NAGATA'S CONTRIBUTIONS TO DIMENSION THEORY

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Jun-iti Nagata published about 20 research papers on dimension theory, a monograph entitled "Modern dimension theory" ([N65] (first edition) and [N83] (second - expanded - edition)), and several survey articles. His work had a great impact on this area of topology, especially on the dimension theory of non-separable metrizable spaces.

The first publications of Nagata on dimension theory appeared in 1956, six years after his characterization of metrizability in terms of locally finite bases, which became classical.

M. Katětov and K. Morita established already a few years earlier some fundamental results on the dimension of non-separable metrizable spaces. In particular, Katětov and Morita proved that, in the realm of metrizable spaces, the notion of dimension based on the order of open covers - the covering dimension dim -, coincides with the inductive notion of dimension, based on separation between pairs of disjoint closed sets - the large inductive dimension Ind (we shall not consider the small inductive dimension in this article).

The papers [N56a], [N56b] and [N56c] by Nagata, published in the same issue of the Proc. Japan Acad., are concerned with characterizations of n-dimensional metrizable spaces in terms of sequences of open covers and special metrics.

[N56c] provides a characterization of dimension in terms of the length of open covers closed under finite intersections, extending to the non-separable case ideas that were introduced by Alexandroff and Kolmogoroff in [AK36]. The most important contributions turned out to be the special metrics that were investigated in [N56b], and which became later known as *Nagata metrics*. The inspiration for these metrics came from a note by J. de Groot and H. de Vries [GV55], who characterized zero-dimensional metrizable spaces as the ones that admit non-Archimedean metrics generating the topology. Nagata improved this greatly by showing that *a metrizable space X has dimension $\leq n$ if and only if its topology can be induced by a metric d such that, in the metric space (X, d), for any $\varepsilon > 0$, whenever $B_1, \ldots, B_{n+2}$ are open ε-balls and the center of B_i is not in B_j for $i \neq j$, no open $\frac{\varepsilon}{2}$-ball intersects all B_i.*

These three papers were rather brief, and a detailed exposition of the results was given by Nagata in [N58a]. The complicated and subtle construction of the metrics was also presented in Nagata's monograph [N65], where he stated the problem of finding a simpler proof. This was addressed by S. Buzási [Bu79] and P. Assouad [As82]. The second paper contained some ideas which turned out to be very fruitful.

Assouad distinguished the following property of metric spaces (X, d) which was implicitly present in Nagata's reasonings: *there is a constant $C \geq 1$ such that, for any $r > 0$, there is a cover $\mathcal{U}$ of X by open sets with diameters $\leq Cr$ such that each open r-ball in (X, d) intersects at most $n + 1$ elements of $\mathcal{U}$.*

Assouad called the Nagata dimension of the metric space (X, d) the minimal n for which this property holds true.

If d is a Nagata metric on X, $r > 0$ and $\mathcal{U}$ is a maximal collection of open $2r$-balls in (X, d) such that for all distinct $B', B'' \in \mathcal{U}$, the center of B' is outside of B'', then $\mathcal{U}$ covers

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X and any open r-ball intersects at most $n+1$ elements of $\mathcal{U}$, i.e., the Nagata dimension of (X, d) is at most n (the constant C being 4).

Assouad used his notion of dimension to get a simpler construction of Nagata metrics. The full importance of the approach was disclosed only 20 years later, when U. Lang and T. Schlichenmaier [LS05] exploited the Nagata dimension in the investigation of the large-scale geometry of metric spaces - a subject of intense recent research which was originated by M. Gromov, cf. [Gr93] (in the current literature, the dimension is called the Assouad-Nagata dimension, cf. Brodskiy and Dydak [BD09]).

In the fifties of the last century, de Groot [G57], motivated by Nagata [N56b], proved that *for separable metrizable X, $\dim X \le n$ if and only if, there is a totally bounded metric d on X inducing the topology, such that for any $n+3$ points $x, y_1, \dots, y_{n+2}$ in X there are indices i, j, k, with $i \neq j$, satisfying $d(y_i, y_j) \le d(x, y_k)$.*

Nagata presented his work on dimension and metrization at the First Prague Symposium on General Topology and its Relations to Modern Analysis and Algebra in 1961, formulating as a conjecture the following non-separable version of the de Groot result: *a metrizable space X has dimension not greater than n if and only if the topology of X is generated by a metric d such that for every $n+3$ points $x, y_1, y_2, \dots, y_{n+2}$ in X, there are indices $i \neq j$ satisfying $d(y_i, y_j) \le d(x, y_j)$.*

The conjecture was confirmed independently by Nagata [N63b], [N64] (the second paper is a detailed version of the first brief note) and P. Ostrand [O65] (where an outline of the proof was given). A simplified, but still rather complicated proof of this result, using some ideas of Ostrand and Buzási, was presented by Nagata in [N83].

It remains, however, an open question, if the de Groot theorem, without requiring that the metric under consideration is totally bounded, extends to the class of all non-separable metrizable spaces. This problem was recalled by Nagata in [N05] as Question 4.6 (attributed to de Groot).

A valuable account of this topic is the article by Y. Hattori and J. Nagata [HN92].

Some other deep results on metrics related to the dimension of metrizable spaces were inspired by Szpilrajn-Marczewski [S37]. One of the results in this paper is that any n-dimensional separable metrizable space X admits a metric d generating the topology such that, in the metric space (X, d), for any $x \in X$, the dimension of the boundary of the open r-ball centered at x is not greater than $n-1$, for almost all $r > 0$ (in the sense of Lebesgue measure).

Using the subtle methods developed in earlier papers, Nagata proved in [N63a] that *any n-dimensional metrizable space X admits a metric d generating the topology such that, in the metric space (X, d), for any $r > 0$, the boundary of any open r-ball has dimension $\le n-1$ and the cover $\mathcal{U}$ of X by open r-balls is closure-preserving, i.e., for any $\mathcal{V} \subset \mathcal{U}$,* $\overline{\bigcup \mathcal{V}} = \bigcup\{\overline{V} : V \in \mathcal{V}\}$ (a special case, for $r = \frac{1}{i}$, was obtained earlier by Nagata in [N60a]).

In particular, in the metric space (X, d), for any closed set $F \subset X$ and $r > 0$, the dimension of the boundary of the set $\{x : \operatorname{dist}(x, F) < r\}$ is not greater than $n-1$.

The paper [N63a] contains also the following elegant result (announced at the Prague Symposium in 1961): *for metrizable X, $\dim X \le n$ if and only if X has an open base $\mathcal{B}$ such that for any $(n+2)$-element collection $\mathcal{C} \subset B$ with $\bigcap \mathcal{C} \neq \emptyset$, there are distinct elements $U, V \in \mathcal{C}$ with $U \subset V$.*

We shall now turn to great achievements of Nagata in dimension theory in other directions - the results related to the classical Nöbeling embedding theorem and countable-dimensional spaces.

The Nöbeling space N^m_n (N^ω_n) is the subspace of the m-dimensional Euclidean cube I^m (the Hilbert cube I^ω, respectively) consisting of all points with at most n rational coordinates. Nöbeling [No31] proved that for a separable metrizable space X, $\dim X \le n$

if and only if X can be embedded in N_n^{2n+1}.

The first paper by Nagata on this subject [N57] is concerned with strongly metrizable spaces, i.e., metrizable spaces which have a base consisting of countably many star-finite coverings. Morita [M54] established that strongly metrizable spaces of weight τ are precisely the ones that can be embedded in $B(\tau) \times I^\omega$, $B(\tau)$ being the countable product of a discrete space of cardinality τ.

Nagata proved that *for a strongly metrizable space X of weight τ, $\dim X \leq n$ if and only if X embeds in $B(\tau) \times N_n^{2n+1}$.*

In [N58b] the investigation is extended to countable-dimensional metrizable spaces, i.e., metrizable spaces that can be expressed as a countable union of zero-dimensional subspaces. Nagata considered the union of Nöbeling spaces $N_\infty^\omega = \bigcup_{n=1}^\infty N_n^\omega$ and proved that *for a strongly metrizable space X of weight τ, X is countable-dimensional if and only if X embeds in $B(\tau) \times N_\infty^\omega$.*

While the first paper [N57] was based on refining methods of Morita and Nöbeling, the second paper [N58b] brought entirely new ideas into the subject. Nagata showed in [N58b] that, in a metrizable space X with given zero-dimensional sets $A_n \subset X$ for $n = 0, 1, \ldots$ and certain collections of pairs of disjoint closed sets, one can find partitions between these sets such that each point in A_n belongs to at most n partitions.

Then, assuming in addition that X is a strongly metrizable space of weight τ, Nagata combined in a subtle way this approach with the classical construction of Urysohn functions to get an embedding $f : X \to B(\tau) \times I^\omega$ such that $f(A_n) \subset B(\tau) \times N_n^\omega$, for $n = 0, 1, \ldots$. This result was new even for $\tau = \omega$, i.e., for the class of all separable metrizable spaces.

The proofs in [N57] and [N58b] were only outlined and a detailed exposition of these results was given by Nagata in two papers published in Fundamenta Mathematicae [N58a] and [N59/60].

These papers also contain the following two remarkable results. The first one is a generalization of Hurewicz's Theorem from [Hu28]: *a metrizable space X of weight τ is countable-dimensional if and only if X is an image of a subspace of $B(\tau)$ under a closed map with finite fibers.*

The second result states that *X is countable-dimensional if and only if X has a base $\mathcal{B} = \bigcup_i \mathcal{B}_i$, where each $\mathcal{B}_i$ is a locally finite cover of X and each point in X belongs to at most finitely many boundaries of elements of $\mathcal{B}$.*

The results of [N58b] were also used by Nagata in [N60b], where the classical Eilenberg-Otto theorem on partitions in n-dimensional spaces is extended to obtain another elegant characterization of countable-dimensionality of metrizable spaces.

The papers [N60c] and [N63c] established some analogues of the Nöbeling embedding theorem for arbitrary metrizable spaces (the second short paper provided some simplifications of the results in the first paper). Key elements of the papers are based on the earlier work by Nagata, and the outcome is the following.

Let $J(\tau)$ be the metric hedgehog with τ spines, i.e., the union of segments in a Hilbert space joining the zero vector with points of a discrete set of cardinality τ in the unit sphere (centered at zero). Let us call a point in $J(\tau)$ *rational* if its norm is a rational number.

The Nagata space $K_n(\tau)$ is the set of points in the product $J(\tau)^\omega$ that have at most n rational coordinates different from zero.

Nagata proved that *in a metrizable space X of weight τ, if $A_n \subset X$ is zero-dimensional for every n, then there exists an embedding $f : X \to J(\tau)^\omega$ such that $f(A_n) \subset K_n(\tau)$ for all n.*

In particular, *$K_\infty(\tau) = \bigcup_{n-1}^\infty K_n(\tau)$ is a universal space for the class of all countable-dimensional metrizable spaces of weight $\leq \tau$.*

By a theorem of Nagata from [N58a], $K_n(\tau)$ embeds in a product of $n+1$ one-dimensional metrizable spaces. Answering a problem by Nagata stated in [N67], S. L. Lipscomb [L75] provided a more transparent construction of 1-dimensional metrizable spaces $L(\tau)$ of weight τ and certain natural n-dimensional analogues of the Nagata spaces in $L(\tau)^{n+1}$ which are universal for n-dimensional metrizable spaces of weight τ.

The monograph [E95] by R. Engelking contains a simplified exposition of the Nagata embedding theorems and a comprehensive account of the topic.

One should mention also some other valuable results of Nagata related to the classical papers by Szpilrajn-Marczewski [S37] and Pontrjagin and Schnirelmann [PS32].

In [N58a], Nagata obtained for strongly metrizable spaces some counterparts of the connections between the Hausdorff measure and the dimension established by Szpilrajn-Marczewski.

The joint paper by Nagata and Bruijning [BN79] was motivated by the results of Pontrjagin and Schnirelmann, who linked the dimension of compact metrizable space X with the cardinality of ε-nets in X with respect to the metrics generating the topology of X.

Nagata and Bruijning gave a similar description of the covering dimension dim of a Tychonoff space X in terms of a new quantity $\Delta_k(X)$ which associates a natural number to the collection of all k-element functionally open covers of X. An illuminating discussion of this subject is given by Nagata in [N05], sec. 3.

Of considerable significance is a series of surveys by Nagata on dimension theory, starting from [N67] and ending with [N92]. These articles provide an outlook of the development of dimension theory during the three decades, with an impressive scope and insight into the subject. The surveys by Nagata strongly influenced the shape of modern dimension theory.

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