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CAN A FREE ULTRAFILTER BE CONNECTEDLY GENERATED?

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Abstract. We prove that a free ultrafilter on a Tychonoff space can not be generated by connected sets. All free ultrafilters can be thought of as connectedly generated on a T_1-space. The obvious questions for T_2 and T_3 spaces are open.

For all undefined notions, we refer to Engelking [1].

See Juhász and van Mill [2] for an example of a paper where set theory and connectivity meet. The aim of this note is to prove a theorem and raise a question that are in the same spirit. Call a free ultrafilter on a space X *connectedly generated* if it has a base consisting of connected subsets of X.

Theorem 1. *If X is Tychonoff, then no free ultrafilter on X is connectedly generated.*

Proof. Let p be a free ultrafilter on X. We may assume that $X \subseteq \mathbb{I}^\kappa$ for some cardinal κ. Let $\mathbf{0}$ be the point in $\mathbb{I}^\kappa$ all whose coordinates are 0. Let us assume first that p converges to $\mathbf{0}$. For $x \in \mathbb{I}^\kappa \setminus \{\mathbf{0}\}$, put

$$\alpha_x = \min\{\alpha < \kappa : x(\alpha) \neq 0\}.$$

Then

$$\mathbb{I}^\kappa \setminus \{\mathbf{0}\} = A_0 \cup A_1,$$

where $A_0 = \{x \neq \mathbf{0} : x(\alpha_x) \text{ is rational}\}$ and $A_1 = \{x \neq \mathbf{0} : x(\alpha_x) \text{ is irrational}\}$. Since p is free, we may assume without loss of generality that $A_0 \cap X \in p$. Striving for a contradiction, assume that $C \in p$ is connected, and $C \subseteq A_0 \cap X$. Put

$$\beta = \min\{\alpha < \kappa : \pi_\alpha(C) \neq \{0\}\},$$

here $\pi_\alpha\colon \mathbb{I}^\kappa \to \mathbb{I}$ is the projection onto the α-th factor. Pick $x \in C$ such that $x(\beta) \neq 0$. Observe that $\alpha_x = \beta$, hence $x(\beta)$ is rational. This means that $\pi_\beta(C)$

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consists of rational points only. Since $\mathbf{0} \in \overline{C}$ we have $0 \in \overline{\pi_\beta(C)}$. Also, $\pi_\beta(C) \neq \{0\}$, and hence we reached a contradiction by connectivity of $\pi_\beta(C)$.

It is clear that this argument can be generalized with $\mathbf{0}$ replaced by any other point of $\mathbb{I}^\kappa$, hence we are done. $\square$

Corollary 2. *Every base of a free ultrafilter on a Tychonoff space has a member with infinitely many connected components.*

Let p be a free ultrafilter on a set X. Then $p \cup \{\emptyset\}$ is a T_1-topology on X and since all nonempty open sets in this topology intersect, they are all connected. We were unable to answer the following natural problems:

Question 3. *Does there exist a free ultrafilter on a T_2-space or a T_3-space which is connectedly generated?*

The case of countable T_2-spaces is in our opinion the most interesting.

References

[1] R. Engelking, *General topology*, Heldermann Verlag, Berlin, second ed., 1989.

[2] I. Juhász and J. van Mill, *Almost disjoint families of connected sets*, Topology Appl. **152** (2005), 209–218.

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