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Negligible sets in Erdős spaces

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ABSTRACT

We show that countable unions of nowhere dense C-sets in complete Erdős space are negligible. This improves on a theorem of Kawamura, Oversteegen, and Tymchatyn that σ-compacta are negligible in that space. We also prove that σ-compacta are negligible in Erdős space.

1. Introduction

Every topological space in this article is assumed to be separable metric. Recall that *Erdős space* $\mathfrak{E}$ consists of all sequences of rational numbers in ℓ^2, the Hilbert space of square summable real sequences. *Complete Erdős space* can be represented as

$$\mathfrak{E}_c = \{(x_i)_{i\in\omega} \in \ell^2\colon\ x_i = 0 \text{ or } 1/x_i \in \mathbb{N} \text{ for each } i \in \omega\}.$$

Both spaces were introduced and shown to be one-dimensional but totally disconnected by Paul Erdős [9] in 1940. This result together with the obvious fact that $\mathfrak{E}$ and $\mathfrak{E}_c$ are homeomorphic to their squares make these spaces important examples in Dimension Theory. Both $\mathfrak{E}$ and $\mathfrak{E}_c$ are universal spaces for the class of almost zero-dimensional spaces; see [7, Theorem 4.15]. A subset of a space is called a *C-set* if it is an intersection of clopen sets. A space is called *almost zero-dimensional* if every point has a neighbourhood basis consisting of C-sets. The spaces $\mathfrak{E}$, $\mathfrak{E}_c$, and also $\mathfrak{E}_c^\omega$ were characterized by Dijkstra and van Mill [5,7,6] and Dijkstra [4]. Complete Erdős space plays a role in complex dynamics (Mayer [11], Aarts and Oversteegen [1]) and it can be represented by, for instance, end-point sets in $\mathbb{R}$-trees (Kawamura, Oversteegen, and Tymchatyn [10]) or Polishable ideals (Dijkstra and van Mill [6]). The most important alternative representation of Erdős space is as the group of homeomorphisms of a topological manifold of dimension at least 2 that leave a countable dense set invariant (Dijkstra and van Mill [7, Theorem 10.2]).

A subset A of a space X is called *negligible* if $X \setminus A$ is homeomorphic to X. Kawamura, Oversteegen, and Tymchatyn [10] proved that σ-compacta and proper closed subsets are negligible in $\mathfrak{E}_c$. The authors proved in [7, Corollary 8.15] that proper closed subsets are negligible in $\mathfrak{E}$. In this paper we show that σ-compacta are also negligible in $\mathfrak{E}$. Dijkstra [4] proved that σ-compacta are negligible in $\mathfrak{E}_c^\omega$. It is not known whether proper closed subsets of $\mathfrak{E}_c^\omega$ are negligible. The main result in this paper improves on the negligibility of σ-compacta in $\mathfrak{E}_c$, as follows.

Theorem 1. *If $\mathcal{A}$ is a countable collection of nowhere dense C-sets in $\mathfrak{E}_c$ then $\bigcup\mathcal{A}$ is negligible in $\mathfrak{E}_c$.*

This theorem also improves upon Proposition 4.5 in van Mill [12], which is used in the construction of a Polish space that is strongly n-homogeneous for every n, but not countable dense homogeneous. According to Dijkstra, van Mill, and

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Steprāns [8] there exist dense G_δ-subsets of $\mathfrak{E}_c$ that are not homeomorphic to $\mathfrak{E}_c$. Thus Theorem 1 is sharp in the sense that we cannot replace the C-set condition by the slightly weaker requirement that elements of $\mathcal{A}$ be closed. Theorem 1 fits nicely between two well-known results: the negligibility of countable unions of nowhere dense closed sets in the space of irrational numbers (Alexandrov and Urysohn [2]) and the negligibility of σZ-sets in Hilbert space (Anderson [3]).

2. Erdős spaces

Let X be a topological space. We say that a (separable metric) topology $\mathcal{W}$ on X *witnesses the almost zero-dimensionality of* X if $\mathcal{W}$ is coarser than the given topology on X, $(X, \mathcal{W})$ is zero-dimensional, and every point of X has a neighbourhood basis in X consisting of sets that are closed in $(X, \mathcal{W})$. A space is almost zero-dimensional if and only if there is a topology $\mathcal{W}$ witnessing this fact; see [7, Remark 2.4]. The standard witness topology on $\mathfrak{E}$ and $\mathfrak{E}_c$ is the topology of coordinate-wise convergence; see [7, Chapter 2].

A space X is called *cohesive* if every point has a neighbourhood that does not contain nonempty clopen subsets of the space. Erdős [9] showed that $\mathfrak{E}$ and $\mathfrak{E}_c$ are not zero-dimensional by proving that they are cohesive.

We shall use the following characterization of $\mathfrak{E}_c$; due to the authors [6].

Theorem 2. *A nonempty space E is homeomorphic to $\mathfrak{E}_c$ if and only if there is a zero-dimensional topology $\mathcal{W}$ on E that is coarser than the given topology on E such that for every $x \in E$ and neighbourhood U of x in E there is a neighbourhood V of x in E with V closed in $(E, \mathcal{W})$, $(V, \mathcal{W})$ topologically complete, and V a nowhere dense subset of $(U, \mathcal{W})$.*

In order to formulate the characterization of $\mathfrak{E}$ that we shall use we need several definitions.

Definition 3. If A is a nonempty set then $A^{<\omega}$ denotes the set of all finite strings of elements of A, including the null string $\emptyset$. If $s \in A^{<\omega}$ then $|s|$ denotes its *length*. In this context the set A is called an *alphabet*. Let A^ω denote the set of all infinite strings of elements of A. If $s \in A^{<\omega}$ and $\sigma \in A^{<\omega} \cup A^\omega$, then we put $s \prec \sigma$ if s is an initial substring of σ, that is, there is a $\tau \in A^{<\omega} \cup A^\omega$ with $s^\frown\tau = \sigma$, where $^\frown$ denotes concatenation of strings. If $\sigma \in A^{<\omega} \cup A^\omega$ and $k \in \omega$, then $\sigma \restriction k \in A^{<\omega}$ is characterized by $\sigma \restriction k \prec \sigma$ and $|\sigma \restriction k| = k$.

Definition 4. A *tree* T on an alphabet A is a subset of $A^{<\omega}$ that is closed under initial segments, i.e., if $s \in T$ and $t \prec s$ then $t \in T$. Elements of T are called *nodes*. An *infinite branch* of T is an element σ of A^ω such that $\sigma \restriction k \in T$ for every $k \in \omega$. The *body* of T, written as $[T]$, is the set of all infinite branches of T. If $s, t \in T$ are such that $s \prec t$ and $|t| = |s| + 1$, then we say that t is an *immediate successor* of s and $\operatorname{succ}(s)$ denotes the set of immediate successors of s in T.

If S and T are trees over A respectively B, then we define the product tree $S * T$ as follows. If $s = a_1 \dots a_l \in S$ and $t = b_1 \dots b_l \in T$ are two strings of equal length, then we define the string $s * t$ over $A \times B$ by $s * t = (a_1, b_1) \dots (a_l, b_l)$. We define $S * T = \{s * t\colon\ s \in S,\ t \in T,\ |s| = |t|\}$ and note that it is a tree over $A \times B$.

Definition 5. Let T be a tree and let $(X_s)_{s \in T}$ be a system of subsets of a space X (called a *scheme*) such that $X_t \subset X_s$ whenever $s \prec t$. A subset A of X is called an *anchor* for $(X_s)_{s \in T}$ in X if for every $\sigma \in [T]$ we have $X_{\sigma \restriction k} \cap A = \emptyset$ for some $k \in \omega$ or the sequence $X_{\sigma \restriction 0}, X_{\sigma \restriction 1}, \dots$ converges to a point in X.

The following characterization of Erdős space is due to the authors [7, Theorem 8.17].

Theorem 6. *A nonempty space E is homeomorphic to $\mathfrak{E}$ if and only if there exists a zero-dimensional topology $\mathcal{W}$ on E that is coarser than the given topology on E and there exist a nonempty tree T over a countable alphabet and subspaces E_s of E that are closed with respect to $\mathcal{W}$ for each $s \in T$ such that*:

(1) *$E_\emptyset = E$ and $E_s = \bigcup\{E_t\colon\ t \in \operatorname{succ}(s)\}$ whenever $s \in T$,*
(2) *for each $s \in T$ and $t \in \operatorname{succ}(s)$ we have that E_t is nowhere dense in E_s, and*
(3) *if x is a point and U is a neighbourhood of x in E, then there is a neighbourhood $V \subset U$ of x in E that is a closed anchor for $(E_s)_{s \in T}$ in $(E, \mathcal{W})$ with the property that whenever an E_s meets V then it also meets $U \setminus V$.*

Remark 7. Let $\mathcal{W}$ and $(E_s)_{s \in T}$ be a witness topology respectively a scheme for $\mathfrak{E}$ as in Theorem 6. Consider a nonempty E_s and the tree $T' = \{t\colon\ s^\frown t \in T\}$. Then it is easily verified that the witness topology that E_s inherits from $(\mathfrak{E}_c, \mathcal{W})$ together with the scheme $(E_{s^\frown t})_{t \in T'}$ also satisfy the conditions of Theorem 6. Thus we may conclude that every E_s is homeomorphic to $\mathfrak{E}$.

Remark 8. Every nonempty open subset of $\mathfrak{E}$ contains a homeomorphic copy of $\mathfrak{E}$ that is closed in $\mathfrak{E}$. This can be seen as follows. Consider a nonempty open subset W of $\mathfrak{E}^{\mathbb{N}}$ and note that by the product topology W contains sets of the form $\{x_1\} \times \cdots \times \{x_n\} \times \mathfrak{E} \times \{x_{n+2}\} \times \cdots$. According to Dijkstra and van Mill [7, Corollary 9.4] $\mathfrak{E}$ is homeomorphic to $\mathfrak{E}^{\mathbb{N}}$. In contrast, every point in $\mathfrak{E}_c$ has a neighbourhood that does not contain a closed copy of $\mathfrak{E}_c$; see Dijkstra, van Mill, and Steprāns [8].

3. Negligibility theorems

In order to prove Theorem 1 we need a lemma about strengthening witness topologies.

Lemma 9. *Let $\mathcal{W}$ be a topology on X that witnesses the almost zero-dimensionality of X and let $\mathcal{C}$ be a countable collection of clopen sets in X such that for each $C \in \mathcal{C}$ also $X \setminus C \in \mathcal{C}$. Then the topology $\mathcal{W}'$ that is generated by the subbasis $\mathcal{W} \cup \mathcal{C}$ is also a witness to the almost zero-dimensionality of X and has the property that for every topologically complete subspace A of $(X, \mathcal{W})$ we have that A is also topologically complete in $(X, \mathcal{W}')$.*

Proof. It is evident that $\mathcal{W}'$ is zero-dimensional and separable metric by the Urysohn Metrization Theorem. Since $\mathcal{W}'$ sits between $\mathcal{W}$ and the topology on X we have that it is a witness topology to the almost zero-dimensionality of X. Let $A \subset X$ be such that $(A, \mathcal{W})$ is topologically complete. Write $\mathcal{C} = \{C_i\colon\ i \in \mathbb{N}\}$ and let for $i \in \mathbb{N}$, A_i be the set A equipped with the topology that is generated by the basis $\{A \cap O \cap C_i, A \cap O \setminus C_i\colon\ O \in \mathcal{W}\}$. Note that A_i is homeomorphic to the topological sum of $A \cap C_i$ and $A \setminus C_i$, both equipped with the weak topology $\mathcal{W}$. Since C_i is clopen in X we have by [7, Remark 2.5] that both C_i and $X \setminus C_i$ are G_δ-sets in $(X, \mathcal{W})$. We may conclude that every A_i is topologically complete and hence so is $\prod_{i=1}^\infty A_i$. It is evident that $(A, \mathcal{W}')$ is homeomorphic to the diagonal $\Delta = \{(x, x, x, \ldots)\colon\ x \in A\}$ of $\prod_{i=1}^\infty A_i$. Since Δ is closed in $\prod_{i=1}^\infty (A, \mathcal{W})$ it is clearly also closed in $\prod_{i=1}^\infty A_i$. Thus we have that Δ and $(A, \mathcal{W}')$ are topologically complete. $\square$

Proof of Theorem 1. Let $\mathcal{W}$ be a witness topology for $\mathfrak{E}_c$ as in Theorem 2. Write $\mathcal{A} = \{A_i\colon\ i \in \mathbb{N}\}$ and $A = \bigcup \mathcal{A}$. For each $i \in \mathbb{N}$ select a countable collection $\mathcal{C}_i$ of clopen subsets of $\mathfrak{E}_c$ such that $A_i = \bigcap \mathcal{C}_i$. Put $\mathcal{C} = \{C, \mathfrak{E}_c \setminus C\colon\ C \in \mathcal{C}_i,\ i \in \mathbb{N}\}$ and let $\mathcal{W}'$ be the topology generated by $\mathcal{W} \cup \mathcal{C}$ as in Lemma 9. Note that every A_i is closed in $(X, \mathcal{W}')$. To show with Theorem 2 that $Y = \mathfrak{E}_c \setminus A$ is homeomorphic to $\mathfrak{E}_c$ let $x \in Y$ and let U be an open neighbourhood of x in $\mathfrak{E}_c$. Since $\mathfrak{E}_c$ is cohesive we may assume that U contains no clopen subsets of $\mathfrak{E}_c$ other than the empty set. Select a neighbourhood $V \subset U$ of x in $\mathfrak{E}_c$ with V closed in $(\mathfrak{E}_c, \mathcal{W})$ and $(V, \mathcal{W})$ topologically complete. Then $V \setminus A$ is closed in $(Y, \mathcal{W}')$ and $(V, \mathcal{W}')$ is topologically complete by Lemma 9. Since A is an F_σ-set with respect to $\mathcal{W}'$ we have that $(V \setminus A, \mathcal{W}')$ is topologically complete.

To prove that $V \setminus A$ has an empty interior in $(U \setminus A, \mathcal{W}')$ let $P \in \mathcal{W}'$ be such that $P \cap (V \setminus A) \neq \emptyset$. Since $\mathcal{W}'$ is zero-dimensional we may assume that P is clopen in $(\mathfrak{E}_c, \mathcal{W}')$ and hence also in $\mathfrak{E}_c$. If $P \cap V = P \cap U$ then $P \cap U$ is a clopen nonempty set in $\mathfrak{E}_c$ which violates the cohesion assumption on U so we may conclude that $P \cap (U \setminus V)$ is not empty. Since $P \cap (U \setminus V)$ is an open subset of the complete space $\mathfrak{E}_c$ it cannot be contained in the first category set A. Thus $P \cap (U \setminus A) \setminus V \neq \emptyset$ and we have that $V \setminus A$ has an empty interior in $(U \setminus A, \mathcal{W}')$. With Theorem 2 we may conclude that Y is homeomorphic to $\mathfrak{E}_c$. $\square$

Theorem 10. *Let $\mathcal{W}$ be a witness topology for $\mathfrak{E}$ as in Theorem 6. If A is a subspace of $\mathfrak{E}$ that is an F_σ-set in $(\mathfrak{E}, \mathcal{W})$ and that does not contain a homeomorphic copy of $\mathfrak{E}$ that is closed in $\mathfrak{E}$ then A is negligible in $\mathfrak{E}$.*

Proof. Let $\mathcal{W}$ and $(E_s)_{s \in T}$ be a witness topology respectively a scheme for $\mathfrak{E}$ as in Theorem 6. Let $Y = \mathfrak{E} \setminus A$ and note that $Y \neq \emptyset$. Write $A = \bigcup_{i=1}^\infty A_i$ with every A_i closed with respect to $\mathcal{W}$. We can write $\mathfrak{E} \setminus A_i = \bigcup_{j=1}^\infty F_{ij}$ with each F_{ij} clopen in $(\mathfrak{E}, \mathcal{W})$. If $s * t$ is a node of the tree $T * \mathbb{N}^{<\omega}$ with $t = j_1 \ldots j_k$, then we define $E'_{s*t} = Y \cap E_s \cap C_t$, where C_t denotes the clopen set $\bigcap_{i=1}^k F_{ij_i}$. Let S be the subtree $\{r \in T * \mathbb{N}^{<\omega}\colon\ E'_r \neq \emptyset\}$. We verify that $(E'_r)_{r \in S}$ satisfies the conditions of Theorem 6 with witness topology $\mathcal{W}' = \{Y \cap O\colon\ O \in \mathcal{W}\}$ for Y.

It is clear that every E'_r is closed in $(Y, \mathcal{W}')$ and that condition (1) is satisfied. For condition (2) Let $s' * t'$ be a successor of an $s * t \in S$ and let O be open in $\mathfrak{E}$ with $E'_{s'*t'} \cap O \neq \emptyset$. Then $E_{s'} \cap C_{t'} \cap O \neq \emptyset$ and hence $P = (E_s \setminus E_{s'}) \cap O \cap C_{t'} \neq \emptyset$. Note that P is an open subset of E_s, which space is homeomorphic to $\mathfrak{E}$ by Remark 7. Then P contains a copy of $\mathfrak{E}$ that is closed in E_s and hence in $\mathfrak{E}$ by Remark 8. Thus P is not contained in A and we have $P \cap Y \neq \emptyset$. Since $C_{t'} \subset C_t$ we now have that $(E'_{s*t} \setminus E'_{s'*t'}) \cap O \neq \emptyset$ and hence that $E'_{s'*t'}$ is nowhere dense in E'_{s*t}.

For condition (3), let $x \in Y$ and let U be an open neighbourhood of x in $\mathfrak{E}$. According to [7, Lemma 8.16] we may choose a neighbourhood $V \subset U$ of x in $\mathfrak{E}$ that is a closed anchor for $(E_s)_{s \in T}$ in $(\mathfrak{E}, \mathcal{W})$ with the property that V contains no nonempty clopen subsets of any E_s. Consider an E'_{s*t} that meets $V \setminus A$. Then $E_s \cap C_t$ is a clopen subset of E_s that meets V. If $E_s \cap C_t \cap V = E_s \cap C_t \cap U$ then it is also a clopen nonempty subset of E_s, which contradicts a property of V. Thus we have $Q = E_s \cap C_t \cap (U \setminus V) \neq \emptyset$. Note that Q is an open subset of E_s and hence by Remarks 7 and 8, Q contains a copy of $\mathfrak{E}$ that is closed in E_s and in $\mathfrak{E}$. So we have that $Q \cap Y = E'_{s*t} \cap (U \setminus V) \neq \emptyset$.

For the anchor property let $\sigma * \tau \in [S]$ be such that $E_{\sigma \restriction k} \cap C_{\tau \restriction k} \cap Y \cap V \neq \emptyset$ for each $k \in \omega$. Then $(E_{\sigma \restriction k})_{k \in \omega}$ converges to an x in $(\mathfrak{E}, \mathcal{W})$ and hence also $(E'_{(\sigma*\tau)\restriction k})_{k \in \omega}$ converges to x. Let $\tau = j_1 j_2 \ldots$ and let $i \in \mathbb{N}$. Then $E'_{(\sigma*\tau)\restriction k} \subset F_{ij_i}$ for each $k \geqslant i$. Since F_{ij_i} is closed with respect to $\mathcal{W}$ we have $x \in F_{ij_i}$ and thus $x \notin A_i$. Thus $x \in Y$ and we have shown $V \setminus A$ to be an anchor for $(E'_r)_{r \in S}$ in $(Y, \mathcal{W}')$. $\square$

A space is called *σ-complete* if it can be written as a countable union of topologically complete subspaces. Since $\mathfrak{E}$ is not σ-complete (see [7, p. 23]) we have:

Corollary 11. *Let $\mathcal{W}$ be a witness topology for $\mathfrak{E}$ as in Theorem 6. If A is a σ-complete subspace of $\mathfrak{E}$ that is an F_σ-set in $(\mathfrak{E}, \mathcal{W})$ then A is negligible in $\mathfrak{E}$.*

An immediate consequence is:

Corollary 12. *σ-Compacta are negligible in $\mathfrak{E}$.*

Example 13. Consider $\mathfrak{E}$ as a subset of ℓ^2. Then the topology of coordinate-wise convergence $\mathcal{W}$ is a witness topology that satisfies the conditions of Theorem 6; see [7, Proposition 8.12]. Note that $\mathfrak{E}_c$ is a subspace of $\mathfrak{E}$ that is closed with respect to $\mathcal{W}$. If A is an arbitrary countable subset of $\mathfrak{E}$ then $A + \mathfrak{E}_c$ is a negligible subset of $\mathfrak{E}$ by Corollary 11.

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