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# On uniquely homogeneous spaces, I

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**Abstract.** It is shown that all uniquely homogeneous spaces are connected. We characterize the uniquely homogeneous spaces that are semitopological or quasitopological groups. We identify two properties of homogeneous spaces called skew-2-flexibility and 2-flexibility that are useful in studying unique homogeneity. We also construct a large family of uniquely homogeneous spaces with only trivial continuous maps.

## 1. Introduction.

*All spaces under discussion are Tychonoff. By a homeomorphism of $X$ we will always mean a homeomorphism of $X$ onto itself. For a function $f\colon X \to Y$ such that $f\colon X \to f(X)$ is a homeomorphism and $f(X) \neq Y$, we use the term embedding.*

A space $X$ is called *uniquely homogeneous* provided that for all $x, y \in X$ there is a unique homeomorphism of $X$ that takes $x$ onto $y$. This concept is due to Burgess [**9**] who asked in 1955 whether there exists a non-trivial uniquely homogeneous metrizable continuum. Ungar [**33**] showed in 1975 that there are no such finite-dimensional metrizable continua and a few years later, Barit and Renaud [**6**] showed that the assumption on finite-dimensionality is superfluous. A somewhat different argument was given by Keesling and Wilson [**19**]. A nontrivial uniquely homogeneous Baire space of countable weight was constructed by van Mill [**23**]. This example is a topological group. There are also uniquely homogeneous spaces that do not admit the structure of a topological group, [**24**]. It is unknown whether there is a non-trivial Polish uniquely homogeneous space.

Topological homogeneity is not well understood outside the class of separable metrizable spaces. In fact, almost all known homogeneous compacta are homeomorphic to a product of dyadic compacta and first countable compacta. See Milovich [**28**] for the only known ***ZFC*** example of a homogeneous compactum that is not of that form. In the light of the above results on unique homogeneity,

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the authors thought about the question of whether there exists a nonmetrizable uniquely homogeneous compactum. Such a space, if it exists, would definitely be completely different from all the known examples of homogeneous compacta. This turned out to be a formidable problem and we unfortunately leave it unanswered.

Our results imply that a non-trivial uniquely homogeneous space is connected, does not admit the structure of a topological group if it is compact, and is not linearly orderable.

Along the way, we identify two properties of homogeneous spaces called skew-2-flexibility and 2-flexibility that are useful in studying unique homogeneity. We prove that every locally compact homogeneous metrizable space is both skew-2-flexible and 2-flexible, and give an example of a homogeneous Polish space that is skew-2-flexible but not 2-flexible. We also show that there are nontrivial products among the uniquely homogeneous spaces.

This paper is organized as follows. We first formulate and prove some results on topological homogeneity. Then we apply these results in our study of uniquely homogeneous spaces. Some pertinent open problems are stated throughout the paper.

## 2. Preliminaries.

For a space $X$ we let $\mathscr{H}(X)$ denote its group of homeomorphisms endowed with the compact-open topology. If $X$ is compact, then $\mathscr{H}(X)$ is easily seen to be a topological group and the natural action

$$\mathscr{H}(X) \times X : (g, x) \mapsto g(x)$$

is continuous. It is not difficult to show that the weight of $\mathscr{H}(X)$ for compact spaces $X$ does not exceed the weight of $X$. If $X$ has countable weight and compact, then $\mathscr{H}(X)$ is a Polish group.

It is a classical result of Arens [**2**] that $\mathscr{H}(X)$ endowed with the compact-open topology is a topological group if $X$ is an arbitrary locally compact and locally connected space. Even if $X$ is locally compact and of countable weight, then the continuity of the inverse may fail. Dijkstra [**11**] generalized the Arens result for spaces $X$ that have the property that every $x \in X$ has a neighborhood that is a continuum. Observe that such spaces are locally compact.

For locally compact spaces there is of course a 'good' topology for $\mathscr{H}(X)$ that works as nicely as the compact-open topology. Just think of $X$ as situated in its Alexandrov 1-point compactification $\alpha X = X \cup \{\infty\}$, and think of $\mathscr{H}(X)$ as the subgroup of $\mathscr{H}(\alpha X)$ consisting of all the homeomorphisms that fix $\infty$. Hence if $X$ is locally compact and of countable weight, then $\mathscr{H}(X)$ with this topology is a

Polish group acting continuously on $X$.

A topological group $G$ is called *$\omega$-narrow* if for every open neighborhood $U$ of the neutral element $e$ of $G$ there is a countable set $F$ in $G$ such that $FU = G$. This class of groups is also called $\aleph_0$-bounded in the literature. It was proved by Guran that a topological group G is $\omega$-narrow if and only if it is topologically isomorphic to a subgroup of a product of groups of countable weight. For a proof, see Uspenskiĭ [**35**].

A *semitopological group* (respectively, *paratopological group*) is a group endowed with a topology for which the product is separately (respectively, jointly) continuous. See [**5**] for conditions guaranteeing that a semitopological group (respectively, paratopological group) is a topological group.

A *quasitopological group* is a semitopological group such that the inverse operation is continuous.

Let $G$ be a semitopological group acting on a space $X$ by a separately continuous action. That is, the maps $\gamma_x = g \mapsto gx : G \to X$ for $x$ in $X$ and $x \mapsto gx : X \to X$ for $g$ in $G$ are all continuous. We say that the action is *micro-transitive* provided that for all open neighborhoods $U$ of the neutral element $e$ of $G$ we have that for every $x \in X$, $Ux$ is open in $X$.

## 3. Homogeneity.

Let $X$ be a uniquely homogeneous space. Fix an element $e \in X$, and let $f_x$ for every $x \in X$ be the unique homeomorphism of $X$ sending $e$ onto $x$. The function $i\colon \mathscr{H}(X) \to X$ defined by $i(g) = g(e)$ is clearly one-to-one and surjective. Hence if we can prove that it is a homeomorphism say for compact $X$, then we know that $X$ is a topological group and then it may be possible to use the existing knowledge of topological groups to draw nontrivial conclusions. That this approach is not as naive as it seems to be at first glance, becomes clear if we consider the methods of Ungar [**33**] and Barit and Renaud [**6**]. For a uniquely homogeneous space $X$ of countable weight that is either compact or locally compact and locally connected, they prove that $i\colon \mathscr{H}(X) \to X$ is indeed a homeomorphism, and from that it can be shown that $X$ contains at most 2 points. Continuity of $i$ is clear from the fact that $\mathscr{H}(X)$ acts continuously on $X$, and the fact that $i$ is open is a consequence of the celebrated Effros Theorem from [**14**] (see also [**1**] and [**26**]). This result says that a transitive and continuous action of a Polish group $G$ on a second category space $X$ of countable weight has the property that all the evaluation functions $\gamma_x\colon G \to X$ defined by $\gamma_x(g) = gx$ are open surjections. In the terminology of Section 2, a transitive action of a Polish group $G$ on a second category space of countable weight is micro-transitive (in fact, this is even true if $G$ is analytic, see [**26**]). So in order to apply the Effros Theorem, all one needs is the existence of a transitive action by a Polish group. By the remarks in Section 2 it follows that the results of Ungar, Barid and Renaud are true for all locally compact spaces of countable weight, the assumption on local connectivity is superfluous. Hence the conclusion is that every locally compact uniquely homogeneous space of countable weight has at most 2 points.

Very little is known about full homeomorphism groups that are compact, or locally compact. The only significant result known to us is due to Keesling [**18**]. He showed that if $X$ is a metric space and $\mathscr{H}(X)$ is locally compact, then it is zero-dimensional. Keesling's proof is based on an elegant result of Beck [**7**] and it is not clear whether it can be generalized. This brings us to our first question.

Question 3.1. *Does there exist an infinite homogeneous compact space $X$ such that $\mathscr{H}(X)$ is (locally) compact?*

The following result is a partial answer to this question.

Theorem 3.2. *There is no compact, homogeneous and infinite space $X$ such that $\mathscr{H}(X)$ is both compact and zero-dimensional.*

Proof. We first claim that $X$ is zero-dimensional. To prove this, fix a point e in $X$ and consider the stabilizer $H_e$, that is, the subgroup of $\mathscr{H}(X)$ consisting of all $f$ in $\mathscr{H}(X)$ such that $f(e) = e$. Clearly, $H_e$ is a closed subgroup of $\mathscr{H}(X)$, and the coset space $F(X) = H(X)/H_e$ can be mapped onto $X$ by a one-to-one continuous function. Hence $F(X)$ is homeomorphic to $X$. But $F(X)$ is zero-dimensional, since the natural quotient map $\mathscr{H}(X) \to F(X)$ is perfect and open.

Observe that $\mathscr{H}(X)$ is $\omega$-narrow. Hence by a result of Uspenskiĭ [**34**], [**37**] (see also [**5**, Section 10.3]), $F(X)$ and hence $X$ is a Dugundji compactum. Suppose first that the weight of $X$ is countable. Then $X$ is by what we just proved a zero-dimensional homogeneous infinite compact space, hence $X$ is homeomorphic to the Cantor set $C$. But it is well-known that $\mathscr{H}(C)$ is homeomorphic to the space of irrational numbers and hence is not compact. Suppose next that the weight of $X$ is uncountable. Since $X$ is homogeneous with respect to pseudo-character, $X$ is homeomorphic to the Cantor space $D^\tau$ for some uncountable cardinal number $\tau$ by Ščepin's Theorem from [**30**] (see also [**10**, Section 8.1]). But $\mathscr{H}(D^\tau)$ clearly contains a closed subgroup topologically isomorphic to $\mathscr{H}(D^\omega)$. Hence we again conclude that $\mathscr{H}(X)$ is not compact. □

We now come to two new concepts in homogeneity that turned out to be very useful in our study of unique homogeneity, see Section 5 for details. Here we study these concepts for their own sakes.

A space X is 2-*flexible* if, for all $a, b \in X$ and open neighborhood $O(b)$ of $b$, there is an open neighborhood $O(a)$ of $a$ such that, for any $z \in O(a)$, there is a homeomorphism $h$ of $X$ satisfying the following conditions: $h(a) = z$ and $h(b) \in O(b)$.

A space $X$ will be called *skew-2-flexible* if, for any $a, b$ in $X$ and any open neighborhood $O(b)$ of $b$, there is an open neighborhood $O(a)$ of $a$ such that, for every $z \in O(a)$, there is a homeomorphism $g$ of $X$ satisfying the following conditions: $g(a) = z$ and $b \in g(O(b))$.

It is clear that every homogeneous zero-dimensional space is both 2-flexible and skew-2-flexible.

PROPOSITION 3.3. *If $X$ is a space on which some semitopological group $G$ acts micro-transitively, then $X$ is 2-flexible. If $G$ is a quasitopological group, then $X$ is skew-2-flexible as well.*

PROOF. Let $a, b \in X$ be given, and let $O(b)$ be an arbitrary open neighborhood of $b$. Let $U$ be an open neighborhood of the neutral element $e$ of $G$ such that $U \subseteq \gamma_b^{-1}(O(b))$. If $G$ is a quasitopological group, then we may additionally assume that $U$ is symmetric. Now put $O(a) = \gamma_a(U)$. We claim that $O(a)$ witnesses that $X$ is 2-flexible, and skew-2-flexible in case $U$ is symmetric. To this end, pick an arbitrary $z \in O(a)$. There is an element $h \in U$ such that $ha = z$. Since $hb \in O(b)$ and $h^{-1}b \in O(b)$ in case $U$ is symmetric, we are done. $\square$

COROLLARY 3.4.

(1) *Every locally compact homogeneous separable metric space is both 2-flexible and skew-2-flexible.*
(2) *Every semitopological group is homogeneous and 2-flexible,*
(3) *Every quasitopological group is homogeneous and both 2-flexible and skew-2-flexible,*
(4) *Every coset space of a topological group is homogeneous and both 2-flexible and skew-2-flexible.*

PROOF. For (1), it suffices to observe that $X$ admits a transitive and continuous action by a Polish group, and that this action is micro-transitive by the Effros Theorem [**14**].

For (2), we let the semitopological group $G$ act on itself in the standard way. Then this action is separately continuous and micro-transitive. Hence we are done by the previous result. And (3) has an identical proof.

For (4), let $G$ be a topological group with closed subgroup $H$. We let $G$ act on $G/H$ in the natural way by

$$G \times G/H \to G/H : (g, xH) \mapsto gxH.$$

A moments reflection shows that this action is transitive and micro-transitive.

Hence in all cases we are done by Proposition 3.3. $\square$

Example 3.5. There is a homogeneous Polish space which is skew-2-flexible but not 2-flexible.

Proof. Consider the product $X = \triangle \times \boldsymbol{R}$, where $\triangle$ is the standard Cantor set. In van Mill [**27**] it was shown that there is a dense subspace $Y$ of $X$ having among other things the following properties:

(1) $Y$ is a homogeneous $G_\delta$-subset of $X$,
(2) the component $C_y$ of every $y = (a, b) \in Y$, has the form $\{a\} \times (u_y, v_y)$, where $-\infty \le u_y < b < v_y \le +\infty$,
(3) the set $A = \{y = (a, b) : C_y = \{a\} \times \boldsymbol{R}\}$ is dense in $Y$,
(4) for every $n \ge 1$, the set $B_n = \{y = (a, b) : v_y - u_y \le 1/n\}$ is dense in $Y$.

We first claim that $Y$ is not 2-flexible. To this end, pick an arbitrary point $y_0 = (a, b_0) \in A$, and let $y_1 = (a, b_1)$, where $b_1 = b_0 + 2$. Let $O(y_1)$ be the strip $\triangle \times (b_1 - 1, b_1 + 1)$. Let $O(y_0)$ be an arbitrary neighborhood of $y_0$. By (4), there is a point $z = (a', b') \in Y$ such that $C_z \subseteq O(y_0) \setminus (\triangle \times [b_1 - 1, b_1 + 1])$. Let $f$ be a homeomorphism of $Y$ such that $f(y_0) = z$. Then $f(C_{y_0}) = C_z$, hence

$$f(\{a\} \times \boldsymbol{R}) \cap \big(\triangle \times (b_1 - 1, b_1 + 1)\big) = \emptyset.$$

We conclude that $O(y_0)$ is not as required in the definition of 2-flexibility for the points $y_0$ and $y_1$.

We next will prove that $Y$ is skew-2-flexible. To this end, pick arbitrary points $y_0 = (a_0, b_0)$ and $y_1 = (a_1, b_1)$ in $Y$. Suppose first that $a_0 \neq a_1$. Let $O(y_1)$ be an arbitrary open neighborhood of $y_1$. Pick a clopen subset $C$ of $\triangle$ such that $a_0 \in C$ but $a_1 \notin C$. By [**27**, Section 3] it follows that $O(y_0) = (C \times \boldsymbol{R}) \cap Y$ is a homogeneous clopen subset of $Y$. Hence for every $z \in O(y_0)$ there is a homeomorphism $f$ of $Y$ such that $f(y_0) = z$ and $f(y_1) = y_1$. So we may assume without loss of generality that for some $a \in \triangle$, $y_0 = (a, b_0)$ and $y_1 = (a, b_1)$. First assume that $b_0 < b_1$. Let $O(y_1)$ be an arbitrary open neighborhood of $y_1$. We assume that $O(y_1)$ is a strip of the form $(\triangle \times (v_1, w_1)) \cap Y$ which misses the strip $O(y_0) = (\triangle \times (v_0, w_0)) \cap Y$. We claim that $O(y_0)$ is as required. Assume first that $z \in O(y_0)$ has the form $z = (a, t)$. Then by [**27**, Section 3] there is a homeomorphism $f$ of $Y$ such that $f(y_0) = z$ and $f(y_1) = y_1$. Assume next that $z$ has the form $(b, t)$, where $a \neq b$. Pick three disjoint nonempty clopen subsets $E, F$ and $G$ of $\triangle$ such that $E \cup F \cup G = \triangle$, $a \in F$ and $b \in G$. Pick a point $e \in E$ such that $(u_e, v_e) \subseteq (v_1, w_1)$. By [**27**, Section 3] there are homeomorphisms $\alpha\colon (E \times \boldsymbol{R}) \cap Y \to (F \times \boldsymbol{R}) \cap Y$, $\beta\colon (F \times \boldsymbol{R}) \cap Y \to (G \times \boldsymbol{R}) \cap Y$ and $\gamma\colon (G \times \boldsymbol{R}) \cap Y \to (E \times \boldsymbol{R}) \cap Y$ such that $\alpha(\{e\} \times (u_e, v_e)) = \{a\} \times (u_{y_0}, v_{y_0})$

and $\beta(y_0) = z$. It is clear that the homeomorphism $f = \alpha \cup \beta \cup \gamma \colon Y \to Y$ is as required. $\square$

It is a natural question whether there is a homogeneous space which is 2-flexible but not skew-2-flexible. Such a space indeed exists, even a uniquely homogeneous one, for details see Arhangel′skii and van Mill [**4**].

Question 3.6. *Is there a homogeneous Polish space which is* 2*-flexible but not skew-*2*-flexible*?

Question 3.7. *Is there a homogeneous compact space which is* 2*-flexible but not skew-*2*-flexible*?

For later use, we formulate and prove the following result.

Proposition 3.8. *If $X$ is a* 2*-flexible space, then the next condition is satisfied*:

(sc) *For any $e, y \in X$, any homeomorphism $f \in \mathscr{H}(X)$, and any open neighbourhood $V$ of $f(y)$, there is an open neighbourhood $U$ of $f(e)$ such that, for every $z \in U$, there exists an element $g \in \mathscr{H}(X)$ satisfying the following conditions*: *$g(e) = z$ and $g(y) \in V$.*

Proof. We put $a = f(e)$, $b = f(y)$, $O(b) = V$ and, using the 2-flexibility of $X$, we find an open neighborhood $O(a)$ of $a$ and a homeomorphism $h$ such that $h(f(e)) = f(z)$ and $h(b) \in V$. Then, obviously, $U = O(a)$ and $g = h \circ f$ is what we need. $\square$

Theorem 3.9. *If $X$ is a homogeneous space such that the group $\mathscr{H}(X)$ contains an Abelian subgroup acting transitively on $X$, then $X$ is* 2*-flexible.*

Proof. Let $G$ be an Abelian subgroup of $\mathscr{H}(X)$ acting transitively on $X$. Take arbitrary $a, b \in X$, and let $g \in G$ be such that $g(a) = b$. For an arbitrary open neighborhood $O(b)$ of $b$, put $O(a) = g^{-1}(O(b))$. For $z \in O(a)$, let $\xi \in G$ be such that $\xi(a) = z$. Then

$$\xi(b) = \xi(g(a)) = g(\xi(a)) \in g(O(a)) = O(b),$$

as required. $\square$

A space $X$ will be called *Abelian* if the elements of $\mathscr{H}(X)$ commute pairwise. We will conclude from Theorem 3.9 in Theorem 5.4 that if $X$ is uniquely homogeneous and $\mathscr{H}(X)$ is Abelian, then X is a semitopological group.

Question 3.10. *Which homogeneous spaces* (*compacta*) *are* 2*-flexible* (*skew-*2*-flexible*)?

Question 3.11. *Is every homogeneous Eberlein* (*Corson*) *compactum* 2*-flexible* (*skew-*2*-flexible*)?

Question 3.12. *Is every homogeneous Dugundji compactum* 2*-flexible* (*skew-*2*-flexible*)?

Question 3.13. *Given a space $X$, is it possible to find a space $Y$ such that $X \times Y$ is homogeneous and* 2*-flexible*?

This is true if $X$ is infinite and zero-dimensional. For let $X$ be any zero-dimensional infinite space. By Uspenskiĭ [**36**], the subspace $Y = \{f \in X^X : (\forall\, x \in X)(|f^{-1}(x)| = |X|)\}$ is homogeneous and satisfies $X \times Y \approx X$. But clearly $Y$ is zero-dimensional, being a subspace of the zero-dimensional space $X^X$. Hence $Y$ is 2-flexible, being homogeneous.

Question 3.14. *Given a space $X$, is it possible to find a space $Y$ such that $X \times Y$ is homogeneous and skew-*2*-flexible*?

Again, this is true if $X$ is infinite and zero-dimensional (see the argument above).

In conclusion of this section, we provide a simple condition in terms of metrics which guarantees that a space with such a metric is 2-flexible and skew-2-flexible.

Suppose that $X$ is a space the topology of which is generated by a metric $\rho$ such that for every $a \in X$ and every positive number $\varepsilon$ there is $\delta > 0$ such that, whenever $z \in X$ and $\rho(a, z) < \delta$, one can find a homeomorphism $h$ of $X$ onto itself satisfying the following conditions: $h(a) = z$ and $\rho(x, h(x)) < \varepsilon$, for every $x \in X$. Then we will say that $X$ is *$\rho$-flexible.* It is easy to verify that every $\rho$-flexible space is 2-flexible and skew-2-flexible. If a space $X$ can be metrized by a metric $\rho$ such that $X$ is $\rho$-flexible, then we call $X$ *flexible.* Observe that not every homogeneous Polish space is flexible. But every homogeneous metrizable compact space is flexible, as was shown by Ungar [**33**].

## 4. Unique homogeneity: part one.

In this section we shift our attention to unique homogeneity. We make some general remarks that will be useful in the sections to come.

If $X$ is uniquely homogeneous, for all $x, y \in X$ we let $f^x_y$ denote the unique homeomorphism of $X$ that sends $x$ to $y$.

The following result is well-known, it is included for completeness sake.

Proposition 4.1.

(1) *A homogeneous space $X$ is uniquely homogeneous if and only if the identity function on $X$ is the unique homeomorphism of $X$ with a fixed-point.*
(2) *If $X$ is uniquely homogeneous, and $e \in X$, then the binary operation $X \times X \to X$ defined by $x{\cdot}y = f^e_x(y)$ is a group operation on $X$ having the property that all left translations of $X$ are homeomorphisms of $X$. That is, $X$ is a left topological group.*

Proof. All statements have routine proofs. For the fact that left translations are homeomorphisms, simply observe that for $a \in X$ the translation $x \mapsto ax$ is nothing but the homeomorphism $f^e_a$. □

We call the group structure in Proposition 4.1 (2) the *standard group structure* on $X$. Every element $x$ in $X$ gives us a group structure on $X$. But all these group structures are topologically isomorphic, it does not matter which neutral element we fix.

Now it is natural to investigate when a uniquely homogeneous space is a (semi)topological group. As we stated in Section 3, this is true for uniquely homogeneous locally compact spaces of countable weight by the result of Barid and Renaud [**6**]. It is not true that every uniquely homogeneous space is a topological group (van Mill [**24**]).

Let $X$ be a uniquely homogeneous space that admits the structure of a left topological group. Since the translation $x \mapsto ax$ is a homeomorphism sending the neutral element $e$ of $X$ onto $a$, this must be the homeomorphism $f^e_a$. The conclusion is that the left topological group structure on $X$ is the standard group structure. Hence the standard group structure on $X$ is unique.

Now assume that the standard group structure on $X$ is semitopological. We already know that $X$ has no homeomorphisms other than translations. This implies that all inner isomorphisms of $X$, i.e., topological isomorphisms of the form $x \mapsto a^{-1}xa$ for some $a \in X$, are trivial (since such a homeomorphism fixes the neutral element). This means that $X$ is Abelian. It does not need to be *Boolean*, that is, every element has order at most 2, see Arhangel′skii and van Mill [**4**]. Call a space $X$ *Boolean* is all of its homeomorphisms are involutions. These remarks motivate the following questions.

Question 4.2.

(1) *Is there a uniquely homogeneous space which is not Abelian*?
(2) *Does every uniquely homogeneous space admit the structure of a semitopological group*?

In Theorem 3.2 we showed that for a compact homogeneous and infinite space $X$, $\mathscr{H}(X)$ cannot both be compact and zero-dimensional. For uniquely homogeneous compacta we do better.

Theorem 4.3. *Let $X$ be an infinite uniquely homogeneous compact space. Then*

(1) *$\mathscr{H}(X)$ is not compact,*
(2) *if $X$ moreover has countable tightness, then $\mathscr{H}(X)$ is not $\omega$-narrow.*

Proof. For (1), assume that $\mathscr{H}(X)$ is compact. Fix an element $e \in X$, and consider the function $\gamma_e \colon \mathscr{H}(X) \to X$. It is one-to-one and surjective by unique homogeneity, hence a homeomorphism by compactness. As a consequence, $X$ is a compact topological group. By a result of Walter Rudin [**29**], every infinite compact group admits a homeomorphism that does not respect Haar measure. Such a homeomorphism is not a translation and hence demonstrates that $X$ is not uniquely homogeneous.

For (2), observe that from the proof of Theorem 3.2 it follows that $X$ is a Dugundji compactum. Since it has countable tightness, it is metrizable. But then $X$ is not uniquely homogeneous by the result of Barit and Renaud [**6**]. □

Question 4.4. *If $G$ is a locally compact topological group, does $G$ admit a homeomorphism that does not respect Haar measure*?

By a result of van Douwen [**13**], there is a compact zero-dimensional homogeneous space $X$ which has a Borel measure $\mu$ such that for arbitrary clopen subsets $E$ and $F$ of $X$ we have that $E$ and $F$ are homeomorphic if and only if $\mu(E) = \mu(F)$. Moreover, up to a multiplicative constant, $\mu$ is the only Borel measure on $X$ which is invariant under all homeomorphisms of $X$. Hence the result of Walter Rudin [**29**] is not true for homogeneous compacta.

It would be interesting to know the answers to the following questions.

Question 4.5.

(1) *Is every uniquely homogeneous Dugundji compactum trivial*?
(2) *Is every uniquely homogeneous Eberlein compactum trivial*?
(3) *Can the product of two non-trivial compacta be uniquely homogeneous*?

Observe that a homogeneous Eberlein compactum is first countable. We will show in Section 6 below that the product of two infinite spaces can be uniquely homogeneous.

Question 4.6. *Is there a nontrivial Polish uniquely homogeneous space*?

## 5. Unique homogeneity: part two.

We continue our study of uniquely homogeneous spaces by proving that they are connected and by characterizing when they have the structure of a semitopological group. Moreover, we prove that no nontrivial uniquely homogeneous space is a subspace of an ordered space.

THEOREM 5.1. *Every uniquely homogeneous space $X$ containing more than two points is connected.*

PROOF. Let $\mathscr{G}$ be the family of all open and closed subsets of $X$, and for $x \in X$ let $Q_x$ be the quasicomponent of $x$ in $X$, that is, $Q_x = \bigcap\{U \in \mathscr{G} : x \in U\}$. Assume that $X$ is not connected. Then we can fix $a, b \in X$ such that $b \notin Q_a$. Clearly, $Q_a$ and $Q_b$ are disjoint, and there is an open and closed set $U$ in $X$ such that $a \in U$ and $b \notin U$. Put $V = X \setminus U$. Then $b \in V$. Fix a homeomorphism $h$ of $X$ onto itself such that $h(a) = b$, and let $W = U \cap h^{-1}(V)$. Obviously, $W$ and $h(W)$ are disjoint open and closed sets. We also have $Q_a \subseteq W$, $Q_b \subseteq h(W)$, and $h(Q_a) = Q_b$. In particular, $Q_a$ and $Q_b$ are homeomorphic.

Case 1: $W = Q_a$. Then $Q_a$ is open. Therefore, it follows that all quasicomponents are open. Since $X$ is homogeneous, all quasicomponents are homogeneous. Taking into account that $Q_a$ and $Q_b$ are open and closed, disjoint, and of the same cardinality, we conclude that $Q_a$ and $Q_b$ are singletons. Hence, all quasicomponents are singletons. Therefore, $X$ is discrete. Since $X$ contains more than two points, it is not uniquely homogeneous, a contradiction.

Case 2: $W \neq Q_a$. Then we can fix $c \in W \setminus Q_a$. Clearly $h(c) \neq c$, since $W$ and $h(W)$ are disjoint. There exists an open and closed set $W_1$ such that $a \in W_1$ and $c$ is not in $W_1$. Define a map $g$ of $X$ onto itself as follows: $g(x) = h(x)$ for all $x \in W_1$, $g(x) = h^{-1}(x)$, for all $x \in h(W_1)$, and $g(x) = x$ for $x$ outside of $W_1 \cup h(W_1)$. Then $g$ is a homeomorphism, $g(a) = b = h(a)$ but $g \neq h$, since $g(c) = c \neq h(c)$. Thus, $X$ is not uniquely homogeneous. $\square$

A space $X$ will be called *$d$-uniquely homogeneous* if, for all non-empty open subsets $U$ and $V$ of $X$ there exist $x \in U$ and $y \in V$ such that $X$ is *uniquely homogeneous at the pair* $(x, y)$, that is, there is exactly one homeomorphism $h$ of $X$ onto itself taking $x$ onto $y$.

Observe that the argument in the proof of Theorem 5.1 actually proves the following more general statement: Every $d$-uniquely homogeneous space $X$ containing more than two points is connected.

Suppose that $Y$ is a subspace of a space $X$. Let us say that $Y$ is *uniquely homogeneous in* $X$ if, for any $y, z \in Y$, there is exactly one homeomorphism $h$ of

$X$ onto itself such that $h(y) = z$ and $h(Y) = Y$.

The following result can be proved by practically the same method as in the proof of Theorem 5.1.

THEOREM 5.2. *Suppose that $X$ is a space containing more than two points, and $Y$ is a dense subspace of $X$ such that $Y$ is uniquely homogeneous in $X$. Then $X$ is connected.*

PROPOSITION 5.3. *Suppose that $Y$ is an infinite uniquely homogeneous first-countable space, and let $\beta Y$ be the Čech-Stone compactification of $Y$. Then*:

(1) *$Y$ is uniquely homogeneous in $\beta Y$*;
(2) *$\beta Y$ is d-uniquely homogeneous*; *and*
(3) *$\beta Y$ is not homogeneous.*

PROOF. Indeed, all of the above follows from the fact that $\beta Y$ is first-countable precisely at the points of $Y$, combined with the fact that first-countability at a point is preserved by homeomorphisms of a space onto itself. $\square$

THEOREM 5.4. *Let $X$ be a uniquely homogeneous space. Then the following statements are equivalent*:

(1) *$X$ is 2-flexible,*
(2) *the standard group structure on $X$ is semitopological,*
(3) *$X$ is homeomorphic to a semitopological group,*
(4) *$X$ is Abelian,*
(5) *the standard group structure on $X$ is semitopological and Abelian,*
(6) *$X$ is homeomorphic to an Abelian semitopological group.*

PROOF. The implication (2) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (2) follows from the fact that a semitopological group structure on $X$, if it exists, must be unique, see Section 4. Since (3) $\Rightarrow$ (1) was proved in Corollary 3.4, to see that (1), (2) and (3) are equivalent statements, all we need to prove is (1) $\Rightarrow$ (2). To this end, fix a neutral element $e \in X$. We will show that the multiplication in $X$ is right-continuous as well.

To this end, fix $x, y \in X$, and let $V$ be an arbitrary open neighborhood of $xy$ in $X$. Let $T_x$ denote the left translation $y \mapsto xy$ of $X$. By Proposition 3.8, $T_x$ satisfies condition (sc). So let $U$ be an open neighborhood of $x$ such as guaranteed by this condition for the points $e$ and $y$. Thus, for every $z \in U$, there exists $g \in \mathscr{H}(X)$ satisfying the following conditions: $g(e) = z$ and $g(y) \in V$. However, for the left translation $T_z$ we also have $T_z(e) = z$. Since $X$ is uniquely homogeneous, it follows that $g = T_z$. Hence, $T_z(y) = g(y) \in V$, that is, $zy \in V$. Since V is an arbitrary open neighbourhood of $xy$, $U$ is an open neighbourhood of $x$, and $z$ is an arbitrary element of $U$, this means that the multiplication in $X$ is right-continuous. the claim is established. Hence, $X$ is a semitopological group.

Observe that (4) $\Rightarrow$ (1) is a consequence of Theorem 3.9. We already know from the remarks in Section 4 that if the standard group structure on a uniquely homogeneous space is semitopological, then it is Abelian. This implies that $X$ is Abelian since all homeomorphisms of $X$ are left translations. So (2) $\Rightarrow$ (4) as well as the rest of the implications are trivial. $\square$

This result allows us to draw some interesting conclusions.

COROLLARY 5.5.

(1) *If a uniquely homogeneous contains a dense Čech-complete subspace and $X$ is 2-flexible, then $X$ is a Čech-complete topological group.*
(2) *A uniquely homogeneous 2-flexible compact space is trivial.*
(3) *A uniquely homogeneous compact coset-space of a topological group is trivial.*

PROOF. For (1), observe that $X$ admits a structure of a semitopological group by Theorem 5.4. But every semitopological group containing a dense Čech-complete subspace is a Čech-complete topological group by Arhangel′skii and Choban [**3**, Section 5].

For (2), we apply (1) to conclude that the space under consideration is a compact topological group. But no nontrivial compact topological group is uniquely homogeneous by Theorem 4.3.

For (3), first observe that coset-space of a topological group is 2-flexible by Corollary 3.4 (3). Hence we are in a position to apply (2). $\square$

Observe that Theorem 5.4 is a partial answer to some of the open problems in Question 4.2. We now approach these problems from a different angle, and again find partial answers.

Suppose that $X$ is a uniquely homogeneous space. Fix a point $e$ in $X$. Then the *standard inverse operation* $i$ on $(X, e)$ is defined as follows. For an arbitrary $x \in X$, we put $i(x) = (f_x^e)^{-1}(e)$.

We claim that $i \circ i$ is the identity mapping on X.

Indeed, take any $x \in X$, and put $y = i(x)$. Then $y = (f_x^e)^{-1}(e)$. Therefore, $f_y^e = (f_x^e)^{-1}$, since $X$ is uniquely homogeneous. Hence, $i(y) = (f_y^e)^{-1}(e) = f_x^e(e) = x$, that is, $(i \circ i)(x) = x$, for every $x \in X$, as claimed.

Since $i$ is a mapping of $X$ to $X$, it follows that $i$ is one-to-one and onto, that is, $i$ is a bijection of $X$.

THEOREM 5.6. *Let $X$ be a uniquely homogeneous space. Then following statements are equivalent.*

(1) *$X$ is skew-2-flexible,*
(2) *$X$ is 2-flexible and skew-2-flexible,*
(3) *the standard group structure on $X$ is quasitopological,*
(4) *$X$ is homeomorphic to a quasitopological group,*
(5) *$X$ is Boolean,*
(6) *the standard group structure on $X$ is quasitopological and Boolean,*
(7) *$X$ is homeomorphic to a Boolean quasitopological group.*

Proof. To begin with, we will prove (1) $\Rightarrow$ (6). To this end, consider the standard inverse operation $i$ on $(X, e)$.

We claim that $i$ is continuous. Take any $a \in X$, and put $b = i(a) = (f_a^e)^{-1}(e)$. Let $O(b)$ be an arbitrary open neighborhood of $b$. Since $(f_a^e)^{-1}$ is continuous, there is an open neighborhood $O(e)$ of $e$ such that $(f_a^e)^{-1}(O(e)) \subseteq O(b)$. Since $X$ is skew-2-flexible, there is an open neighborhood $O(a)$ of $a$ such that, for every $z \in O(a)$, there is a homeomorphism $g$ of $X$ satisfying the following conditions: $g(a) = z$ and $e \in g(O(e))$. Consider $i(z) = (f_z^e)^{-1}(e)$. We have: $g(f_a^e(e)) = z = f_z^e(e)$. Since $X$ is uniquely homogeneous, it follows that $f_z^e = g \circ f_a^e$. Hence, $(f_z^e)^{-1}(e) = (g \circ f_a^e)^{-1}(e) = (f_a^e)^{-1}(g^{-1}(e)) \in (f_a^e)^{-1}(O(e)) \subseteq O(b)$. Thus, $i(z) = (f_z^e)^{-1}(e) \in O(b)$ for every $z \in O(a)$, that is, the mapping $i$ is continuous.

Recall that $i \circ i$ is the identity mapping on $X$, that is, $i^{-1} = i$. It follows that $i^{-1}$ is continuous. Therefore, $i$ is a homeomorphism of $X$.

Since clearly $i(e) = e$, we get by unique homogeneity that $i$ is the identity function on $X$. In other words, $f_x^e = f_e^x$ for every $x \in X$. Hence, $xx = f_x^e(x) = f_e^x(x) = e$ for every $x \in X$, that is, the standard group structure on $X$ is Boolean, and hence, Abelian. But this means that every left translation is a right translation, i.e., $X$ is a semitopological group. But even more is true since inversion is continuous: $X$ is a Boolean quasitopological group.

The other implications follow from Corollary 3.4 and Theorem 5.4. $\square$

Remark 5.7. It is surprising that in the presence of unique homogeneity, skew-2-flexibility implies 2-flexibility, as was shown in Theorem 5.6. The converse is not true, as was shown in Arhangel′skii and van Mill [**4**]. In Example 3.5 we presented an example of a Polish space that is homogeneous, skew-2-flexible but not 2-flexible. Hence the assumption on unique homogeneity in Theorem 5.6 is essential.

A source of homogeneous compact spaces that are not metrizable are various classes of compact ordered spaces. In his thesis, Maurice [**22**] constructed a family of $\omega_1$ infinite homogeneous nonmetrizable compact ordered spaces. This bound was improved by van Douwen [**13**] to $2^{\mathfrak{c}}$. A compact homogeneous ordered space is easily seen to be zero-dimensional, hence none of these examples is uniquely homogeneous by Theorem 5.1. But there are also many noncompact connected ordered spaces that are homogeneous. Connected ordered spaces that have an order reversing homeomorphism are not uniquely homogeneous since such a homeomorphism must have a fixed-point. But there are many homogeneous connected ordered spaces that are not reversible. The first (real) example of an order homogeneous non-reversible ordered continuum was constructed by Shelah [**31**]. See Hart and van Mill [**17**] for another construction. Such a space with its minimum and maximum removed is homogeneous and a good candidate for a uniquely homogeneous linearly ordered space. One can also identify the minimum and the maximum of an order-homogeneous ordered continuum, thus obtaining a homogeneous generalized 1-sphere. We will show that this approach leads nowhere, by establishing the following result.

THEOREM 5.8. *If $X$ is an infinite subspace of an ordered space, then $X$ is not uniquely homogeneous.*

PROOF. Suppose that $X$ is an infinite uniquely homogeneous subspace of an ordered space. Then $X$ is connected by Theorem 5.1. So $X$ is itself linearly orderable. Let $<$ be a linear order on $X$ generating its topology. It is clear that $X$ does neither has a first not a last element. For such points would violate homogeneity.

CLAIM 1. *If $f$ and $g$ are homeomorphisms of $X$ such that $f(e) < g(e)$ for some $e$ in $X$, then $f(x) < g(x)$, for every $x$ in $X$.*

Indeed, the sets $M = \{x \in X : f(x) < g(x)\}$ and $L = \{x \in X : g(x) < f(x)\}$ are open, since $g$ and $f$ are continuous. Clearly, $M$ and $L$ are disjoint, and $e \in M$. Since $X$ is uniquely homogeneous, it follows from $f(e) < g(e)$ that $f(x) \neq g(x)$ for every $x \in X$. Therefore, $X = M \cup L$. Since $X$ is connected and $M \neq \emptyset$, we get that $L = \emptyset$. Hence, $X = M = \{x \in X : f(x) < g(x)\}$, as required.

CLAIM 2. *If $g$ is a homeomorphism of $X$ such $e < g(e)$ for some $e$ in $X$, then $x < g(x)$, for every $x$ in $X$.*

Just apply Claim 1 with the identity function on $X$ in the role of $f$.

CLAIM 3. *Every homeomorphism of $X$ is strictly increasing.*

Let $f$ be an arbitrary homeomorphism of $X$ which is not the identity function on $X$. First assume that there is an $e \in X$ such that $e < f(e)$. Then $x < f(x)$ for every $x \in X$ by Claim 2. Assume that there exist $x < y$ in $X$ such that $f(y) < f(x)$. Observe that since $f$ is a homeomorphism, by connectivity we consequently get that $f([x,\to)) = (\leftarrow, f(x)]$. Since $X$ does not have a last element, we may pick $z > f(x)$. But then on the one hand $f(z) > z$ and on the other hand $f(z) < f(x) < z$, which is a contradiction. So we conclude that $f$ is indeed strictly increasing. Assume next that there is an $e \in X$ such that $f(e) < e$. Then for $e' = f(e)$ we have $e' < f^{-1}(e')$. Hence $f^{-1}$ is strictly increasing by what we just proved and so $f$ is strictly increasing as well.

Now fix $e \in X$, and consider the standard inverse operation $i$ on $(X, e)$ defined by $i(x) = (f^e_x)^{-1}(e)$ for every $x \in X$. Clearly, $i(e) = e$.

Claim 4. *The mapping $i$ is a bijection of $X$ reversing the order.*

Pick $a \in X$ such that $e \le a$, and consider $f^e_a$. Then $e \le f^e_a(e)$ and hence $x \le f^e_a(x)$, for every $x \in X$ by Claim 2. Hence $i(a) \le e$. It follows similarly that $i(b) \ge e$ for every $b \le e$.

Now take arbitrary $a, b \in X$ such that $a < b$. If $a \le e < b$ then $i(b) < i(a)$ by what we just proved. Assume first that $e < a < b$.

Case 1: $i(b) = i(a)$.

Then $f^e_a(i(b)) = f^e_a(i(a)) = e = f^e_b(i(b))$. So we conclude that $f^e_a = f^e_b$ by unique homogeneity. But his is impossible since $a = f^e_a(e)$, $b = f^e_b(e)$, and $a \ne b$.

Case 2: $i(a) < i(b)$.

Then $f^e_b(i(a)) < f^e_b(i(b)) = e$ since $f^e_b$ is strictly increasing (Claim 3). But $a = f^e_a(e) < f^e_b(e) = b$, hence $f^e_a(x) < f^e_b(x)$ for every $x \in X$ (Claim 1), from which it follows that $e = f^e_a(i(a)) < f^e_b(i(a))$, which is a contradiction.

So we conclude that $i(b) < i(a)$. If $a < b < e$ then it follows by an identical reasoning that $i(b) < i(a)$. This proves Claim 4.

So we conclude that $i$ is a homeomorphism of $X$. Since $i(e) = e$, and $X$ is uniquely homogeneous, $i$ must be the identity. But this is a contradiction since $i$ reverses the order and $X$ is infinite. $\square$

## 6. Many uniquely homogeneous spaces.

In Question 4.5 (3) we asked whether a product of nondegenerate compact spaces can be uniquely homogeneous. The aim of this section is among other things to show that compactness is essential in this problem.

In [**12**], van Douwen proved the following result:

Proposition 6.1. *Let $\Pi$ denote the product of an indexed family $\{X_\gamma : \gamma \in \Gamma\}$ of spaces. The following are equivalent*:

(1) *if $\gamma, \delta \in \Gamma$ are distinct, then every continuous map $X_\gamma \to X_\delta$ is constant*; *and*
(2) *every continuous map $f\colon \Pi \to \Pi$ has the form $\Pi_\gamma f_\gamma$, with each $f_\gamma$ a continuous*

*map $X_\gamma \to X_\gamma$.*

Corollary 6.2. *Let $\Pi$ denote the product of an indexed family $\{X_\gamma : \gamma \in \Gamma\}$ of uniquely homogeneous spaces such that*

(1) *if $\gamma, \delta \in \Gamma$ are distinct, then every continuous map $X_\gamma \to X_\delta$ is constant*; *and*
(2) *for $\gamma \in \Gamma$, every embedding $X_\gamma \to X_\gamma$ is surjective.*

*Then $\Pi$ is uniquely homogeneous.*

Proof. Let $f : \Pi \to \Pi$ be a homeomorphism with a fixed-point. By Proposition 6.1, we may write $f$ as $\Pi_\gamma f_\gamma$, where $f_\gamma : X_\gamma \to X_\gamma$ is continuous for every $\gamma$. Clearly, $f_\gamma$ is an embedding for every $\gamma$ since $f$ is a homeomorphism. By our assumptions, this means that for every $\gamma$, $f_\gamma$ is a homeomorphism of $X_\gamma$ with a fixed-point and hence is the identity function on $X_\gamma$ by unique homogeneity. Hence $f$ is the identity function on $\Pi$. $\square$

So our aim is now to construct a large family of uniquely homogeneous spaces that satisfy these conditions. In Theorem 6.3 below we construct a family of $2^{\mathfrak{c}}$ such spaces. So there is a uniquely homogeneous space that is a product of $2^{\mathfrak{c}}$ nontrivial spaces.

As usual $\lambda$ denotes Lebesgue measure on $\boldsymbol{R}$. Let $\mathscr{M}$ denote the collection of Lebesgue measurable subsets of $\boldsymbol{I}$ and let $\mathscr{N}$ be the ideal of null-sets. The quotient algebra $\mathscr{M}/\mathscr{N}$ will be denoted by $\mathscr{G}$. If $A \in \mathscr{M}$, then $[A]$ denotes the $\mathscr{N}$-equivalence class of $A$. Metrize $\mathscr{G}$ by $d([A],[B]) = \lambda(A \triangle B)$. As is well-known, $\mathscr{G}$ with the topology induced by $d$ is Polish, [**16**, Exercise 40.1]. It is easy to see that $d$ is convex, [**16**, Exercise 40.8], hence $\mathscr{G}$ is connected. In fact, $\mathscr{G}$ is homeomorphic to the separable Hilbert space $\ell^2$, as was shown by Bessaga and Pełczyński, [**8**, VI 7.2]. If we define an operation $+$ on $\mathscr{G}$ by $[A] + [B] = [A \triangle B]$, then $\mathscr{G}$ with this operation is a Boolean group.

Hence there is a topological group $G$ having the following properties:

( i ) $G$ is nontrivial and Polish,
( ii ) $G$ is Boolean,
(iii) $G$ is connected and locally connected.

The example of a uniquely homogeneous space constructed in van Mill [**23**] is a subgroup of $G$. The examples that we construct here will all be subgroups of $G$ as well.

Theorem 6.3. *There is a family $\mathscr{A}$ of $2^{\mathfrak{c}}$ Boolean groups having the following properties*:

(1) *every $A \in \mathscr{A}$ is nontrivial, has countable weight and is connected and locally*

*connected,*

(2) *if $A, B \in \mathscr{A}$ are distinct, then every continuous function $f\colon A \to B$ is constant,*
(3) *for every $A \in \mathscr{A}$, every continuous function $f\colon A \to A$ is either constant or a translation.*

That there is a single group having the property that all continuous self maps are either constant or a translation, was claimed without giving a proof at the end of van Mill [**25**]. The proof here combines ideas in de Groot [**15**] and van Mill [**23**], [**25**].

**6.1. Proof of Theorem 6.3.**

A subset $X$ of $G$ is a *bi-Bernstein set* in $G$, abbreviated *BB-set*, if $X$ as well as $G \setminus X$ intersects every Cantor set in $G$.

If $A \subseteq G$, then $\langle\!\langle A \rangle\!\rangle$ denotes the subgroup generated by $A$. We say that a subset $A$ of $G$ is *independent* if for all subsets $B$ of $A$ we have that

$$\langle\!\langle B \rangle\!\rangle \cap \langle\!\langle A \setminus B \rangle\!\rangle = \{0\}.$$

If $A \subseteq G$ and $f\colon A \to G$ is a function, then a subset $B$ of $A$ is said to be *$f$-independent* provided that the following conditions are satisfied:

(1) $f{\restriction}B$ is injective,
(2) $B \cap f(B) = \emptyset$,
(3) $B \cup f(B)$ is independent.

The construction depends on the following result.

LEMMA 6.4. *Let $A$ be a Polish subspace of $G$, and let $f\colon A \to G$ be continuous. If $A$ contains an uncountable $f$-independent set, then $A$ contains an $f$-independent Cantor set.*

PROOF. Identical to the proof of van Mill [**25**, Proposition 3.4]. $\square$

COROLLARY 6.5. *Every uncountable $G_\delta$-subset $A$ of $G$ contains an independent Cantor set.*

PROOF. Let $B$ be a maximal independent subset of $A$. If $B$ is countable, then $A \subseteq \langle\!\langle B \rangle\!\rangle$ which is countable. This contradicts $A$ being uncountable. Now apply Lemma 6.4. $\square$

Let $\mathscr{K}$ denote the collection of all homeomorphisms $h\colon K_1 \to K_2$ between disjoint Cantor sets in $G$ such that $K_1 \cup K_2$ is independent. Clearly, $|\mathscr{K}| = \mathfrak{c}$.

Indeed, that $|\mathscr{K}| \le \mathfrak{c}$ is clear. That $|\mathscr{K}| \ge \mathfrak{c}$ follows by observing on the one hand that $G$ contains an algebraically independent Cantor set by Corollary 6.7, and on the other hand by the fact that every Cantor set can be spit into a family of $\mathfrak{c}$ pairwise disjoint Cantor subsets.

List $\mathscr{K}$ as $\{g_\alpha : 1 \le \alpha < \mathfrak{c}, \alpha \text{ odd}\}$. Let $V_0 = W_0 = \emptyset$. By transfinite induction on $\alpha < \mathfrak{c}$, we will construct subsets $V_\alpha$ and $W_\alpha$ in $G$ having the following properties:

(1) $|V_\alpha| \le |\alpha|{\cdot}\omega$, $|W_\alpha| \le |\alpha|{\cdot}\omega$,
(2) if $\beta < \alpha$, then $V_\beta \subseteq V_\alpha$ and $W_\beta \subseteq W_\alpha$,
(3) $V_\alpha$ is independent, and $\langle\!\langle V_\alpha \rangle\!\rangle \cap \langle\!\langle W_\alpha \rangle\!\rangle = \{0\}$,
(4) if $\alpha$ is odd, then there exists $x \in \operatorname{dom}(g_\alpha) \cap (V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta)$ such that $g_\alpha(x) \in W_\alpha \setminus \{0\}$,
(5) if $\alpha$ is even and $\alpha > 0$, then $V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta \ne \emptyset$.

Suppose that for some $\alpha < \mathfrak{c}$ we constructed the sets $V_\beta$ and $W_\beta$ for all $\beta < \alpha$. Put $V' = \bigcup_{\beta<\alpha} V_\beta$ and $W' = \bigcup_{\beta<\alpha} W_\beta$, respectively. Observe that by (1) we have $|V| \le |\alpha|{\cdot}\omega$ and $|W| \le |\alpha|{\cdot}\omega$. Moreover, by (2) and (3), $V' = \bigcup_{\beta<\alpha} V_\beta$ is independent and $V \cap W = \{0\}$, where $V = \langle\!\langle V' \rangle\!\rangle$ and $W = \langle\!\langle W' \rangle\!\rangle$. If $\alpha$ is even, pick any $x \in G \setminus (V + W)$, and put $V_\alpha = V' \cup \{x\}$ and $W_\alpha = W$, respectively. So assume that $\alpha$ is odd, and let

$$H = \bigl\{x \in \operatorname{dom}(g_\alpha) : \langle\!\langle \{x\} \cup V \rangle\!\rangle \cap \langle\!\langle \{g_\alpha(x)\} \cup W \rangle\!\rangle \ne \{0\}\bigr\}.$$

We will show that $|H| < \mathfrak{c}$. Since $|V + W| < \mathfrak{c}$ we have $|g_\alpha^{-1}(V + W)| < \mathfrak{c}$. Pick any point $x \in S = \operatorname{dom}(g_\alpha) \setminus ((V + W) \cup g_\alpha^{-1}(V + W))$. Then $(x + V) \cap W = \emptyset$ and $(g_\alpha(x) + W) \cap V = \emptyset$. Since $V \cap W = \{0\}$, for $x \in H \cap S$ there consequently exist $v_x \in V$ and $w_x \in W$ such that

$$x + v_x = g_\alpha(x) + w_x.$$

If $|H \cap S| = \mathfrak{c}$, then there are distinct $x, y \in H \cap S$ and $v \in V$ and $w \in W$ such that

$$x + v = g_\alpha(x) + w, \quad y + v = g_\alpha(y) + w.$$

Hence $x + y + g_\alpha(x) + g_\alpha(y) = 0$. But this contradicts the fact that $\operatorname{dom}(g_\alpha) \cup \operatorname{range}(g_\alpha)$ is independent. Hence $|H \cap S| < \mathfrak{c}$ and so $|H| < \mathfrak{c}$. Now pick any $x \in \operatorname{dom}(g_\alpha) \setminus (H \cup V)$ such that $g_\alpha(x) \ne 0$, and define $V_\alpha = V' \cup \{x\}$ and $W_\alpha = W' \cup \{g_\alpha(x)\}$, respectively. Then these choices clearly satisfy our inductive hypotheses.

Let $\boldsymbol{E}$ and $\boldsymbol{O}$ denote the sets of even and odd ordinals less than $\mathfrak{c}$, respectively.

Let $\{Z_\kappa : \kappa < 2^{\mathfrak{c}}\}$ be a family of subsets of $\boldsymbol{E} \setminus \{0\}$ such that

(5) $|Z_\kappa| = \mathfrak{c}$ for all $\kappa < 2^{\mathfrak{c}}$,
(6) for distinct $\kappa, \mu < 2^{\mathfrak{c}}$, $|Z_\kappa \setminus Z_\mu| = \mathfrak{c}$.

The existence of this family follows from [**20**, Lemma 3 on p. 424]. For every $\alpha \in \boldsymbol{E}$, pick an element $x_\alpha \in V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$. Similarly, for every $\alpha \in \boldsymbol{O}$, pick an arbitrary element $x_\alpha \in V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$ such that $x_\alpha \in \operatorname{dom}(g_\alpha)$ and $g_\alpha(x_\alpha) \in W_\alpha \setminus \{0\}$. Observe that this is guaranteed by (4). Now for $\kappa < 2^{\mathfrak{c}}$, put

$$G_\kappa = \langle\!\langle \{x_\alpha : \alpha \in Z_\kappa\} \cup \{x_\alpha : \alpha \in \boldsymbol{O}\} \rangle\!\rangle.$$

Lemma 6.6. *For every $\kappa < 2^{\mathfrak{c}}$, $G_\kappa$ is a BB-set in $G$.*

Proof. Let $K$ be an arbitrary Cantor set in $G$. By Corollary 6.7 we may assume that $K$ is independent. Write $K$ as $K_0 \cup K_1$, where both $K_0$ and $K_1$ are Cantor sets and $K_0 \cap K_1 = \emptyset$. Now let $g\colon K_0 \to K_1$ be any homeomorphism. Then $g \in \mathscr{K}$ and hence there exists $\alpha \in \boldsymbol{O}$ such that $g = g_\alpha$. Then

$$x_\alpha \in G_\kappa \cap \operatorname{dom}(g_\alpha) = G_\kappa \cap K_0,$$

as required. □

Corollary 6.7. *For every $\kappa < 2^{\mathfrak{c}}$, $G_\kappa$ is a connected and locally connected dense subgroup of $G$.*

Proof. This follows from van Mill [**23**, 3.4]. □

Lemma 6.8. *If $\kappa, \mu < 2^{\mathfrak{c}}$ and $\kappa \neq \mu$, then $G_\kappa \setminus G_\mu \neq \emptyset$.*

Proof. First observe that the set $\{x_\alpha : 1 \le \alpha < \mathfrak{c}\}$ is independent. Since $\kappa \neq \mu$, we may pick by (6) an index $\gamma \in Z_\kappa \setminus Z_\mu$. But then

$$x_\gamma \notin \{x_\alpha : \alpha \in Z_\mu\} \cup \{x_\alpha : \alpha \in \boldsymbol{O}\},$$

i.e., $x_\gamma \in G_\kappa \notin G_\mu$ by independence. □

W now come to the crucial properties of our family of subgroups of $G$.

Proposition 6.9. *Let $\kappa, \mu < 2^{\mathfrak{c}}$. If $f\colon G_\kappa \to G_\mu$ is continuous, then $f$ is constant if $\kappa \neq \mu$, and either constant or a translation if $\kappa = \mu$.*

Proof. Let $f\colon G_\kappa \to G_\mu$ be an arbitrary continuous function. By Lavrentieff's Theorem from [**21**], there exists a $G_\delta$-subset $S$ of $G$ such that $f$ can be extended to a continuous function $\bar{f}\colon S \to G$.

Suppose first that $S$ contains an uncountable $\bar{f}$-independent set. Then $S$ contains by Lemma 6.4 an $\bar{f}$-independent Cantor set, say $K$. Hence $g = \bar{f}\restriction K$ belongs to $\mathscr{K}$. Hence there exists $\alpha \in \boldsymbol{O}$ such that $g = g_\alpha$. Then

$$x_\alpha \in G_\kappa \cap \mathrm{dom}(g_\alpha) = G_\kappa \cap K.$$

Since $\bar{f}$ extends $f$, we have $f(x_\alpha) = \bar{f}(x_\alpha) = g(x_\alpha) = g_\alpha(x_\alpha) \in G_\kappa$. But by construction, $g_\alpha(x_\alpha) \in W_\alpha \setminus \{0\}$. But this implies by (3) that $g_\alpha(x_\alpha) \notin G_\mu$, which is a contradiction.

So every $\bar{f}$-independent set is countable. Let $Q$ be a maximal $\bar{f}$-independent subset of $S$. So for an arbitrary $x \in G_\kappa \setminus Q$ we have that $Q \cup \{x\}$ is not $\bar{f}$-independent.

Put $P_0 = \langle\!\langle Q \cup \bar{f}(Q)\rangle\!\rangle$, and $P_{n+1} = \langle\!\langle P_n \cup \bar{f}(P_n \cap S)\rangle\!\rangle$ for $n \geq 1$. We claim that $\bar{f}(x) \in \langle\!\langle \{x\} \cup P\rangle\!\rangle = (\{x\} + P) \cup P$ for every $x \in S$, where $P = \bigcup_{n<\omega} P_n$. To get rid of some trivial cases first, observe that if $\bar{f}(x) \in P$, then there is nothing to prove. If $x \in P$, then $x \in P_n \cap S$ for some $n$, hence $\bar{f}(x) \in P_{n+1} \subseteq P$, and so in this case there is also nothing to prove. So assume that $x$ is an arbitrary element from $S \setminus P$ such that $\bar{f}(x) \notin P$. Then $x \notin Q$, hence $Q \cup \{x\}$ is not $\bar{f}$-independent. There are several cases to be considered.

Clearly $\bar{f}\restriction(Q \cup \{x\})$ is injective since $\bar{f}(x) \notin P \supseteq \bar{f}(Q)$.

Suppose that $(Q \cup \{x\}) \cap \bar{f}(Q \cup \{x\}) \neq \emptyset$. Since $Q \cap \bar{f}(Q) = \emptyset$, there are three cases to consider. If $x = \bar{f}(x)$, then we have nothing to check. The other cases are that $x \in \bar{f}(Q) \subseteq P$ or $\bar{f}(x) \in Q \subseteq P$, but this violates our choice of $x$.

So we are left with the case that $\bar{f}\restriction(Q \cup \{x\})$ is injective, $(Q \cup \{x\}) \cap \bar{f}(Q \cup \{x\}) = \emptyset$, but $(Q \cup \{x\}) \cup \bar{f}(Q \cup \{x\})$ is not independent. Hence $\{x, \bar{f}(x)\} \cup (Q \cup \bar{f}(Q))$ is not independent. Since $\{x, \bar{f}(x)\} \cap P = \emptyset$, this implies that for some $y \in Q \cup \bar{f}(Q) \subseteq P$ we have that $x + \bar{f}(x) + y = 0$. This completes the proof of our claim.

For every $p \in P$ we put $S_p = \{x \in S : f(x) = p\}$ and $T_p = \{x \in S : f(x) = x + p\}$. The countable collection of closed sets

$$\mathscr{B} = \{S_p : p \in P\} \cup \{T_p : p \in P\}$$

covers $S$. Fix two distinct elements $p$ and $q$ in $P$ for a moment. Then clearly, $S_p \cap S_q = \emptyset$, $S_p \cap \bigcup_{p' \in P} T_{p'} \subseteq P$, $T_p \cap T_q = \emptyset$, and $T_p \cap \bigcup_{p' \in P} S_p \subseteq P$. Now, $G \setminus S$ is a countable union of closed sets in $G$ which all have to be countable. For otherwise, one of them would contain a Cantor set which would intersect $G_\kappa$ and hence $S$ by Lemma 6.6. So we conclude that for the countable set $P' = (G \setminus S) \cup P$

we have that

$$\mathscr{B}' = \{S_p \setminus P' : p \in P\} \cup \{T_p \setminus P' : p \in P\}$$

is pairwise disjoint. But $G$ is homeomorphic to $\ell^2$, hence $G$ minus any countable set is path-connected (this can also be verified directly). Hence the Sierpiński Theorem from [**32**] stating that no continuum can be partitioned in at most countably many pairwise disjoint nonempty closed sets, gives us that there is a unique element of $\mathscr{B}'$ that is nonempty.

Suppose that for $p \in P$ we have that $S_p \setminus P'$ is the unique nonempty element in $\mathscr{B}'$. Observe that $G_\kappa \setminus S_p$ is countable. Define the continuous function $g\colon G_\kappa \to G$ by $g(x) = p + f(x)$. Then the range of $g$ is countable and hence a single point since $G_\kappa$ is connected (Corollary 6.7). Since 0 is in the range of $g$ since $G_\kappa$ is uncountable, we conclude that $G$ is the constant function with value 0, i.e., $f$ is the constant function with value $p$.

Suppose next that for $p \in P$ we have that $T_p \setminus P'$ is the unique nonempty element in $\mathscr{B}'$. Observe as above that $G_\kappa \setminus T_p$ is countable. Define the continuous function $h\colon G_\kappa \to G$ by $g(x) = x + f(x)$. The range of this function is countable, so it is constant by connectivity of $G_\kappa$. Since $T_p \setminus P'$ is nonempty, the function $g$ is therefore the constant function with value $p$. But this implies that $f$ is the translation $x \mapsto x + p$. So if $\kappa = \mu$ then we are done. If $\kappa \neq \mu$, pick an arbitrary $x \in G_\kappa \setminus G_\mu$ (Lemma 6.8). Then $f(0) = p$ and $f(x) = x + p$ both belong to $G_\mu$, hence $p + x + p = x$ belongs to $G_\mu$, a contradiction. □

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