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Polish Topology*

1. Introduction[1]

Krzysztof Ciesielski invited me to write something about *Polish Topology* for the special issue of the journal *Wiadomości Matematyczne* that will be published on the occasion of the Kraków European Congress of Mathematics in July 2012. I am extremely pleased to do so.

Thinking about what Polish Topology is, mathematical giants such as Kuratowski, Mazurkiewicz, Hurewicz, Sierpiński, Knaster, and Borsuk come to mind. I met Professor Kuratowski at the first conference I ever attended, the 4th International Conference on General Topology and its Relations to Modern Analysis and Algebra in Prague, 1976. I was still a PhD student then, and meeting the mathematicians whose papers I was studying and struggling with was fascinating. Moreover, I met Roman Pol there with whom I developed a lifelong cooperation. I lectured in Knaster's seminar in Wrocław and he asked me to deliver my lecture in French, but unfortunately I was not prepared for that. I regret to never have met Borsuk; when I visited Warsaw, he was ill.

Polish Topology is by definition the topology that the aforementioned mathematical giants have created. Their topological and geometric conjectures and themes stimulated research for more than a century. But not only their legacy determines the role of Polish Topology in the world. Their work was continued by very strong researchers that were influenced by their

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[1] Throughout, unless stated otherwise, all topological spaces under discussion are separable and metrizable.

papers in journals such as *Fundamenta Mathematicae* and the numerous strong books on topology that were written by Polish mathematicians, e.g., [10, 12, 13, 22–24, 33–35].

In this note I will concentrate on a few such themes that are of particular interest to me. So this is a highly personal account of my involvement with Polish Topology, and is therefore very far from complete. I will for example not say anything about shape theory, set theoretic topology, continuum theory, not much about function spaces, descriptive set theory and topological dynamics, and almost nothing about A(N)R-theory.

2. Infinite-dimensional topology

Let s and ℓ^2 denote the countable infinite product of real lines and separable Hilbert space, respectively. These spaces are not topologically isomorphic as linear spaces, because one of them is normed and the other one is not. Fréchet [25] in 1928 and Banach [7] in 1932, asked whether all infinite-dimensional Fréchet spaces are *topologically* homeomorphic. Banach [7] wrote that Stanisław Mazur had solved the problem, but this claim turned out to be incorrect.

By several ad hoc methods, homeomorphy of many linear spaces were established. In a series of papers, Kadec developed an interesting "renorming technique" for separable Banach spaces and finally proved in 1965 that all infinite-dimensional separable Banach spaces are homeomorphic (see Kadec [28]). His proof used several results of Bessaga and Pełczyński [8, 9] and showed that the homeomorphy of s and ℓ^2 would imply a positive answer to the problem of Fréchet and Banach, i.e., the homeomorphy of all separable infinite-dimensional Fréchet spaces. Results and techniques from functional analysis, and especially the geometry of Banach spaces, with various ingenious constructions from general topology were essential in the arguments leading to this conclusion.

In [5], Anderson proved that for any space X one can delete any σ-compact subspace from the space $X \times s$, in that the remaining subspace is homeomorphic to $X \times s$. A new field in topology was born: it was called *infinite-dimensional topology*.

Anderson was motivated by purely intrinsic topological questions. It turned out, quite unexpectedly, that his methods could be used to solve the above mentioned classical open problem. He solved it in the affirmative in [6] by using the results from his previous paper [5] by showing that s and ℓ^2

are indeed homeomorphic. A good/complete exposition of the problem of Fréchet and Banach can be found in Bessaga and Pełczyński [10].

In the early seventies, Chapman began the study of spaces modeled on the Hilbert cube Q, the so called *Hilbert cube manifolds* or *Q-manifolds*. Certain delicate finite-dimensional obstructions turned out not to appear in Q-manifold theory. In some vague sense, Q-manifold theory is a "stable" PL n-manifold theory. It was known from previous work that if P is a polyhedron then $P \times Q$ is a Q-manifold. Chapman [16] proved the converse, namely that all Q-manifolds can be triangulated, i.e., are of this form. This result turned out later to be of fundamental importance. Some truly spectacular results were the result of Chapman's efforts. In 1974 he proved the invariance of Whitehead torsion. This is the statement that any homeomorphism between compact polyhedra is a simple homotopy equivalence.

Borsuk [12] asked whether every compact ANR has the homotopy type of a compact polyhedron. This problem stayed a mystery for a long time. It was laid to rest by West [53] who showed that for every compact ANR X there are a compact Q-manifold M and a cell-like map from M onto X (a cell-like map between compact ANR's is a homotopy equivalence). As we have seen above, the Q-manifold M is homeomorphic to $P \times Q$ for some compact polyhedron P and so X has the same homotopy type as P.

In 1974, Edwards [16] improved West's result by showing that $X \times Q$ is a Q-manifold if and only if X is a locally compact ANR. In his proof, a crucial role was played by shrinkable maps.

In 1980, Toruńczyk [51] was able to characterize topologically the Q-manifolds among the locally compact ANR's. From Edwards's Theorem it was already known that if X is a locally compact ANR then $X \times Q$ is a Q-manifold. Toruńczyk studied the question when the projection $\pi\colon X \times Q \to X$ is shrinkable, and came to an astounding conclusion. This map is shrinkable if and only if X has the following property: given $n \in \mathbb{N}$ and two maps $f, g\colon \mathbb{I}^n \to X$ and $\varepsilon > 0$ there exist maps $\xi, \eta\colon \mathbb{I}^n \to X$ such that $\xi(\mathbb{I}^n) \cap \eta(\mathbb{I}^n) = \emptyset$ while moreover

$$d(f, \xi) < \varepsilon \quad \text{and} \quad d(\eta, g) < \varepsilon.$$

For obvious reasons this property is called the *disjoint cells-property.* So one arrives at the following conclusion, which is called Toruńczyk's Theorem:

Theorem 2.1. *A space is a manifold modeled on Q if and only if it is a locally compact* ANR *with the disjoint-cells property.*

In [52], Toruńczyk characterized the topology of Hilbert space manifolds in much the same way as he characterized the topology of the Hilbert cube. In this characterization the disjoint-cells property is replaced by the *discrete approximation property*; this property states that for every open cover $\mathcal{U}$ of the space X and every map f from the topological sum $\bigoplus_{n=1}^{\infty} \mathbb{I}^n$ to X there is another map g from $\bigoplus_{n=1}^{\infty} \mathbb{I}^n$ to X that is $\mathcal{U}$ close to f and is such that the family $\{g(\mathbb{I}^n) : n \in \mathbb{N}\}$ is discrete. The characterization reads:

Theorem 2.2. *A space is a manifold modeled on ℓ^2 if and only if it is a completely metrizable* ANR *with the discrete approximation property.*

This result implies that if X is a complete ANR as well as a topological group, then X is Lie group or an ℓ^2-manifold, [18].

Ingenious counterexamples, such as the ones by Taylor [50] and Dranišnikov [20] (see also [21]), nicely complement these results and indicate the boundaries of what is possible. For example, Dranišnikov's result was the main ingredient in Cauty's construction in [14] of a linear space that is not an ANR. By this example we know that even among the complete linear spaces, the discrete approximation property does not detect the ℓ^2-manifolds.

In recent years, it became clear that there are finite-dimensional spaces that behave very much like the infinite-dimensional spaces Q and ℓ^2. Bestvina [11] characterized the k-dimensional universal Menger compacta. Moreover, Ageev [1–3], Levin [37] and Nagórko [43] characterized the Nöbeling manifolds. These characterizations of finite-dimensional spaces are all in the same spirit as the ones that we discussed above.

Interesting applications of infinite-dimensional topology were obtained in the field of function spaces. For a space X, let $C_p(X)$ denote the set of all continuous real valued functions on X endowed with the topology of pointwise convergence. If X is discrete, then $C_p(X)$ is the product $\mathbb{R}^X$. What about nondiscrete X? It is not difficult to show that $C_p(X)$ is metrizable if and only if it is separable and metrizable if and only if X is countable. If X is a countable metrizable space, then $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X \equiv s$. The converse of this result is not true. Let σ denote $\{x \in \mathbb{R}^\infty : x_i = 0$ for all but finitely many $i\}$. In [19], Dobrowolski, Marciszewski and Mogilski proved:

Theorem 2.3. *Let X be a countable nondiscrete completely regular space such that $C_p(X)$ is an absolute $F_{\sigma\delta}$. Then $C_p(X)$ is homeomorphic to σ^∞.*

By [17], $C_p(X)$ is not an absolute $G_{\delta\sigma}$ if X is not discrete. Thus, Theorem 2.3 gives a complete topological classification of the spaces $C_p(X)$

which are absolute Borel sets of the class not higher than 2. Interestingly, the obvious conjecture that for a countable space X, the Borel class of $C_p(X)$ determines its topological type, turned out to be incorrect, as was shown by Cauty [15].

3. Dimension theory

A (finite or infinite) collection $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed sets of a space X is called *essential* if there exist closed sets $L_1, L_2, \ldots$ such that for every partition L_i between A_i and B_i we have that $\bigcap_i L_i \neq \emptyset$. If the collection is not essential, then it is called *inessential*.

The collection of opposite faces of the cubes $\mathbb{I}^n (n \leq \infty)$ are essential, as follows easily from the the Brouwer Fixed-Point Theorem. One may define a space X to be at most n-dimensional if every family consisting of $n+1$ pairs of disjoint closed sets is inessential.

If every sequence of pairs of disjoint closed sets in a space X is inessential, then we say that X is *weakly infinite-dimensional*. A space that is not weakly infinite-dimensional is called *strongly infinite-dimensional*.

Hence every finite-dimensional space is weakly infinite-dimensional, and Q is strongly infinite-dimensional. The topological sum $\bigoplus_{n=1}^{\infty} \mathbb{I}^n$ of n-cubes is weakly infinite-dimensional, but not finite-dimensional.

It is well-known that a space X is at most n-dimensional if it can be written as the union of a family consisting of $n+1$ 0-dimensional subspaces. A space X is *countable-dimensional* if it can be written as the union of countably many 0-dimensional subspaces. It is not difficult to show that a strongly infinite-dimensional space is not countable-dimensional, and that a countable-dimensional space is weakly infinite-dimensional.

In [27], Hurewicz and Wallman published the state of the art of classical dimension theory in 1941. In their book they mainly deal with finite-dimensional spaces. By now there is a well-developed theory for infinite-dimensional spaces with vital contributions by Polish mathematicians. For example, it was asked by Alexandroff [4] in 1951 whether every compact weakly infinite-dimensional space is countable-dimensional. This was answered in the negative by R. Pol [45] in 1981.

Theorem 3.1. *There is a compact weakly infinite-dimensional space that is not countable-dimensional.*

Interestingly, Pol's space is a C-space (the C-spaces which were introduced by Haver [26], form a natural and useful class of spaces situated between the countable-dimensional spaces and the weakly infinite-dimensional spaces). The question whether every weakly-infinite-dimensional compact space is a C-space is still open as far as I know.

In Pol's construction, a very fruitful idea is used that turned out to be the key in obtaining many more results in dimension theory. It goes back to a construction by Rubin, Schori and Walsh [48] which is related to a construction by Knaster [29] and its modification by Lelek [36]. Let C denote the standard Cantor middle-third set in $\mathbb{I}$. It can be shown that for every $n \leq \infty$ there are a compact space X_n, a continuous surjection $f_n\colon X_n \to C$, such that X_n is n-dimensional if $n < \infty$ and X_∞ is strongly infinite-dimensional, while moreover every subset $S \subseteq X_n$ with $f_n(S) = C$ has the same dimensional properties as X_n. Hence by taking a point from each fiber of the map $f_\infty\colon X_\infty \to C$, one obtains a strongly infinite-dimensional totally disconnected space. Krasinkiewicz [30] gave an interesting different construction of such spaces based on Bing partitions of the Hilbert cube. As I said, this idea is behind many results in dimension theory. For example, it was used in the construction of easy weakly n-dimensional spaces in [39] (an n-dimensional space is *weakly n-dimensional* if the set of its points at which it is n-dimensional has dimension at most $n-1$; the first weakly 1-dimensional space was constructed by Sierpiński [49] and subsequently Mazurkiewicz [38] provided subtle examples of weakly n-dimensional spaces for each $n > 1$), the construction of a complete C-space whose square is strongly infinite-dimensional in [40] (improving a result in [44]), in the construction of two subspaces A and B in $\mathbb{R}^4$ with $\dim(A \cup B) > \dim(A \times B) + 1$, [41], in the analysis of the dimensional structure of hereditarily indecomposable continua, [46], and in the construction of n-dimensional complete spaces whose countable infinite product remains n-dimensional, [31].

As is well-known, the three classical dimension functions take the same values in the class of separable metrizable spaces. Even in the class of all metrizable spaces, this need not hold, as the famous example of Roy [47] demonstrates. His space $\triangle$ is metrizable and 0-dimensional, but its covering dimension is 1. It is a natural question whether there are 0-dimensional such spaces with arbitrarily large covering dimension. This question sounds innocent, but has turned out to be a formidable one. The first significant progress on it was made by Mrówka [42] under the set theoretic hypothesis $\mathrm{S}(\aleph_0)$. Then, Kulesza [32] proved that under the assumption $\mathrm{S}(\aleph_0)$, the covering dimension of the n-th power of Mrówka's space $\nu\mu_0$ is always n. This gives a provisional answer to the above problem. The result is provisional in that $\mathrm{S}(\aleph_0)$ is a large cardinal assumption.

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