrak{E}_c$. A novel application states that if I is a Polishable F_σ-ideal on ω, then I with the Polish topology is homeomorphic to either $\mathbb{Z}$, the Cantor set 2^ω, $\mathbb{Z} \times 2^\omega$, or $\mathfrak{E}_c$. This last result answers a question that was asked by Stevo Todorčević.

1 Introduction

We present a number of topological characterizations of complete Erdős space $\mathfrak{E}_c$. As an application we determine the class of factors of $\mathfrak{E}_c$, and we prove that $\mathfrak{E}_c$ has the curious property that whenever a product $\prod_{i=0}^{\infty} X_i$ is homeomorphic to $\mathfrak{E}_c$, then at least one but no more than finitely many of the X_i's are homeomorphic to $\mathfrak{E}_c$. In another application we determine precisely which of the spaces that can be constructed in the Banach spaces ℓ^p according to the 'Erdős method' [20] are homeomorphic to $\mathfrak{E}_c$; see Theorem 4.1. A new type of application can be found in §4.4 and states that if I is a Polishable F_σ-ideal on ω, then I with the Polish topology is homeomorphic to either $\mathbb{Z}$, the Cantor set 2^ω, $\mathbb{Z} \times 2^\omega$, or $\mathfrak{E}_c$; see Theorem 4.15. This last result answers a question that was posed to us by S. Todorčević. We also show by example that Polishable ideals that are not F_σ can be either homeomorphic to $\mathfrak{E}_c$ or not homeomorphic to $\mathfrak{E}_c$ in the Polish topology, even if that topology is one-dimensional; see Example 4.20.

Consider the Hilbert space ℓ^2 consisting of the square summable sequences $x = (x_0, x_1, \dots)$ of real numbers. Erdős [20] introduced the closed subspace of ℓ^2 consisting of all $x \in \ell^2$ such that every coordinate x_i is in the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. This space is now known as *complete Erdős space*. Kawamura, Oversteegen, and Tymchatyn [22] represented complete Erdős space as $\{x \in \ell^2 : x_i \in \mathbb{R} \setminus \mathbb{Q} \text{ for all } i\}$. It is known that this space is homeomorphic to Erdős' original model; see Dijkstra [9] and Remark 4.3. It is proved in [22] that complete Erdős space is homeomorphic to the end-point set of a Lelek fan as constructed in [25]. We find it convenient to use the latter representation for $\mathfrak{E}_c$. Since the Lelek fan was shown to be topologically unique by Charatonik [6] and Bula and Oversteegen [5], we have that $\mathfrak{E}_c$ is well defined.

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The paper [22] contains a characterization of $\mathfrak{E}_c$. However, this characterization is quite technical and it is not topological but metric in nature. In [29, Question 7.1] Oversteegen asks whether there is a simple characterization of complete Erdős space. We believe that the following result fits the bill.

Theorem 1.1 **(Characterization)** *A nonempty space E is homeomorphic to $\mathfrak{E}_c$ if and only if there is a zero-dimensional topology $\mathcal{W}$ on E that is coarser than the given topology on E such that for every $x \in E$ and neighbourhood U of x in E there is a neighbourhood V of x in E with V closed in $(E, \mathcal{W})$, $(V, \mathcal{W})$ topologically complete, and V a nowhere dense subset of $(U, \mathcal{W})$.*

2 Preliminaries

Unless otherwise stated all topological spaces in this paper are assumed to be separable metric.

Definition 2.1 A subset A of a space X is called a *C-set in X* if A can be written as an intersection of clopen subsets of X. A space is called *almost zero-dimensional* if every point of the space has a neighbourhood basis consisting of C-sets of the space. If Z is a set that contains X, then we say that a (separable metric) topology $\mathcal{T}$ on Z *witnesses the almost zero-dimensionality of X* if $\dim(Z, \mathcal{T}) \leq 0$, $O \cap X$ is open in X for each $O \in \mathcal{T}$, and every point of X has a neighbourhood basis in X consisting of sets that are closed in $(Z, \mathcal{T})$. We will also say that the space $(Z, \mathcal{T})$ is a witness to the almost zero-dimensionality of X.

Remark 2.2. Observe that every C-set is closed and that the property is preserved under finite unions and intersections. The concept of an almost zero-dimensional space is due to Oversteegen and Tymchatyn [30]. The definition given here is easier to use than the original one in [30] and was shown to be equivalent in Dijkstra, van Mill, and Steprāns [15]. Note that almost zero-dimensionality is hereditary.

Clearly, a space X is almost zero-dimensional if and only if there is a topology on X witnessing this fact. Let Z be a witness to the almost zero-dimensionality of some space X and let O be open in X. Then since X is separable metric, we can write O as a union of countably many sets that are closed in Z. So every open set of X is F_σ in the witness topology.

A function $\varphi\colon X \to [-\infty, \infty]$ is called *upper semi-continuous* (*USC*) if $\{x \in X : \varphi(x) < t\}$ is open in X for every $t \in \mathbb{R}$. φ is called *lower semi-continuous* (*LSC*) if $-\varphi$ is USC.

Definition 2.3 Let $\varphi\colon X \to [0, \infty)$ and define

$$G_0^\varphi = \{(x, \varphi(x))\colon x \in X \text{ and } \varphi(x) > 0\}$$

and

$$L_0^\varphi = \{(x, t)\colon x \in X \text{ and } 0 \leq t \leq \varphi(x)\}$$

both equipped with the topology inherited from $X \times \mathbb{R}$. We say that φ is a *Lelek function* if X is zero-dimensional, φ is USC, and G_0^φ is dense in L_0^φ.

Remark 2.4. The following facts can be found in Lelek [25]. Lelek functions with compact domain C exist and C must be homeomorphic to the Cantor set. If φ is a Lelek function with a compactum C as domain and we identify the set $C \times \{0\}$ with a point in L_0^φ, then we obtain a *Lelek fan*. The end-point set of a Lelek fan G_0^φ is one-dimensional and topologically complete. As mentioned in the introduction we will use G_0^φ as our standard model for $\mathfrak{E}_c$. Since the Lelek fan is unique by [6] and [5], any Lelek function φ with compact domain will do.

The following result links witness topologies with USC functions and was taken from [14, Lemma 4.11]; see also [2, Corollary 5].

Lemma 2.5 *Let X be a space and let Z be zero-dimensional space that contains X as a subset (but not necessarily as a subspace). Then the following statements are equivalent:*

(1) *Z is a witness to the almost zero-dimensionality of X.*
(2) *There exists a USC function $\varphi\colon Z \to [0, \infty)$ such that the map $h\colon X \to G_0^\varphi$ defined by the rule $h(x) = (x, \varphi(x))$ is a homeomorphism.*

Definition 2.6 A space is called *nowhere zero-dimensional* if no point of the space has a clopen neighbourhood basis. A space X is called *cohesive* if every point of the space has a neighbourhood that does not contain nonempty clopen subsets of X.

Every cohesive space is clearly nowhere zero-dimensional, but the converse is not true even for homogeneous spaces; see Dijkstra [10]. The following observation is trivial but useful.

Proposition 2.7 *A product $\prod_{i=0}^\infty X_i$ is cohesive if and only if some X_k is cohesive.*

Lelek [25] proved that $\mathfrak{E}_c$ can be turned into a connected space through the addition of just one point, which means that $\mathfrak{E}_c$ is cohesive. Also Erdős proved in [20] that his representations of complete Erdős space are cohesive, cf. Remark 4.3.

The following result from [14, Lemma 5.9] gives the connection between cohesion and Lelek functions.

Lemma 2.8 *Let φ be a USC function from a zero-dimensional space X to $[0, \infty)$ such that G_0^φ is cohesive and $\{x \in X\colon \varphi(x) > 0\}$ is dense in X. Then there exists a Lelek function $\chi\colon X \to \mathbb{R}^+$ such that $\chi \le \varphi$, the natural bijection h from the graph of φ to the graph of χ is continuous, and the restriction $h \restriction G_0^\varphi\colon G_0^\varphi \to G_0^\chi$ is a homeomorphism.*

Definition 2.9 Let $\varphi\colon X \to [0, \infty]$ be a function and let X be a subset of a metric space (Y, d). We define $\operatorname{ext}_Y \varphi\colon Y \to [0, \infty]$ by

$$(\operatorname{ext}_Y \varphi)(y) = \lim_{\varepsilon \searrow 0} (\sup\{\varphi(z) : z \in X \text{ with } d(z, y) < \varepsilon\}) \quad \text{for } y \in Y,$$

where we use the convention $\sup \varnothing = 0$.

Note that the metric on Y is mentioned strictly for the sake of convenience and that the definition of $\operatorname{ext}_Y \varphi$ does not depend on the choice of d. It is easily seen that $\operatorname{ext}_Y \varphi$ is always USC and that it extends φ whenever φ is USC.

3 Characterization and Stability

The following result includes Theorem 1.1.

Theorem 3.1 *For a nonempty space E, the following statements are equivalent.*

(1) *E is homeomorphic to $\mathfrak{E}_c$.*

(2) *E is cohesive and there is a zero-dimensional topology $\mathcal{W}$ on E such that $\mathcal{W}$ is coarser than the given topology and that has the property that every point in E has a neighbourhood basis consisting of sets that are compact with respect to $\mathcal{W}$.*

(3) *There is a zero-dimensional topology $\mathcal{W}$ on E that is coarser than the given topology on E such that for every $x \in E$ and neighbourhood U of x in E there is a neighbourhood V of x in E with V closed in $(E, \mathcal{W})$, $(V, \mathcal{W})$ topologically complete, and V a nowhere dense subset of $(U, \mathcal{W})$.*

(4) *There is a topology $\mathcal{W}$ on E such that $\mathcal{W}$ witnesses the almost zero-dimensionality of E, every point in E has a neighbourhood that is topologically complete in $(E, \mathcal{W})$, and every open subset O of E is first category in $(O, \mathcal{W})$.*

(5) *E is cohesive, there is a topology $\mathcal{W}$ on E such that $\mathcal{W}$ witnesses the almost zero-dimensionality of E, and every point in E has a neighbourhood that is topologically complete in $(E, \mathcal{W})$.*

Proof (1) $\Rightarrow$ (2). Let $\mathfrak{E}_c = G_0^{\varphi}$ for some Lelek function φ with compact domain K. $\mathfrak{E}_c$ is known to be cohesive. Let $\pi\colon K \times \mathbb{R} \to K$ denote the projection. Let Z be the graph of φ with the topology that is lifted from K, that is, $\pi\restriction Z\colon Z \to K$ is a homeomorphism. According to Lemma 2.5 the space Z witnesses the almost zero-dimensionality of G_0^{φ}. If $\mathcal{W}$ is the topology that G_0^{φ} inherits from the compact space Z, then it is clear that $\mathcal{W}$ satisfies (2).

(2) $\Rightarrow$ (3). Assume that E satisfies (2) and let U be a neighbourhood of some point x in E. Since E is cohesive, we may select an open neighbourhood W of x in E such that $W \subset U$ and W contains no nonempty clopen subsets of E. Select a neighbourhood V of x in E such that $V \subset W$ and $(V, \mathcal{W})$ is compact. Then V is closed and topologically complete in $(E, \mathcal{W})$. Suppose that V has a nonempty interior in $(U, \mathcal{W})$. Since $\dim(E, \mathcal{W}) = 0$, we have that V contains a nonempty set C that is clopen in $(U, \mathcal{W})$. Then C is closed in $(V, \mathcal{W})$ and hence it is closed in $(E, \mathcal{W})$ and E. On the other hand, C is open in $(W, \mathcal{W})$ and hence open in W and therefore also in E. Thus we have that V and W contain a nonempty clopen subset C of E. Since this contradicts the cohesion assumption, we have shown that V is nowhere dense in $(U, \mathcal{W})$.

(3) $\Rightarrow$ (4). Assume that E satisfies (3) and note that it suffices to prove that every open subset of E is first category in itself with respect to $\mathcal{W}$. Let O be an arbitrary open subset of E. Choose for each $x \in O$ a neighbourhood U_x of x in O that is nowhere dense in $(O, \mathcal{W})$. Since E is separable metric, we can find a countable set $A \subset O$ with $O = \bigcup\{U_x : x \in A\}$. We have that $(O, \mathcal{W})$ is first category in itself.

(4) $\Rightarrow$ (5). Assume that E satisfies (4) and note that it suffices to prove that E is cohesive. Let $x \in E$ be arbitrary and select a neighbourhood U of x in E such that $(U, \mathcal{W})$ is topologically complete. Let C be a clopen subset of E that is contained in U. By Remark 2.2 we have that $E \setminus C$ is an F_σ-set in $(E, \mathcal{W})$. Thus C is a G_δ-subset of the complete space $(U, \mathcal{W})$ and hence $(C, \mathcal{W})$ is topologically complete. On the other

hand, we have by assumption that $(C, \mathcal{W})$ is first category in itself thus $C = \varnothing$ by the Baire Category Theorem. We have shown that E is cohesive.

(5) $\Rightarrow$ (1). Assume that E is some space that satisfies condition (5). Let Z denote E equipped with the witness topology $\mathcal{W}$. Let K be a zero-dimensional compactification of Z. Let $\mathcal{B}$ be a countable collection of closed and topologically complete subsets of Z such that for each $x \in E$ and each neighbourhood U of x in E there is a $B \in \mathcal{B}$ that is a neighbourhood of x in E that is contained in U. We define

$$Y = K \setminus \bigcup\{\overline{B} \setminus B : B \in \mathcal{B}\},$$

where $\overline{B}$ stands for the closure in K. Note that Y is a G_δ-subset of K that contains Z and that Y is a witness to the almost zero-dimensionality of E. With Lemma 2.5 we can find a USC function $\varphi\colon Y \to [0,1]$ such that E is homeomorphic to G_0^φ. Since E is cohesive, we can find a Lelek function $\chi\colon Y \to [0,1]$ using Lemma 2.8 such that G_0^χ is homeomorphic to G_0^φ. Let $\widetilde{\chi} = \mathrm{ext}_K\, \chi$ and note that G_0^χ is dense in $L_0^{\widetilde{\chi}}$ by [14, Lemma 4.8.b] and hence $\widetilde{\chi}$ is a Lelek function just as χ. Because the domain of $\widetilde{\chi}$ is compact we have $\mathfrak{E}_\mathrm{c} = G_0^{\widetilde{\chi}}$. Note that $K \setminus Y$ is σ-compact and $G_0^{\widetilde{\chi}\restriction K\setminus Y}$ is a first category set in $G_0^{\widetilde{\chi}}$ because its complement is G_0^χ. According to [22, Theorem 6] we have that G_0^χ is homeomorphic to $G_0^{\widetilde{\chi}}$; see also [16, Theorem 26]. We now have $\mathfrak{E}_\mathrm{c} = G_0^{\widetilde{\chi}} \approx G_0^\chi \approx G_0^\varphi \approx E$. ■

If X is a nonempty space, then Y is called an *X-factor* if there is a space Z such that $Y \times Z$ is homeomorphic to X.

Theorem 3.2 **(Stability)** *For a nonempty space E the following statements are equivalent:*

(1) *$E \times \mathfrak{E}_\mathrm{c}$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$.*
(2) *E is an $\mathfrak{E}_\mathrm{c}$-factor.*
(3) *E is homeomorphic to a retract of $\mathfrak{E}_\mathrm{c}$.*
(4) *E admits an imbedding as a C-set in $\mathfrak{E}_\mathrm{c}$.*
(5) *E admits a closed imbedding into $\mathfrak{E}_\mathrm{c}$.*
(6) *E is homeomorphic to G_0^φ where φ is some USC function with a complete zero-dimensional domain.*
(7) *E is almost zero-dimensional as witnessed by a topology $\mathcal{W}$ such that every point of E has a neighbourhood that is complete in $(E, \mathcal{W})$.*

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial, and we have (3) $\Leftrightarrow$ (4) by [14, Theorem 4.18].

(5) $\Rightarrow$ (6). Assume that E is a closed subset of G_0^φ for some Lelek function φ with compact domain K. Let $\pi\colon K \times \mathbb{R} \to K$ denote the projection. Let Y be the graph of φ as a subspace of $K \times \mathbb{R}$ and let Z be the graph of φ with the topology that is lifted from K, that is, $\pi\restriction Z\colon Z \to K$ is a homeomorphism. Applying Lemma 2.5 to the function $1 + \varphi$ we find that the space Z witnesses the almost zero-dimensionality of Y. Since G_0^φ is open in Y, we have that $G_0^\varphi \setminus E$ is also open in Y, and hence by Remark 2.2 this set is an F_σ-set in Z. Thus we have that $X = K \setminus \pi(G_0^\varphi \setminus E)$ is a G_δ-set in K and topologically complete. Note that $G_0^{\varphi\restriction X} = E$, which proves this case.

(6) $\Rightarrow$ (7) This implication follows by the same argument as in the proof of the case (1) $\Rightarrow$ (2) for Theorem 3.1 (just replace "compact" by "complete").

(7) $\Rightarrow$ (1) Assume (7) and let $Z = (E, \mathcal{W})$. By Proposition 2.7 we have that $E \times \mathfrak{E}_c$ is cohesive. Let $Z' = (\mathfrak{E}_c, \mathcal{W}')$ be a witness to $\mathfrak{E}_c$ that satisfies property (5) of Theorem 3.1. Then, trivially, the topology on $Z \times Z'$ is a witness to the almost zero-dimensionality of $E \times \mathfrak{E}_c$ that also satisfies that property. Apply Theorem 3.1 to find $E \times \mathfrak{E}_c \approx \mathfrak{E}_c$. ■

Remark 3.3. In particular, we have that every nonempty and zero-dimensional complete space is an $\mathfrak{E}_c$-factor. This result follows also from [22]. It is also shown in [22] that every nonempty open subset of $\mathfrak{E}_c$ is homeomorphic to $\mathfrak{E}_c$. Note that this result also follows immediately from Theorem 1.1. The paper [15] features a non-homogeneous dense G_δ-subset G of $\mathfrak{E}_c$ such that $G \times \mathfrak{E}_c \approx G$.

The example G was presented in [15] to give a negative answer to the question in [22] whether every cohesive dense G_δ-subset of $\mathfrak{E}_c$ is homeomorphic to $\mathfrak{E}_c$; see also [14, Proposition 5.4]. In connection to this question we have the following positive result.

Proposition 3.4 *Let $\mathcal{W}$ be a witness topology on $\mathfrak{E}_c$ such that every point of $\mathfrak{E}_c$ has a neighbourhood that is complete in $(\mathfrak{E}_c, \mathcal{W})$. If X is a dense subset of $\mathfrak{E}_c$ that is a G_δ-set in $(\mathfrak{E}_c, \mathcal{W})$, then X is homeomorphic to $\mathfrak{E}_c$.*

Proof By the same argument used for the implication (2) $\Rightarrow$ (3) in the proof of Theorem 3.1 we have that $\mathcal{W}$ satisfies the requirements as formulated in Theorem 1.1. We show that the restriction of $\mathcal{W}$ to X also satisfies Theorem 1.1 whence $X \approx \mathfrak{E}_c$. Let $x \in X$ and let U be an open set in $\mathfrak{E}_c$ that contains x. Note that $X \cap U$ is dense in U and hence dense in $(U, \mathcal{W})$ because $\mathcal{W}$ is weaker. Let V be a neighbourhood of x in $\mathfrak{E}_c$ such that V is closed in $(\mathfrak{E}_c, \mathcal{W})$, $(V, \mathcal{W})$ is complete, and V is a nowhere dense subset of $(U, \mathcal{W})$. Then $X \cap V$ is closed in $(X, \mathcal{W})$, and $X \cap V$ is a G_δ-subset of $(V, \mathcal{W})$ thus complete in the topology $\mathcal{W}$. Since $X \cap U$ is dense, we have that $X \cap V$ is nowhere dense in $(X \cap U, \mathcal{W})$. ■

If we combine Theorem 3.2 with Theorem 3.1, we find:

Theorem 3.5 *A nonempty space is homeomorphic to $\mathfrak{E}_c$ if and only if it is cohesive and it satisfies one of the seven equivalent conditions of Theorem* 3.2.

Corollary 3.6 *A nonempty space is homeomorphic to $\mathfrak{E}_c$ if and only if it is homeomorphic to G_0^φ where φ is some Lelek function with a complete domain.*

Proof Let φ be a Lelek function with a complete domain. Proposition 4.4 in [14] shows that G_0^φ is cohesive. Now use Theorem 3.5. ■

Lemma 3.7 *A closed subset of $\mathfrak{E}_c$ is cohesive if and only if it is nowhere zero-dimensional.*

Proof Let E be a nowhere zero-dimensional closed subset of $\mathfrak{E}_c$. Let $x \in E$ be arbitrary. According to Dijkstra, van Mill, and Steprāns [15, Theorem 3.1] there exists a neighbourhood U of x in $\mathfrak{E}_c$ such that the empty set is the only closed nowhere zero-dimensional subspace of U. Let C be a clopen subset of E such that $C \subset E \cap U$. Then C is just as E nowhere zero-dimensional and closed in U. Thus C is empty, and we have shown that E is cohesive. ■

With Lemma 3.7, we can improve upon Theorem 3.5 as follows.

Theorem 3.8 *A nonempty space is homeomorphic to $\mathfrak{E}_c$ if and only if it is nowhere zero-dimensional and it satisfies one of the seven equivalent conditions of Theorem* 3.2.

Remark 3.9. If X is a nonempty space, then $\mathrm{Fact}(X)$ stands for the class of X-factors and $\mathrm{Stab}(X)$ is the subclass of $\mathrm{Fact}(X)$ that consists of the spaces Y such that $Y \times X \approx X$. If $\mathrm{Fact}(X) = \mathrm{Stab}(X)$, then clearly $X \approx X \times X$ (and hence $X \approx X^n$ for $n \in \mathbb{N}$). If $X \approx X^\omega$ and $Y \in \mathrm{Fact}(X)$, then for some Z, $Y \times X \approx Y \times X^\omega \approx Y \times Y^\omega \times Z^\omega \approx Y^\omega \times Z^\omega \approx X$. Thus we have

$$X \approx X^\omega \quad \Rightarrow \quad \mathrm{Fact}(X) = \mathrm{Stab}(X) \quad \Rightarrow \quad X \approx X^2.$$

Since Dijkstra, van Mill, and Steprāns [15] proved that $\mathfrak{E}_c \not\approx \mathfrak{E}_c^\omega$, complete Erdős space is one of the examples that shows that the first implication cannot be reversed. Unlike simpler examples such as $\mathbb{Z}$, $\mathbb{Q}$, and $2^\omega \times \mathbb{Q}$, we have that $\mathfrak{E}_c$ and $\mathfrak{E}_c^\omega$ belong to the same Borel class. Interestingly, $\mathrm{Stab}(\mathfrak{E}_c^\omega)$ consists of all nonempty complete almost zero-dimensional spaces; see Dijkstra [11]. According to Trnková [34] there is a space T such that $T \not\approx T^2$ yet $T \approx T^3$. Then $T \in \mathrm{Fact}(T^2) \setminus \mathrm{Stab}(T^2)$, so also the second implication cannot be reversed.

The following results show that the property $\mathrm{Fact}(\mathfrak{E}_c) = \mathrm{Stab}(\mathfrak{E}_c)$ is valid in a very strong way.

Theorem 3.10 *If $\prod_{i\in\omega} X_i$ is homeomorphic to $\mathfrak{E}_c$, then $\{i \in \omega : X_i \approx \mathfrak{E}_c\}$ is finite and nonempty.*

Proof Let $\prod_{i\in\omega} X_i \approx \mathfrak{E}_c$ thus every X_i is an $\mathfrak{E}_c$-factor. By Proposition 2.7 we have that some X_k is cohesive. Consequently, $X_k \approx \mathfrak{E}_c$ by Theorem 3.5. Now assume that infinitely many X_i's are homeomorphic to $\mathfrak{E}_c$. Then by Theorem 3.2 we have $\mathfrak{E}_c \approx \prod_{i\in\omega} X_i \approx \mathfrak{E}_c^\omega$ in contradiction to [15, Corollary 3.2]. ■

Corollary 3.11 *The product of two spaces is homeomorphic to $\mathfrak{E}_c$ if and only if one space is homeomorphic to $\mathfrak{E}_c$ and the other space is an $\mathfrak{E}_c$-factor.*

Example 3.12 Let $X_0 = \mathfrak{E}_c$ and let X_i for $i \in \mathbb{N}$ be the union of $\mathfrak{E}_c$ with an isolated point. Then every X_i is an $\mathfrak{E}_c$-factor but only X_0 is homeomorphic to $\mathfrak{E}_c$. If $X = \prod_{i\in\omega} X_i \approx \mathfrak{E}_c$, then X contains $\mathfrak{E}_c^\omega$ as a closed subspace and hence by Theorem 3.2 we would have that $\mathfrak{E}_c^\omega \approx \mathfrak{E}_c^\omega \times \mathfrak{E}_c \approx \mathfrak{E}_c$ in contradiction to [15, Corollary 3.2]. Thus X is not homeomorphic to $\mathfrak{E}_c$ making the natural converse of Theorem 3.10 invalid.

4 Representations of $\mathfrak{E}_c$

It follows from work of Mayer [26] and Aarts and Oversteegen [1] that $\mathfrak{E}_c$ is homeomorphic to the end-point sets of Julia sets of certain exponential maps. In this section we discuss several other types of representations of $\mathfrak{E}_c$. We will use our characterization theorems to improve upon known results concerning representations of $\mathfrak{E}_c$ as Erdős type sets in ℓ^p, end-point sets of $\mathbb{R}$-trees, and line-free groups in Banach spaces. We also present a novel application concerning Polishable ideals on ω.

4.1 Erdős Type Spaces in ℓ^p

We now discuss the class of representations of $\mathfrak{E}_c$ that is responsible for the name "complete Erdős space". Let $p > 0$ and consider the (quasi-)Banach space ℓ^p. This space consists of all sequences $x = (x_0, x_1, \dots)$ of real numbers such that $\sum_{i=0}^{\infty} |x_i|^p < \infty$. The topology on ℓ^p is generated by the (quasi-)norm $\|x\| = (\sum_{i=0}^{\infty} |x_i|^p)^{1/p}$. Now let $E_0, E_1, \dots$ be a fixed sequence of subsets of $\mathbb{R}$ and let

$$\mathcal{E} = \{x \in \ell^p : x_n \in E_n \text{ for every } n \in \omega\}$$

be a corresponding subspace of some fixed ℓ^p. If we choose $p = 2$ and $E_n = \mathbb{Q}$ for every n, then $\mathcal{E}$ is called *Erdős space* $\mathfrak{E}$; see Erdős [20]. We characterized $\mathfrak{E}$ in [12] and [14].

The following result generalizes Dijkstra [9, Theorem 3 and Corollary 4].

Theorem 4.1

(a) *The space $\mathcal{E}$ is an $\mathfrak{E}_c$-factor if and only if $\mathcal{E} \neq \varnothing$ and every E_n is a zero-dimensional G_δ-set in $\mathbb{R}$.*

(b) *The space $\mathcal{E}$ is homeomorphic to $\mathfrak{E}_c$ if and only if* $\dim \mathcal{E} > 0$ *and every E_n is a zero-dimensional G_δ-set in $\mathbb{R}$.*

Proof According to Dijkstra [9] $\dim \mathcal{E} \neq 0$ if and only if $\mathcal{E}$ is cohesive, thus it suffices to prove part (a).

Let $\mathcal{E}$ be an $\mathfrak{E}_c$-factor and thus $\mathcal{E}$ is totally disconnected and complete. Since $\mathcal{E} \neq \varnothing$, every E_n is clearly imbeddable as a closed subset of $\mathcal{E}$. Thus, just as $\mathfrak{E}_c$, every E_n is totally disconnected and hence zero-dimensional as a subset of $\mathbb{R}$. Moreover, E_n is topologically complete and thus a G_δ-set in $\mathbb{R}$.

For the 'if' part consider the (weaker) topology $\mathcal{W}$ that $\mathcal{E}$ inherits from the zero-dimensional and topologically complete product space $\prod_{n=0}^{\infty} E_n$. Noting that for every $x \in \mathcal{E}$ the closed ball $\{y \in \mathcal{E} : \|y - x\| \leq \varepsilon\}$ is also a closed subset of $\prod_{n=0}^{\infty} E_n$, we have that condition (7) of Theorem 3.2 is satisfied. ■

Dijkstra [9] contains useful criteria for the property $\dim \mathcal{E} > 0$ and also the following easily verified sufficient condition. Recall that if $A_0, A_1, \dots$ is a sequence of subsets of a space X, then $\limsup_{n\to\infty} A_n = \bigcap_{n=0}^{\infty} \overline{\bigcup_{k=n}^{\infty} A_k}$.

Lemma 4.2 *If 0 is a cluster point of* $\limsup_{n\to\infty} E_n$*, then every nonempty clopen subset of $\mathcal{E}$ is unbounded* (*and hence* $\dim \mathcal{E} \neq 0$).

Remark 4.3. It is clear from Theorem 4.1 and Lemma 4.2 that

$$\{x \in \ell^2 : x_i \notin \mathbb{Q} \text{ for each } i \in \omega\}$$

is homeomorphic to $\mathfrak{E}_c$, a fact that was first established by Kawamura, Oversteegen, and Tymchatyn [22]. It is also clear that both

$$\{x \in \ell^2 : 1/x_i \in \mathbb{N} \text{ for each } i \in \omega\}$$

and its closure in ℓ^2 are homeomorphic to $\mathfrak{E}_c$. Both spaces were introduced and shown to be one-dimensional by Erdős in [20].

Remark 4.4. If we put $E_i = 2^{-i}\mathbb{Z}$ for $i \in \omega$ then the corresponding space $\mathcal{E}$ is homeomorphic to $\mathfrak{E}_c$ for each $p > 0$. Consider now the Banach space c which consists of all convergent sequences of real numbers with the supremum norm. We proved in [13] that the space $\{x \in c : x_i \in 2^{-i}\mathbb{Z} \text{ for each } i \in \omega\}$ is one-dimensional but not homeomorphic to $\mathfrak{E}_c$.

4.2 End-Point Sets of $\mathbb{R}$-Trees

In [22] Kawamura, Oversteegen, and Tymchatyn sketch a proof that the end-point set of the separable universal $\mathbb{R}$-tree as constructed in [27] is homeomorphic to $\mathfrak{E}_c$. We present with Theorem 4.5 a generalization of that result as an application of Theorem 1.1.

An *arc* is a space that is homeomorphic to the interval $[0, 1]$, and an *open arc* is homeomorphic to $(0, 1)$. An $\mathbb{R}$-*tree* $(\mathbb{T}, \rho)$ is a metric space that is arcwise connected such that every arc in $\mathbb{T}$ is isometric to an interval in $\mathbb{R}$ (such a ρ is called a *convex* metric). Mayer and Oversteegen [28] proved that, topologically, the $\mathbb{R}$-trees are precisely the spaces that are uniquely arcwise connected and locally arcwise connected.

Let X be a uniquely arcwise connected space. If $x, y \in X$ with $x \neq y$, then $[x, y]$ denotes the unique arc in X that has x and y as end-points, and $[x, x]$ denotes the singleton $\{x\}$. We shall also use $(x, y) = [x, y] \setminus \{x, y\}$. We define the set of *interior* points of X by $\mathfrak{i}X = \bigcup\{(x, y) : x, y \in X\}$. The set of *end-points* of X is $\mathfrak{e}X = X \setminus \mathfrak{i}X$.

Theorem 4.5 *Let $(\mathbb{T}, \rho)$ be a nonempty $\mathbb{R}$-tree and let $\varepsilon > 0$ be such that every open arc A in $\mathbb{T}$ with $\operatorname{diam} A < \varepsilon$ has a compact closure in $\mathbb{T}$ and such that for each $x \in \mathbb{T}$ every component C of $\mathbb{T} \setminus \{x\}$ has the property $\operatorname{diam} C > \varepsilon$. If $\mathfrak{e}\mathbb{T}$ is dense in $\mathbb{T}$ then $\mathfrak{e}\mathbb{T}$ is homeomorphic to $\mathfrak{E}_c$.*

Proof For $x \in \mathbb{T}$ and $\delta > 0$ let $B_\delta(x)$ denote the closed ball $\{y \in \mathbb{T} : \rho(x, y) \leq \delta\}$. According to [14, Lemmas 3.1 and 3.2] the collection

$$\mathcal{S} = \{C \colon C \text{ is a component of } \mathbb{T} \setminus \{x\} \text{ for some } x \in \mathfrak{i}\mathbb{T}\}$$

is a subbasis for a separable metric topology $\mathcal{W}$ on $\mathbb{T}$, called the *weak topology*, that is weaker than the ρ-topology such that every $B_\delta(x)$ is $\mathcal{W}$-closed and the restriction of $\mathcal{W}$ to $\mathfrak{e}\mathbb{T}$ is zero-dimensional. Observe that if $x \in \mathfrak{e}\mathbb{T}$, $y \in \mathbb{T} \setminus \{x\}$, and $a_1, a_2, \ldots$ is a sequence in (x, y) that converges to x then $\{C : C \text{ is the component of } x \text{ in some } \mathbb{T} \setminus \{a_i\}\}$ is a neighbourhood basis for x in $(\mathbb{T}, \mathcal{W})$. Let $x \in \mathfrak{e}\mathbb{T}$ and let $\delta > 0$ be such that $2\delta < \varepsilon$. Define $V = \mathfrak{e}\mathbb{T} \cap B_\delta(x)$ and note that V is $\mathcal{W}$-closed in $\mathfrak{e}\mathbb{T}$. Let $y \in V$ and let C be a basic $\mathcal{W}$-neighbourhood of y in $\mathbb{T}$, that is, C is the component of y in $\mathbb{T} \setminus \{a\}$ for some $a \in \mathfrak{i}\mathbb{T}$. Since $\operatorname{diam} C > \varepsilon > 2\delta$, there is a $z \in C \setminus B_\delta(x)$. Select a $b \in [y, z] \subset C$ such that $\delta < \rho(x, b) < 2\delta$. Since $\mathfrak{e}\mathbb{T}$ is dense, we may approximate b by a $b' \in \mathfrak{e}\mathbb{T} \cap C \cap (B_{2\delta}(x) \setminus B_\delta(x))$. This proves that V is nowhere dense in $\mathfrak{e}\mathbb{T} \cap B_{2\delta}(x)$ with respect to the weak topology.

Note that if D is a countable dense subset of $\mathfrak{i}\mathbb{T}$, then $\mathfrak{i}\mathbb{T} = \bigcup_{a,b \in D}[a, b]$ and hence $\mathfrak{i}\mathbb{T}$ is σ-compact in both topologies. Let $F_1, F_2, \ldots$ be a sequence of compacta in $(\mathbb{T}, \mathcal{W})$ such that $\bigcup_{i=1}^{\infty} F_i = \mathfrak{i}\mathbb{T}$. For $i \in \mathbb{N}$ define $\mathcal{U}_i = \{C \in \mathcal{S} : \overline{C} \cap F_i = \varnothing\}$ and note that $\mathcal{U}_i$ is an open cover of $\mathfrak{e}\mathbb{T}$ in $(\mathbb{T}, \mathcal{W})$. Let d be a metric for the weak topology on $\mathfrak{e}\mathbb{T}$ such that for each $i \in \mathbb{N}$ there is a $\gamma > 0$ with the property that the collection of all open γ-balls with respect to d refines $\mathcal{U}_i$. To apply Theorem 1.1 it now suffices to show that d is complete on V. Let $x_1, x_2, \ldots$ be a d-Cauchy sequence in V. Let a be a fixed point in $\mathfrak{i}\mathbb{T} \cap B_\delta(x)$ and note that for each $i \in \mathbb{N}$, $\bigcap_{j=i}^{\infty}[a, x_j]$ has the form $[a, c_i]$ for some $c_i \in \mathbb{T}$. Since ρ is convex, we have that every $[a, c_i]$ is contained in $B_\delta(x)$. Let $A = \bigcup_{i=1}^{\infty}[a, c_i]$ and note that $\operatorname{diam} A \leq 2\delta < \varepsilon$. Since $[a, c_i] \subset [a, c_{i+1}]$ for each $i \in \mathbb{N}$, we have that the closure of A has the form $[a, c]$ for some $c \in B_\delta(x)$. Note that $\lim_{i \to \infty} c_i = c$ in both topologies. Let $i \in \mathbb{N}$ and note that there is an $N \in \mathbb{N}$ and a $C \in \mathcal{U}_i$ such that $x_n \in C$ for each $n \geq N$. Then there is a $b \in \mathfrak{i}\mathbb{T}$ such that C is a component of $\mathbb{T} \setminus \{b\}$ and $\overline{C} = C \cup \{b\} \subset \mathbb{T} \setminus F_i$. Note that for every $n \geq N$, $c_n \in C \cup \{b\}$ thus $c \in C \cup \{b\} \subset \mathbb{T} \setminus F_i$. We may conclude that $c \in B_\delta(x) \setminus \bigcup_{i=1}^{\infty} F_i = V$. If some c_i equals c, then for each $k \geq i$, $c \in [a, x_k]$ thus $c = x_k$ because $c \in \mathfrak{e}\mathbb{T}$. So we may assume that $c \neq c_i$ for every $i \in \mathbb{N}$. Let U_i be the component of $\mathbb{T} \setminus \{c_i\}$ that contains c and recall that the U_i's form a $\mathcal{W}$-neighbourhood basis at c. Let $i \in \mathbb{N}$ and select a $c_j \in (c_i, c) \subset U_i$. Note that $x_k \in U_i$ for each $k \geq j$, thus we may conclude that $\lim_{i \to \infty} x_i = c$ in the weak topology. ■

Remark 4.6. There are many $\mathbb{R}$-trees with $\mathfrak{E}_c$ as end-point set that do not satisfy the premises of Theorem 4.5. For instance, the constructions in [30, Theorem 2] and [14, Lemma 3.5] show that every almost zero-dimensional space can be represented as a closed end-point set of an $\mathbb{R}$-tree.

Note that the proof of Theorem 4.5 allows us to weaken the premise that for some $\varepsilon > 0$ every open arc with diam $< \varepsilon$ is relatively compact to the topological condition:

($*$) Each $x \in \mathfrak{e}\mathbb{T}$ has a neighbourhood U in $\mathbb{T}$ such that every open arc in U is relatively compact in $\mathbb{T}$.

By the same argument we have:

Proposition 4.7 *If $\mathbb{T}$ is an $\mathbb{R}$-tree that satisfies condition ($*$), then $\mathfrak{e}\mathbb{T}$ is an $\mathfrak{E}_c$-factor or $\mathfrak{e}\mathbb{T} = \varnothing$.*

4.3 Line-Free Groups in Banach Spaces

Let $(X, |\cdot|)$ be a normed vector space and let $(X^*, |\cdot|)$ denote its dual. In this subsection we do not assume a priori that a vector space is separable. If $\varepsilon > 0$, then B_ε denotes the closed ball $\{x \in X : |x| \leq \varepsilon\}$. Let $F = \{f_0, f_1, \ldots\}$ be a countable subset of X^* and define the linear continuous map $T_F \colon X \to \mathbb{R}^\omega$ by $T_F(x) = (f_0(x), f_1(x), \ldots)$ for $x \in X$. Assume that F is *total*, that is, that T_F is an injection. The F-topology on

X is obtained by pulling back the product topology of the Fréchet space $\mathbb{R}^\omega$. We say that a subset of A is *relatively complete in* X with respect to the F-topology if there is an A' such that $A \subset A' \subset X$ and $T_F(A')$ is closed in $\mathbb{R}^\omega$. This is equivalent to saying that for any or all invariant metrics d on X that generate the F-topology we have that every Cauchy sequence that is contained in A converges in X with respect to d. Define $G_F = T_F^{-1}(\mathbb{Z}^\omega)$ and note that G_F is a line-free group in X that is closed in the F-topology. Dobrowolski and Grabowski [17] have shown that every weakly closed line-free subgroup of a separable X can be represented as G_F for some total sequence in the dual.

Dobrowolski, Grabowski, and Kawamura present the following statement as [18, Main Theorem].

Claim 4.8 *Let $(X, |\cdot|)$ be a Banach space and let F be a total sequence of functionals from the dual X^*. Assume that the norm bounded subsets of G_F are relatively complete in the F-topology. If G_F is separable, then G_F is either discrete or homeomorphic to $\mathfrak{E}_c$.*

Remark 4.9. Unfortunately, the proof given in [18] does not fully support Claim 4.8. The problem is that at the beginning of the proof it is asserted that the total sequence F admits a Kadec norm. A *Kadec norm* $\|\cdot\|$ *for* F is a norm on X that is equivalent to $|\cdot|$ with the property that whenever a sequence $x_1, x_2, \ldots$ converges to a point x in X in the F-topology and $\lim_{i\to\infty} \|x_i\| = \|x\|$, then $\lim_{i\to\infty} \|x_i - x\| = 0$. The sequence F is called *norming* if there is an equivalent norm on X that is LSC with respect to the F-topology or, equivalently, there is a bounded neighbourhood U of $\mathbf{0}$ in X that is closed in the F-topology. Every Kadec norm for F is LSC with respect to the F-topology; see [4, p. 176]. In fact, F admits a Kadec norm if and only if F is norming; see Davis and Johnson [7]. It is known that a separable Banach space X admits a non-norming total sequence of functionals if and only if $\dim(X^{**}/X) = \infty$; see Davis and Lindenstrauss [8]. In view of these considerations the word "total" in Claim 4.8 should be replaced by "norming". Observe that the important Corollaries 1 and 2 in [18] still follow from the corrected version of Claim 4.8 and hence the negative fall-out from our observation should be limited. We do not know whether Claim 4.8 is valid.

The same problem also affects Theorem 3.1 and Proposition 3.2 in Ancel, Dobrowolski, and Grabowski [3]. Thus also in these results "norming" needs to be substituted for merely "total". Again we do not know whether Theorem 3.1 and Proposition 3.2 in [3] are valid as written.

Remark 4.10. We observe that if a bounded subset A of X is relatively complete in the F-topology then A is relatively compact in the F-topology. Thus the use of the word "complete" in Claim 4.8 (and "closed" in Theorem 4.11) suggests a level of generality that is not actually present.

Let $A \subset A' \subset X$ be such that $A \subset B_M$ for some $M \in \mathbb{N}$ and $T_F(A')$ is closed in $\mathbb{R}^\omega$. We may assume that A' is the closure of A in the F-topology. Note that $T_F(A)$ is contained in the compactum $C = \prod_{i\in\omega}[-M|f_i|, M|f_i|]$ and that C therefore contains $T_F(A')$ which is the closure of $T_F(A)$. We have that $T_F(A')$ is compact which means that A' is compact in the F-topology.

The following generalization of the corrected Claim 4.8 follows in a straightforward manner from Theorem 1.1.

Theorem 4.11 *Let $(X, |\cdot|)$ be a normed vector space and let F be a total sequence of functionals from the dual X^*. Let U be a bounded neighbourhood of $\mathbf{0}$ in G_F such that $T_F(U)$ is closed in $\mathbb{R}^\omega$. If G_F is separable, then G_F is either discrete or homeomorphic to $\mathfrak{E}_c$.*

With Remark 4.10 we see that the condition that $T_F(U)$ is closed in $\mathbb{R}^\omega$ is equivalent to requiring that U be compact (or complete) with respect to the F-topology on X.

There exists a total sequence of functionals F on the separable Banach space c such that G_F is one-dimensional but not homeomorphic to $\mathfrak{E}_c$; see Remark 4.4 and [13].

Proof Let X, F, and U be as in the premise and assume that G_F is separable and non-discrete. For Theorem 1.1 we let $\mathcal{W}$ be the restriction of the F-topology to G_F. Note that $\mathcal{W}$ is zero-dimensional and coarser than the norm topology because $\dim \mathbb{Z}^\omega = 0$ and T_F is continuous. Let $n \in \mathbb{N}$ be such that $B_{1/n} \cap G_F \subset U \subset B_n$. Since both the norm-topology and $\mathcal{W}$ are compatible with the group structure, it suffices to verify the conditions of Theorem 1.1 for the point $x = \mathbf{0}$. Let $m \in \mathbb{N}$ and consider $B_{1/m}$. Define $V = G_F \cap \frac{1}{2mn} U$. Note that $V \subset B_{1/2m}$ and that $T_F(V) = \mathbb{Z}^\omega \cap \frac{1}{2mn} T_F(U)$ thus V with the F-topology is homeomorphic to a closed subset of $\mathbb{R}^\omega$ and topologically complete. Since $V \supset G_F \cap \frac{1}{2mn}(G_F \cap B_{1/n}) = G_F \cap B_{1/2mn^2}$, we have that V is a neighbourhood of $\mathbf{0}$ in G_F.

Now let $y \in V$ and $k \in \omega$ be arbitrary and consider the basic $\mathcal{W}$-neighbourhood $C = \{z \in G_F : f_i(z) = f_i(y) \text{ for } 0 \le i \le k\}$ of y. Since G_F is non-discrete, there is an a with $y + a \in C$ and $0 < |a| < 1/2m$. Since V is bounded, we may choose l to be the least element of ω such that $y + la \notin V$. Since $y \in V$, we have $y + (l-1)a \in V \subset B_{1/2m}$ thus $y + la \in (G_F \cap B_{1/m}) \setminus V$. Note that $y + la \in C$ because $f_i(a) = f_i(y+a) - f_i(y) = 0$ for every $i \le k$ thus we have shown that V is nowhere dense in $G_F \cap B_{1/m}$ with respect to the F-topology. ■

4.4 Polishable Ideals on ω

We now turn to an application of Theorem 1.1 to ideals on ω, and we thank S. Todorčević for bringing these spaces to our attention; see [33] for background information. The following definitions and Theorems 4.12, 4.13, and 4.14 are taken from Solecki [31, 32] where the reader can find more details and references.

Let $D = \{x_n : n \in \omega\}$ be a countable infinite set that is enumerated such that $x_n \neq x_k$ if $n \neq k$. Consider the power set $\mathcal{P}(D)$ with the symmetric difference $\triangle$ group structure. We equip $\mathcal{P}(D)$ with the standard Cantor set topology that comes with identification with 2^D. An *ideal* I on D is a subset of $\mathcal{P}(D)$ such that I contains the finite sets $B \in I$ whenever $B \subset A \in I$, and $A \cup B \in I$ whenever $A, B \in I$. A function $\varphi \colon \mathcal{P}(D) \to [0, \infty]$ is a *submeasure on* D if $\varphi(\varnothing) = 0$, $\varphi(X) \le \varphi(X \cup Y) \le \varphi(X) + \varphi(Y)$ for any $X, Y \subset D$, and $0 < \varphi(\{x\}) < \infty$ for any $x \in D$. With a submeasure φ we associate two ideals on D:

$$\mathrm{Exh}(\varphi) = \{A \subset D \colon \lim_{m\to\infty} \varphi(\{x_n \in A : n \ge m\}) = 0\},$$

$$\mathrm{Fin}(\varphi) = \{A \subset D \colon \varphi(A) < \infty\}.$$

Observe that $\operatorname{Exh}(\varphi) \subset \operatorname{Fin}(\varphi)$. If φ is a measure rather than just a submeasure and φ is LSC as a function from 2^D to $[0, \infty]$, then $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$. An ideal is clearly a subgroup of 2^D. An ideal I is *Polishable* if there exists a Polish group topology τ on I such that the family of Borel sets with respect to τ is equal to the family of Borel sets of I with respect to the topology inherited from 2^D. This class of ideals was first studied by Kechris and Louveau [24]. If such a Polish topology exists, then it is unique; see [23, Theorem 9.10].

Theorem 4.12 *If φ is an LSC submeasure on ω, then*

$$d(A, B) = \varphi(A \triangle B) \text{ for } A, B \subset \omega$$

restricts to an invariant, complete, separable metric on $\operatorname{Exh}(\varphi)$.

Observe that the topology on $I = \operatorname{Exh}(\varphi)$ generated by d is stronger than the subspace topology that I inherits from 2^ω. So for ideals of the form $\operatorname{Exh}(\varphi)$ this describes in an explicit way a Polish topology on I that witnesses that I is Polishable. Note that in general the d-topology on $\operatorname{Fin}(\varphi)$ may be nonseparable. The following results provide useful context; see also van Engelen [19].

Theorem 4.13 *If I is a Polishable ideal on ω, then it is homeomorphic to $\mathbb{Q}$, 2^ω, $\mathbb{Q} \times 2^\omega$, or $\mathbb{Q}^\omega$.*

Theorem 4.14 *Let I be an ideal on ω. Then the following statements hold* (*where φ stands for an LSC submeasure on ω*):

(1) *I is Polishable if and only if $I = \operatorname{Exh}(\varphi)$ for some finite φ.*
(2) *I is F_σ in 2^ω if and only if $I = \operatorname{Fin}(\varphi)$ for some φ.*
(3) *I is Polishable and F_σ if and only if $I = \operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$ for some φ.*

As an application of Theorem 1.1 we have:

Theorem 4.15 *Let I be a Polishable F_σ-ideal on ω and let φ be an LSC submeasure with $I = \operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$. If τ denotes the Polish topology on I that is generated by φ, then the following statements are equivalent.*

(1) *(I, τ) is homeomorphic to $\mathfrak{E}_c$.*
(2) *$\dim(I, \tau) > 0$.*
(3) *(I, τ) is not σ-compact.*
(4) *(I, τ) is not locally compact.*
(5) *(I, τ) is not homeomorphic to $\mathbb{Z}$, 2^ω, or $\mathbb{Z} \times 2^\omega$.*
(6) *There is no $B \subset \omega$ with $I = \{A \subset \omega : A \cap B \text{ is finite}\}$.*
(7) *For every $\varepsilon > 0$ we have $\{n \in \omega : \varphi(\{n\}) \leq \varepsilon\} \notin I$.*
(8) *There is a $B \in 2^\omega \setminus I$ with $\lim_{n\to\infty} \varphi(\{n\} \cap B) = 0$.*

Proof The implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (5) are obvious. For (2) $\Rightarrow$ (3), note that τ is stronger than the zero-dimensional topology thus (I, τ) is totally disconnected. Consequently, σ-compactness implies zero-dimensionality.

(5) $\Rightarrow$ (6). Let B be such that $I = \{A \subset \omega : A \cap B \text{ is finite}\}$. Then the sets $\{F \cup A : A \in C\}$, where F is a finite subset of B and C is a clopen subset of the

compactum $2^{\omega\setminus B}$, form a basis for a separable metric topology τ' on I that is locally compact and hence Polish. Noting that the basis sets are closed in 2^ω and that τ' is compatible with the group structure on I we have $\tau = \tau'$. Observe that $(I,\tau) \approx 2^\omega$ if B is finite, $(I,\tau) \approx \mathbb{Z}$ if $\omega \setminus B$ is finite, and $(I,\tau) \approx \mathbb{Z} \times 2^\omega$ otherwise.

(6) $\Rightarrow$ (7). Assume that there is an $\varepsilon > 0$ with

$$B = \{k \in \omega : \varphi(\{k\}) \leq \varepsilon\} \in I.$$

If $A\setminus B$ is finite, then $A\setminus B \in I$ thus $A \in I$. If $A \in I$, then $A \in \mathrm{Exh}(\varphi)$ and $\varphi(A\setminus n) \leq \varepsilon$ for some $n \in \omega$. Consequently, $A \setminus n \subset B$ and $A \setminus B$ is finite. We have shown that $I = \{A \subset \omega : A \cap (\omega \setminus B) \text{ is finite}\}$.

(7) $\Rightarrow$ (8). Assume (7) and let $k \in \omega$. If we define $A_k = \{n \in \omega : \varphi(\{n\}) \leq 2^{-k}\}$ then by assumption $\varphi(A_k) = \infty$. Since φ is LSC, we can find an $m_k \in \omega$ such that $\varphi(A_k \cap m_k) > k$. Define $B = \bigcup_{k=0}^{\infty} A_k \cap m_k$ and note that $\varphi(B) = \infty$ thus $B \notin I$. If $k \in \omega$, let $M_k = \max\{m_0, \ldots, m_k\}$ and note that for each $n > M_k$ we have $\varphi(\{n\} \cap B) \leq 2^{-k}$. Condition (8) is verified.

(8) $\Rightarrow$ (1). Assume condition (8) so there is a B with $\varphi(B) = \infty$ and $\lim_{n\to\infty} \varphi(\{n\} \cap B) = 0$. We shall use Theorem 1.1 where $\mathcal{W}$ is the topology that I inherits from 2^ω. Let U be some τ-neighbourhood of an $X \in I$. Since both topologies are compatible with the group structure, it suffices to consider the case that $X = \varnothing$. Let $\varepsilon > 0$ be such that $\{A : \varphi(A) \leq 2\varepsilon\} \subset U$. Note that $V = \{A : \varphi(A) \leq \varepsilon\}$ is a subset of U that is closed in the compactum 2^ω because φ is LSC. Thus V is certainly closed and complete with respect to $\mathcal{W}$.

It remains to show that V is nowhere dense in $(U, \mathcal{W})$. Let m be such that $\varphi(\{n\} \cap B) \leq \varepsilon$ for all $n \geq m$. Let $W_n = \{A' \subset \omega : A' \cap n = A \cap n\}$ for $n > m$ be a basic neighbourhood of some $A \in V$ in 2^ω. Define for each $k \geq n$,

$$A_k = (A \cap n) \cup (B \cap (k \setminus n))$$

and note that $A_k \in W_n \cap I$ for every $k \geq n$ and $A_n \subset A$. Since $\lim_{k\to\infty} B \cap (k \setminus n) = B \setminus n$ in 2^ω and φ is LSC, we have

$$\liminf_{k\to\infty} \varphi(A_k) \geq \liminf_{k\to\infty} \varphi(B \cap (k \setminus n)) \geq \varphi(B \setminus n) \geq \varphi(B) - \varphi(B \cap n) = \infty.$$

Let l be the first index with $\varphi(A_l) > \varepsilon$. Since $\varphi(A_n) \leq \varphi(A) \leq \varepsilon$, we have $l > n > m$ and

$$\varphi(A_l) \leq \varphi(A_{l-1}) + \varphi(\{l-1\} \cap B) \leq 2\varepsilon.$$

Thus $A_l \in W_n \cap (U \setminus V)$ and we have that V is nowhere dense in $(U, \mathcal{W})$. ■

Remark 4.16. The equivalence (4) $\Leftrightarrow$ (6) is already contained in Solecki [31, 32].

Remark 4.17. Let φ be an LSC submeasure on ω and let τ be the topology that is generated on $\mathrm{Fin}(\varphi)$ by the metric $d(A,B) = \varphi(A \triangle B)$. By the same argument used for the implication (8) $\Rightarrow$ (1) in Theorem 4.15 we find that every point in $\mathrm{Fin}(\varphi)$ has a τ-neighbourhood basis consisting of sets that are closed in 2^ω. This implies that $(\mathrm{Exh}(\varphi), \tau)$ is almost zero-dimensional. Also, if it is given that $(\mathrm{Fin}(\varphi), \tau)$ is separable, then the space is almost zero-dimensional with a compactum as witness. Consequently, by Theorem 3.2 we have that τ is Polish and hence $\mathrm{Fin}(\varphi)$ is Polishable (and of course F_σ).

Example 4.18 We first consider a simple example of an ideal that is homeomorphic to $\mathfrak{E}_c$. Define the following LSC measure on ω:

$$\eta(A) = \sum_{n \in A} \frac{1}{n+1}.$$

Since η is a measure, we have $\mathrm{Exh}(\eta) = \mathrm{Fin}(\eta)$, and we call this ideal the *harmonic ideal* I_{harm}. Note that $I_{\mathrm{harm}} \approx \mathbb{Q} \times 2^\omega$. Since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, condition (8) of Theorem 4.15 is satisfied and we have that I_{harm} with its Polish topology is homeomorphic to $\mathfrak{E}_c$. Alternatively, we can consider the following imbedding of I_{harm} in the Banach space ℓ^1:

$$(\alpha(A))_n = \begin{cases} 1/(n+1), & \text{if } n \in A; \\ 0, & \text{if } n \in \omega \setminus A. \end{cases}$$

Then $\eta(A \triangle B) = \|\alpha(A) - \alpha(B)\|$ and α is a homeomorphism between I_{harm} with its Polish topology and the closed Erdős type subset $\alpha(I_{\mathrm{harm}})$ of ℓ^1. In Dijkstra [9] $\alpha(I_{\mathrm{harm}})$ is shown to be homeomorphic to $\mathfrak{E}_c$ and is called *harmonic Erdős space.*

Remark 4.19. Consider a φ as in Theorem 4.15. Define a USC function $\widetilde{\varphi}\colon 2^\omega \to [0,1]$ by $\widetilde{\varphi}(A) = 1/(1+\varphi(A))$, and note that with the same method as employed in the proof of the implication (8) $\Rightarrow$ (1) in Theorem 4.15 one can show that $\widetilde{\varphi}$ is a Lelek function thus $\mathfrak{E}_c = G_0^{\widetilde{\varphi}}$. Clearly, $G_0^{\widetilde{\varphi}}$ is homeomorphic to the graph G of $\varphi \restriction I$ with the product topology from $2^\omega \times \mathbb{R}$. Then $h(A) = (A, \varphi(A))$ defines a continuous bijection between I with the Polish topology and G, which is a copy of $\mathfrak{E}_c$. The question arises why we use Theorem 1.1 to prove Theorem 4.15 rather than directly linking I with $\mathfrak{E}_c$. The reason is that h is in general not a homeomorphism.

Let us consider a simple example. Using Example 4.18, define for each $A \subset \omega$,

$$\varphi(A) = \begin{cases} \eta(A), & \text{if } 0 \notin A; \\ \max\{2, \eta(A)\}, & \text{if } 0 \in A. \end{cases}$$

It is easily verified that φ is also an LSC submeasure with $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi) = I_{\mathrm{harm}}$ and that η and φ generate the same Polish topology τ on I_{harm}. Consider now the open neighbourhood $V = \{A : \eta(\{0\} \triangle A) < 1\}$ of $\{0\}$ in $(I_{\mathrm{harm}}, \tau)$ and hence $\dim V = 1$. Note that $\varphi \restriction V$ is constant, thus $h(V)$ carries the zero-dimensional topology that V inherits from 2^ω. We have that h is not a homeomorphism.

Example 4.20 We look at the case that I is Polishable but not F_σ.

Consider first the following LSC submeasure on $\omega \times \omega$:

$$\varphi_1(A) = \max\{2^{-n} : (n,m) \in A\},$$

where $A \subset \omega \times \omega$ and $\max \varnothing = 0$. Note that

$$I_1 = \mathrm{Exh}(\varphi_1) = \{A \subset \omega \times \omega : A \cap (\{n\} \times \omega) \text{ is finite for } n \in \omega\}$$

and that I_1 is homeomorphic to $\mathbb{Q}^\omega$. It is easily seen that I_1 with the Polish topology generated by φ_1 is homeomorphic to $\mathbb{Z}^\omega \approx \mathbb{R} \setminus \mathbb{Q}$.

Secondly, we consider the ideal:

$$I_{\mathrm{EU}} = \left\{ A \subset \omega \colon \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

This ideal is the most important of the Erdős–Ulam density ideals, see Farah [21]. It is well known that this ideal is equal to $\mathrm{Exh}(\psi)$, where

$$\psi(A) = \sup_{n\in\omega} \frac{|A \cap [2^n - 1, 2^{n+1} - 2]|}{2^n}$$

for $A \subset \omega$. Note that $\psi(I_{\mathrm{EU}}) \subset \mathbb{Q}$, thus I_{EU} is zero-dimensional in the Polish topology. Since this topology is clearly not locally compact, we have that I_{EU} is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ in the Polish topology.

Define an ideal I_2 on $\omega \cup (\omega \times \omega)$ by $I_2 = \{A \cup B : A \in I_{\mathrm{harm}} \text{ and } B \in I_1\}$. Then we have that I_2 is homeomorphic to $I_{\mathrm{harm}} \times I_1 \approx \mathbb{Q}^\omega$. The Polish topology is generated by $\varphi_2(A \cup B) = \eta(A) + \varphi_1(B)$ for $A \subset \omega$ and $B \subset \omega \times \omega$. We have that I_2, with its Polish topology, is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\mathbb{R} \setminus \mathbb{Q}) \approx \mathfrak{E}_{\mathrm{c}}$.

Finally, consider the LSC submeasure φ_3 on $\omega \times \omega$ that is given by

$$\varphi_3(A) = \sum_{k=0}^{\infty} \min\{2^{-k}, \eta(\{n : (k,n) \in A\})\}.$$

Then we have

$$I_3 = \mathrm{Exh}(\varphi_3) = \{A \subset \omega \times \omega : \{n : (k,n) \in A\} \in I_{\mathrm{harm}} \text{ for } k \in \omega\}.$$

Note that I_3 is homeomorphic to $(I_{\mathrm{harm}})^\omega \approx \mathbb{Q}^\omega$ and that I_3 with the Polish topology is homeomorphic to $\mathfrak{E}_{\mathrm{c}}^\omega$. It was proved by Dijkstra, van Mill, and Steprāns [15] that $\mathfrak{E}_{\mathrm{c}}^\omega$ is not homeomorphic to $\mathfrak{E}_{\mathrm{c}}$, so we have found a new topological type for Polishable ideals.

References

[1] J. M. Aarts and L. G. Oversteegen, *The geometry of Julia sets.* Trans. Amer. Math. Soc. **338**(1993), no. 2, 897–918.

[2] M. Abry and J. J. Dijkstra, *On topological Kadec norms.* Math. Ann. **332**(2005), no. 4, 759–765.

[3] F. D. Ancel, T. Dobrowolski, and J. Grabowski, *Closed subgroups in Banach spaces.* Studia Math. **109**(1994), no. 3, 277–290.

[4] C. Bessaga and A Pełczyński, *Selected Topics in infinite-dimensional topology.* Monografie Matematyczne 58, PWN–Polish Scientific Publishers, Warsaw, 1975.

[5] W. D. Bula and L. G. Oversteegen, *A characterization of smooth Cantor bouquets.* Proc. Amer. Math. Soc. **108**(1990), no. 2, 529–534.

[6] W. J. Charatonik, *The Lelek fan is unique.* Houston J. Math. **15**(1989), no. 1, 27–34.

[7] W. J. Davis and W. B. Johnson, *A renorming of nonreflexive Banach spaces.* Proc. Amer. Math. Soc. **37**(1973), 486–488.

[8] W. J. Davis and J. Lindenstrauss, *On total nonnorming subspaces.* Proc. Amer. Math. Soc. **31**(1972), 109–111.

[9] J. J. Dijkstra, *A criterion for Erdős spaces.* Proc. Edinb. Math. Soc. (2) **48**(2005), no. 3, 595–601.
[10] ———, *A homogeneous space that is one-dimensional but not cohesive.* Houston J. Math. **32**(2006), no. 4, 1093–1099.
[11] ———, *Characterizing stable complete Erdős space.* www.few.vu.nl/~dijkstra/research/papers/stable.pdf
[12] J. J. Dijkstra and J. van Mill, *Homeomorphism groups of manifolds and Erdős space.* Electron. Res. Announc. Amer. Math. Soc. **10**(2004), 29–38.
[13] ———, *A counterexample concerning line-free groups and complete Erdős space.* Proc. Amer. Math. Soc. **134**(2006), no. 8, 2281–2283.
[14] ———. *Erdős space and homeomorphism groups of manifolds.* Mem. Amer. Math. Soc., to appear.
[15] J. J. Dijkstra, J. van Mill, and J. Steprāns, *Complete Erdős space is unstable*, Math. Proc. Cambridge Philos. Soc. **137**(2004), 465–473.
[16] J. J. Dijkstra, J. van Mill, and K. I. S. Valkenburg, *On nonseparable Erdős spaces.* J. Math. Soc. Japan **60**(2008), no. 3, 793–818.
[17] T. Dobrowolski and J. Grabowski, *Subgroups of Hilbert spaces.* Math. Z. **211**(1992), no. 4, 657–669.
[18] T. Dobrowolski, J. Grabowski, and K. Kawamura, *Topological type of weakly closed subgroups in Banach spaces.* Studia Math. **118**(1996), no. 1, 49–62.
[19] A. J. M. van Engelen, *Homogeneous zero-dimensional absolute Borel sets*, CWI Tract, Vol. 27, Centre for Mathematics and Computer Science, Amsterdam, 1986.
[20] P. Erdős, *The dimension of the rational points in Hilbert space.* Ann. of Math. **41**(1940), 734–736.
[21] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers.* Mem. Amer. Math. Soc. **148**(2000), no. 702.
[22] K. Kawamura, L. G. Oversteegen, and E. D. Tymchatyn, *On homogeneous totally disconnected 1-dimensional spaces.* Fund. Math. **150**(1996), no. 2, 97–112.
[23] A. S. Kechris, *Classical descriptive set theory.* Graduate Texts in Mathematics 156, Springer-Verlag, New York, 1995.
[24] A. S. Kechris and A. Louveau, *The classification of hypersmooth Borel equivalence relations.* J. Amer. Math. Soc. **10**(1997), no. 1, 215–242.
[25] A. Lelek, *On plane dendroids and their end points in the classical sense.* Fund. Math. **49**(1960/1961), 301–319.
[26] J. C. Mayer, *An explosion point for the set of endpoints of the Julia set of* $\lambda\exp(z)$. Ergodic Theory Dynam. Systems **10**(1990), no. 1, 177–183.
[27] J. C. Mayer, J. Nikiel, and L. G. Oversteegen, *Universal spaces for* $\mathbb{R}$*-trees.* Trans. Amer. Math. Soc. **334**(1992), no. 1, 411–432.
[28] J. C. Mayer and L. G. Oversteegen, *A topological characterization of* $\mathbb{R}$*-trees.* Trans. Amer. Math. Soc. **320**(1990), no. 1, 395–415.
[29] ———, *Continuum theory.* In: Recent Progress in General Topology, North-Holland, Amsterdam, 1992, pp. 453–492.
[30] L. G. Oversteegen and E. D. Tymchatyn, *On the dimension of certain totally disconnected spaces.* Proc. Amer. Math. Soc. **122**(1994), no. 3, 885–891.
[31] S. Solecki, *Analytic ideals.* Bull. Symbolic Logic **2**(1996), no. 3, 339–348.
[32] ———, *Analytic ideals and their applications.* Ann. Pure Appl. Logic **99**(1999), no. 1-3, 51–72.
[33] S. Todorčević, *Analytic gaps.* Fund. Math. **150**(1996), no. 1, 55–66.
[34] V. Trnková, X^m *is homeomorphic to* X^n *iff* $m \sim n$ *where* $\sim$ *is a congruence on natural numbers.* Fund. Math. **80**(1973), no. 1, 51–56.

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