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HOMEOMORPHISM GROUPS AND METRISATION OF MANIFOLDS

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Abstract. We prove that a manifold M is metrisable if and only if its group of homeomorphisms $\mathscr{H}(M)$ endowed with the compact-open topology is a q-space. We also discuss pseudo-character and tightness.

All spaces under discussion are Tychonoff.

1. Introduction

In this note we continue the search for topological properties that in general are weaker than metrisability but for manifolds are equivalent to metrisability. See Gauld [**7**] and Gauld and Mynard [**8**] for details and references. By a *manifold* we mean a connected space which is locally homeomorphic to Euclidean space. Our focus here is on the homeomorphism group $\mathscr{H}(M)$ of a manifold M. That is, we identify topological properties $\mathscr{P}$ such that $\mathscr{H}(M)$ has $\mathscr{P}$ if and only if M is metrisable. Here $\mathscr{H}(M)$ is endowed with the compact-open topology. Our main result is that a manifold M is metrisable if and only if $\mathscr{H}(M)$ is a q-space. We also discuss pseudo-character and tightness.

2. Preliminaries

For a space X we let $\mathscr{H}(X)$ denote the group of homeomorphisms of X endowed with the compact-open topology. The neutral element of $\mathscr{H}(X)$, that is, the identity function on X, will be denoted by e. For subsets A and B of X we define $[A, B] = \{f \in \mathscr{H}(X) : f(A) \subseteq B\}$, and we recall that the topology on $\mathscr{H}(X)$ is generated by the subbase

$$\mathscr{S}_X = \{[K, O] : K, O \subseteq X, K \text{ compact, } O \text{ open}\}.$$

The space $\mathscr{H}(X)$ is *homogeneous*, i.e., all points of it are topologically identical. To prove this, first consider for a given $f \in \mathscr{H}(X)$ the translation λ defined by $\lambda_f(g) = f \circ g$. Then λ_f is continuous, since $\lambda_f^{-1}([K, O]) = \{g \in \mathscr{H}(X) : f\big(g(K)\big) \subseteq O\} = \{g \in \mathscr{H}(X) : g(K) \subseteq f^{-1}(O)\} = [K, f^{-1}(O)]$, for all $K, O \subseteq X$. Hence λ_f is a homeomorphism of $\mathscr{H}(X)$ since its inverse is the translation $\lambda_{f^{-1}}$. This easily implies that $\mathscr{H}(X)$ is homogeneous. Our argument is well-known of course.

Observe that for a locally compact space X, the weight of $\mathscr{H}(X)$ does not exceed the weight of X. Hence if X is a locally compact separable metrisable space, then $\mathscr{H}(X)$ is separable and metrisable as well.

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If X is compact, then $\mathscr{H}(X)$ is easily seen to be a topological group and the natural action

$$\mathscr{H}(X) \times X : (g, x) \mapsto g(x)$$

is continuous. It is a classical result of Arens [**1**] that $\mathscr{H}(X)$ endowed with the compact-open topology is a topological group if X is an arbitrary locally compact and locally connected space. Even if X is locally compact and of countable weight, then the continuity of the inverse may fail. Dijkstra [**4**] generalized the Arens result for spaces X that have the property that every $x \in X$ has a neighbourhood that is a continuum. Observe that such spaces are locally compact.

For $x \in X$ we let $\gamma_x\colon \mathscr{H}(X) \to X$ be the evaluation at the point x, that is, $\gamma_x(f) = f(x)$ for every $f \in \mathscr{H}(X)$. Observe that the functions γ_x are continuous. Simply observe that for every open subset O of X, we have that $\gamma_x^{-1}(O)$ is nothing but the subbasic open set $[\{x\}, O]$.

We say that X is *strongly locally homogeneous* (abbreviated SLH) if it has a base $\mathscr{B}$ such that for all $B \in \mathscr{B}$ and $x, y \in B$ there is an element $f \in \mathscr{H}(X)$ that is supported on B (that is, f is the identity outside B) and moves x to y. Clearly, every manifold is SLH.

A sequence $\mathscr{V}_0, \mathscr{V}_1, \ldots$ of open covers of a space X is called *normal* if each $\mathscr{V}_n$ is a star-refinement of $\mathscr{V}_{n-1}$.

A space X is a *q-space* if for every point $x \in X$ there exists a sequence $\{U_n : n < \omega\}$ of open neighbourhoods of x in X such that for every choice $x_n \in U_n$, the sequence $\{x_n : n < \omega\}$ has a cluster point.

3. Lemmas

Lemma 3.1. *Let X be SLH and homogeneous. Then for every $x \in X$ we have that the function $\gamma_x\colon \mathscr{H}(X) \to X$ is a continuous, open surjection.*

Proof. It is clear that $\gamma_x\colon \mathscr{H}(X) \to X$ is a continuous surjection, due to the fact that X is homogeneous. So the only thing to be proved is that every γ_x is open.

To begin with, let U be an arbitrary open neighbourhood of the neutral element e of $\mathscr{H}(X)$.

Claim 1. For every $x \in X$, x is in the interior of $\gamma_x(U)$.

There are compact sets $K_1, \ldots, K_m$ and open sets $O_1, \ldots, O_m$ in X such that

$$e \in \bigcap_{i=1}^{m} [K_i, O_i] \subseteq U.$$

Let V be an open neighbourhood containing x such that for every $i \leq m$, $V \subseteq O_i$ if $x \in K_i$ and $V \cap K_i = \emptyset$ if $x \notin K_i$. Since X is SLH, we may assume that V has the property that for every $y \in V$ there is a homeomorphism f of X which sends x onto y and is supported on V. Fix an arbitrary $y \in V$ and let f be such a homeomorphism for y. Then f sends every K_i into O_i, hence $f \in U$. But this means that $y \in \gamma_x(U)$, i.e., $V \subseteq \gamma_x(U)$.

Now let V be an arbitrary open subset of $\mathscr{H}(X)$.

Claim 2. For every $x \in X$, $\gamma_x(V)$ is open.

Pick an arbitrary point $y \in \gamma_x(V)$. Let $f \in V$ be such that $f(x) = y$. There exists an open neighbourhood U of the neutral element e of $\mathscr{H}(X)$ such that $fU \subseteq V$. By Claim 1, there is an open subset W of X containing x which is contained in $\gamma_x(U)$. Now consider the open neighbourhood $Z = f(W)$ of y. For $z \in Z$, pick $w \in W$ such that $f(w) = z$. There exists $g \in U$ such that $g(x) = w$. Hence $f \circ g(x) = z$, and $f \circ g \in fU \subseteq V$. We conclude that $z \in \gamma_x(V)$, i.e., $y \in Z \subseteq \gamma_x(V)$. □

Lemma 3.2. *Let X be* SLH *and homogeneous, and let $(U_n)_n$ be a sequence of symmetric open neighbourhoods of the neutral element e of $\mathscr{H}(M)$ such that $U_{n+1}^3 \subseteq U_n$, for every n. For every n let $\mathscr{V}_n = \{\gamma_x(U_n) : x \in X\}$. Then the sequence of open covers $\mathscr{V}_0, \mathscr{V}_1, \ldots$ is normal.*

Proof. That every $\mathscr{V}_n$ is an open cover of X follows from Lemma 3.1. Let $P = \gamma_x(U_n)$ for some $x \in X$. Take an arbitrary $p \in X$ such that $\gamma_p(U_n) \cap P \neq \emptyset$, say $z \in \gamma_p(U_n) \cap \gamma_x(U_n)$. Let w be an arbitrary element of $\gamma_p(U_n)$. There are $\alpha, \beta, \gamma \in U_n$ such that $\alpha(p) = w$, $\beta(p) = z$ and $\gamma(x) = z$. Then

$$\alpha \circ \beta^{-1} \circ \gamma(x) = w$$

and $\alpha \circ \beta^{-1} \circ \gamma \in U_{n-1}$ since U_n is symmetric and $U_n^3 \subseteq U_{n-1}$. Hence we conclude that $w \in \gamma_x(U_{n-1})$. Hence $\mathsf{St}(P, \mathscr{V}_{n-1}) \subseteq \gamma_x(U_{n-1}) \in \mathscr{V}_{n-1}$, as required. □

Lemma 3.3. *Let M be a manifold, and let $(U_n)_n$ be a sequence of neighbourhoods of the neutral element e in $\mathscr{H}(M)$. Then there exists a closed and separable subset K of M such that if $f \in \mathscr{H}(M)$ and f restricts to the identity on K, then $f \in \bigcap_{n<\omega} U_n$.*

Proof. For a fixed n we may choose a basic neighbourhood of e of the form

$$[K_1, O_1] \cap \cdots \cap [K_i, O_i]$$

that is contained in U_n. Hence any homeomorphism of M that is supported on a set that misses the compact set $\bigcup_{j=1}^{i} K_j$ belongs to U_n. We conclude that there exists a σ-compact set A in M such that any homeomorphism that fixes A pointwise belongs to $\bigcap_{n<\omega} U_n$. Observing that A is separable we see that $K = \overline{A}$ is as required. □

4. Main Results

We are looking for conditions on $\mathscr{H}(M)$ for a manifold M that ensure that M is metrisable.

Here is our first 'duality' theorem.

Theorem 4.1. *Let M be a manifold. Then the following statements are equivalent:*
(1) *M is separable,*
(2) *$\mathscr{H}(M)$ has countable pseudo-character.*

Proof. Let us first assume that M is separable, and fix a countable dense subset D of M. For every $d \in D$ let $\mathscr{U}_d$ be a countable local base at d. It is easily seen that the collection

$$\{[\{d\}, U] : d \in D, U \in \mathscr{U}_d\}$$

is a countable pseudo-base at the neutral element e of $\mathscr{H}(M)$. Since $\mathscr{H}(M)$ is homogeneous, it consequently has countable pseudo-character. (Since $\mathscr{H}(X)$ is homogeneous for every space X, the proof shows that we actually proved the following more general statement: if X has a countable dense set D such that X is first countable at every point of D, then $\mathscr{H}(X)$ has countable pseudo-character.)

Now assume that $\mathscr{H}(M)$ has countable pseudo-character, and let $(U_n)_n$ be a countable pseudo-base at the neutral element e of $\mathscr{H}(M)$. Let K be the separable closed set we get from Lemma 3.3. We claim that $K = M$. Suppose not, and pick a nonempty euclidean open ball B that misses K. There clearly is a homeomorphism f of M which is supported on B and moves a point of B. Hence f is not the identity, yet $f \in \bigcap_{n<\omega} U_n = \{e\}$. This is a contradiction. $\square$

By Nyikos [**11**, Example 3.7] there is a separable nonmetrisable Moore manifold M. Hence $\mathscr{H}(M)$ has countable pseudo-character by Theorem 4.1, yet M is not metrisable. See also Rudin and Zenor [**12**] for another example. This shows that countable character in Theorem 4.2 below cannot be replaced by pseudo-character.

It is not true that M has a G_δ-diagonal if and only if $\mathscr{H}(M)$ has one. To see this, consider the Prüfer manifold, [**11**, Example 3.7]. It is a Moore space so it has a G_δ-diagonal. But it also has cellularity $\mathfrak{c}$, hence it is not separable. Hence $\mathscr{H}(M)$ does not have a G_δ-diagonal since it is not even of countable pseudo-character by Theorem 4.1.

We now come to our main 'duality' result. The equivalence of (1) and (5) answers a question posed by Alexandre Gabard.

Theorem 4.2. *For a manifold M, the following statements are equivalent:*
(1) *M is metrisable,*
(2) *M is separable and metrisable,*
(3) *$\mathscr{H}(M)$ is first countable,*
(4) *$\mathscr{H}(M)$ is a q-space,*
(5) *$\mathscr{H}(M)$ is metrisable,*
(6) *$\mathscr{H}(M)$ is separable and metrisable.*

Proof. Some of the implications are trivial. That a first countable topological group is metrisable is well-known, see e.g., [**10**, Theorem 8.3]. We already observed that $\mathscr{H}(M)$ has countable weight if M is metrisable. It consequently suffices to prove that (3) implies (1) and (4) implies (3).

For (3) $\Rightarrow$ (1), assume that $\mathscr{H}(M)$ is first countable. By the Arens result quoted above, $\mathscr{H}(M)$ is a topological group. Hence there is a sequence of symmetric neighbourhoods $(U_n)_n$ of the neutral element e of $\mathscr{H}(M)$ such that
(1) $\{U_n : n < \omega\}$ is a local base at e in $\mathscr{H}(M)$,
(2) $U_{n+1}^3 \subseteq U_n$, for every n.
Let $\mathscr{V}_0, \mathscr{V}_1, \ldots$ be the sequence of open covers we get from $(U_n)_n$ as in Lemma 3.2. We claim that this sequence is a strong development for M which suffices by the Moore Metrisation Theorem, see [**5**, 5.4.2]. For that it suffices to prove that $\mathscr{V} = \bigcup_{n<\omega} \mathscr{V}_n$ is a base. But this is trivial. To see this, let $x \in M$, and let O be an arbitrary open neighbourhood of x. Then $\gamma_x^{-1}(O)$ is an open neighbourhood of e in $\mathscr{H}(M)$, hence there exists n such that $U_n \subseteq \gamma_x^{-1}(O)$. But then $x \in \gamma_x(U_n) \subseteq O$.[1]

[1]Added in proof: We are indebted to Konstantin Kozlov for pointing out to us that implication 3) to 1) in Theorem 4 could also be proved using Corollary 4 in [**2**]. In this case one needs also to appeal to our Lemma 3.1.

For (4) $\Rightarrow$ (3), assume that $\mathscr{H}(M)$ is a q-space. We claim that M is separable. Assume the contrary, and let $(U_i)_i$ be any sequence of neighbourhoods of the neutral element e of $\mathscr{H}(M)$ which makes e a q-point of $\mathscr{H}(M)$. Let K be the separable closed subset of M that we get from Lemma 3.3. Let U be a nonempty euclidean open subset of M that misses K. Let B be a nonempty open ball whose compact closure $\overline{B}$ is contained in U. The group of homeomorphisms of $\overline{B}$ that fix the boundary ∂B of B pointwise is not compact (see e.g., Geoghegan [**9**]). Hence there is a sequence $(f_n)_n$ of homeomorphisms of $\overline{B}$ each element of which fixes ∂B pointwise and which has no limit point. Extend each of these homeomorphisms to a homeomorphism of M by requiring it to be the identity outside $\overline{B}$. The sequence of homeomorphisms thus obtained is contained in $\bigcap_{i<\omega} U_i$ but has no limit point in $\mathscr{H}(M)$. This is a contradiction.

Hence M is separable, which implies by Theorem 4.1 that $\mathscr{H}(M)$ has countable pseudo-character. But a q-space of countable pseudo-character is first countable. This is implicit in Chiba [**3**]. For completeness sake, we will present the easy proof. Let $(U_i)_i$ be a sequence of open neighbourhoods of the neutral element e of $\mathscr{H}(M)$ that makes e a q-point of $\mathscr{H}(M)$. Since $\mathscr{H}(M)$ has countable pseudo-character, we may additionally assume that $\overline{U}_{i+1} \subseteq U_i$ for every i and $\bigcap_{i<\omega} U_i = \{e\}$. We claim that $(U_i)_i$ is a neighbourhood base at e in $\mathscr{H}(M)$. To see this, let O be any open neighbourhood of e in $\mathscr{H}(M)$. If $U_i \nsubseteq O$ for every i, then pick an element $f_i \in U_i \setminus O$. For a limit point f of the sequence $(f_i)_i$ we have $f \in \bigcap_{i<\omega} \overline{U}_i = \{e\}$. Hence infinitely many terms of the sequence $(f_i)_i$ belong to O, a contradiction. □

5. Tightness

Recall that a space X has *countable tightness* if for every $x \in X$ and subset A of X such that $x \in \overline{A}$ there is a countable subset B of A such that $x \in \overline{B}$. It is an interesting question whether countable tightness of $\mathscr{H}(M)$ implies that M is metrisable.

See Nyikos [**11**] for the definition of the open long ray $\mathbb{L}_+$.

Example 5.1. The homeomorphism group of the open long ray $\mathbb{L}_+$ does not have countable tightness.

There is clearly a homeomorphism h of $\mathbb{L}_+$ such that $h{\restriction}\omega_1 = e$ and $h{\restriction}[\alpha, \alpha{+}1] \neq e$ for every $\alpha \in \omega_1 \setminus \{0\}$. Now let $h_\alpha \colon \mathbb{L}_+ \to \mathbb{L}_+$ for every $\alpha \in \omega_1 \setminus \{0\}$ be defined as follows:

$$h_\alpha(\beta) = \begin{cases} \beta & (0 < \beta \leq \alpha), \\ h(\beta) & (\alpha < \beta < \omega_1). \end{cases}$$

Then for $S = \{h_\alpha : \alpha \in \omega_1 \setminus \{0\}\}$ we have $e \in \overline{S}$ but $e \notin \overline{T}$ for every countable $T \subseteq S$.

Theorem 5.2. *Let M be an ω-bounded manifold containing two disjoint closed subsets A and B such that for every compact subset K of M, some component of $M \setminus K$ meets both A and B. Then $\mathscr{H}(M)$ does not have countable tightness.*

Proof. Since M is ω-bounded, we can write it as $\bigcup_{\alpha<\omega_1} K_\alpha$, where K_α is compact for every $\alpha < \omega$ and $K_\alpha \subseteq K_\beta$ if $\alpha < \beta$. For every $\alpha < \omega_1$, let U_α be a component of $M \setminus K_\alpha$ meeting both A and B, say in the points a_α and b_α, respectively. By using simple chains of connected Euclidean open sets connecting a_α and b_α, it is easy to construct for every $\alpha < \omega_1$ a connected Euclidean open subset V_α of M containing both a_α and b_α while moreover $V_\alpha \subseteq U_\alpha$. Now for every $\alpha < \omega_1$ let, $h_\alpha \in \mathscr{H}(M)$ be supported on V_α while moreover $h_\alpha(a_\alpha) = b_\alpha$. Put $S = \{h_\alpha : \alpha < \omega_1\}$.

Claim 1. $e \in \overline{S}$.

Indeed, let $\mathscr{K}$ be an arbitrary finite family of compact subsets of M, and put $K = \bigcup \mathscr{K}$. Since K is separable, there is an $\alpha < \omega_1$ such that $K \subseteq K_\alpha$. Hence h_β for $\beta \geq \alpha$ restricts to the identity on every member of $\mathscr{K}$. This is clearly as required.

Claim 2. For every $\alpha < \omega_1$, $e \notin \overline{\{h_\beta : \beta < \alpha\}}$.

Let A_α be the closure of the set $\{a_\beta : \beta < \alpha\}$. Then A_α is compact, and $e \in [A_\alpha, M \setminus B]$. But $h_\beta \notin [A_\alpha, M \setminus B]$ for every $\beta < \alpha$. □

There are of course many manifolds to which the theorem applies, for example, the square of the open long ray as well as many long pipes. We do not know whether the homeomorphism groups of the Prüfer manifolds have countable tightness; for a nice description of these manifolds we refer to Gabard [**6**, Section 3].

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