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GENERAL TOPOLOGY

On the Separation Dimension of K_ω

by

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Summary. We prove that $\operatorname{trt} K_\omega > \omega + 1$, where trt stands for the transfinite extension of Steinke's separation dimension. This answers a question of Chatyrko and Hattori.

1. Introduction. *All spaces under discussion are separable and metrizable.* In [5], Steinke introduced a (topologically invariant) dimension function t called the *separation dimension.* If X is a topological space, then $\mathrm{t}X = -1$ iff $X = \emptyset$; $\mathrm{t}X = 0$ if $|X| = 1$; whenever $|X| > 1$, $n \geq 0$, and for each subset M of X with $|M| > 1$ there exist different points $x, y \in M$ and a partition L_M in the subspace M of X between x and y such that $\mathrm{t}L_M \leq n-1$, then we say that $\mathrm{t}X \leq n$.

The separation dimension of a space is determined by the separation dimension of its components in the sense that if $\mathcal{C}$ is the collection of components of a nonempty space X, then $\mathrm{t}X = \sup\{\mathrm{t}C : C \in \mathcal{C}\}$. Moreover, a nonempty space X is hereditarily disconnected if and only if $\mathrm{t}X = 0$. Finally, $\mathrm{t}X \leq \dim X$ for every space X, and $\dim X \leq \mathrm{t}X$ if X is locally compact. Hence for every locally compact space X we have $\mathrm{t}X = \dim X$. These facts are all due to Steinke [5].

In [1], Arenas, Chatyrko and Puertas considered the natural transfinite extension trt of t, and showed that every compact space with $\operatorname{trt} X \neq \infty$ must be a C-space. They also showed that for the well-known space K_ω, the subspace of the Hilbert cube $Q = \prod_{n=1}^{\infty} [0,1]_n$ consisting of all points all but finitely many coordinates of which are 0, we have $\operatorname{trt} K_\omega > \omega$, and asked whether $\operatorname{trt} K_\omega = \infty$. In Chatyrko and Hattori [3, Problem 4.2] it was asked whether there is a countable-dimensional space X with $\operatorname{trt} X > \omega + 1$. The aim of the present note is to show that K_ω has this property.

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2. Preliminaries. Let A and B be disjoint closed subsets of a space X. We say that a closed subset E of X is a *partition* between A and B if we can write $X \setminus E$ as $U \cup V$, where U and V are disjoint open subsets of X such that $A \subseteq U$ and $B \subseteq V$. The partition E is said to be *irreducible* if no proper closed subset of E is a partition between A and B.

It is well-known that in a locally path-connected space every partition E between the disjoint closed sets A and B contains an irreducible partition. A proof of this can be found in Lemma 3.10.2 and Exercise 3.10.2 in [4] (it was proved there for locally connected Polish spaces, but an inspection of the proof shows that it is true with an identical proof for locally path-connected spaces).

The space K_ω is a well-known object in infinite-dimensional topology. In Bessaga and Pełczyński [2, Chapter V, §5] it was shown that K_ω is a so-called skeletoid for the finite-dimensional Z-sets in Q. This implies the following nontrivial fact:

(K) If X is a compact finite-dimensional space, and $A \subseteq X$ is closed, then any continuous map $f\colon X \to K_\omega$ that restricts to an embedding on A can be approximated arbitrarily closely by an embedding $g\colon X \to K_\omega$ such that $f{\restriction}A = g{\restriction}A$.

For proofs of these facts we refer to Bessaga and Pełczyński [2]. Observe that K_ω is an absolute retract, being a convex subset of Q.

3. The construction. Striving for a contradiction, assume that $\operatorname{trt} K_\omega = \omega + 1$. There consequently exist distinct points x_1 and y_1 in K_ω and a partition L_1 between x_1 and y_1 in K_ω such that $\operatorname{trt} L_1 \le \omega$. By the remarks in §2 we may assume that L_1 is an irreducible partition between x_1 and y_1. Write $K_\omega \setminus L_1$ as $U_1 \cup V_1$, where both U_1 and V_1 are open in K_ω, $x_1 \in U_1$ and $y_1 \in V_1$. It is clear that L_1 is not a singleton. Hence there exist distinct points x_2 and y_2 in L_1 and a partition L_2 between x_2 and y_2 in L_1 such that $n = \operatorname{trt} L_2 < \omega$. Write L_2 as $\bigcup_{i<\omega} X_i$, where each X_i is compact. Then $\operatorname{trt} X_i = \operatorname{t} X_i \le n$ for all i, hence $\dim X_i \le n$ by the result of Steinke that we quoted in the introduction. Hence we conclude that $\dim L_2 \le n$ by the Countable Closed Sum Theorem [4, 3.2.8].

Write $L_1 \setminus L_2$ as $U_2 \cup V_2$, where both U_2 and V_2 are open in L_1, $x_2 \in U_2$ and $y_2 \in V_2$. By irreducibility of L_1 we deduce that $V_2 \cup L_2$ is not a partition in K_ω between x_1 and y_1. Hence some component C of $K_\omega \setminus (V_2 \cup L_2)$ contains both x_1 and y_1. Since C is connected and locally path-connected, C is path-connected. Hence, we may pick a continuous function $\alpha_1\colon \mathbb{I} \to K_\omega \setminus (V_2 \cup L_2)$ such that $\alpha_1(0) = x_1$ and $\alpha_1(1) = y_1$. Observe that $\alpha_1(\mathbb{I}) \cap L_1 \subseteq U_2$. There similarly exists a continuous function $\alpha_2\colon \mathbb{I} \to K_\omega \setminus (U_2 \cup L_2)$ such that $\alpha_2(0) = x_1$ and $\alpha_2(1) = y_1$ and $\alpha_2(\mathbb{I}) \cap L_1 \subseteq V_2$.

Consider the square $\mathbb{I}^2$ and its opposite faces

$$A_1 = \{0\} \times \mathbb{I}, \quad B_1 = \{1\} \times \mathbb{I}, \quad A_2 = \mathbb{I} \times \{0\}, \quad B_2 = \mathbb{I} \times \{1\}.$$

Let $\partial\mathbb{I}^2$ be the geometric boundary of $\mathbb{I}^2$. By using the functions α_1 and α_2, it is immediate that there exists a continuous function $\beta\colon \partial\mathbb{I}^2 \to K_\omega$ such that

$$\beta(A_1) = \{x_1\}, \quad \beta(B_2) = \alpha_1(\mathbb{I}), \quad \beta(B_1) = \{y_1\}, \quad \beta(A_2) = \alpha_2(\mathbb{I}).$$

Since K_ω is an absolute retract, we may extend β to a continuous function $\bar{\beta}\colon \mathbb{I}^2 \to K_\omega$. Consider the projection $\pi\colon \mathbb{I}^{n+1} \times \mathbb{I}^2 \to \mathbb{I}^2$, and the composition

$$\bar{\beta} \circ \pi\colon \mathbb{I}^{n+1} \times \mathbb{I}^2 \to K_\omega.$$

By (K) in §2, there exists an embedding $\gamma\colon \mathbb{I}^{n+1} \times \mathbb{I}^2 \to K_\omega$ such that

$$\begin{aligned} &\gamma(\pi^{-1}(A_1)) \subseteq U_1, && \gamma(\pi^{-1}(B_1)) \subseteq V_1, \\ &\gamma(\pi^{-1}(B_2)) \cap L_1 \subseteq U_2, && \gamma(\pi^{-1}(A_2)) \cap L_1 \subseteq V_2. \end{aligned}$$

By abuse of notation, we identify $K = \mathbb{I}^{n+1} \times \mathbb{I}^2$ and $\gamma(\mathbb{I}^{n+1} \times \mathbb{I}^2)$, hence we pretend that γ is the identity function. Observe that $S_1 = L_1 \cap K$ is a partition in K between its opposite faces $\pi^{-1}(A_1)$ and $\pi^{-1}(B_1)$. Moreover, $L_2 \cap K$ is a partition in $L_1 \cap K$ between the 'reduced' opposite faces

$$\pi^{-1}(A_2) \cap L_1 \quad \text{and} \quad \pi^{-1}(B_2) \cap L_1.$$

Hence there is by [4, 3.1.5] a partition S_2 in K between the opposite faces $\pi^{-1}(A_2)$ and $\pi^{-1}(B_2)$ such that $S_2 \cap S_1 \subseteq L_2 \cap K$. Hence $\dim(S_1 \cap S_2) \leq n$. But in an $(n+3)$-dimensional cube, the intersection of any two partitions between pairs of distinct opposite faces has to be at least $(n+1)$-dimensional by [4, Theorem 3.1.9]. This is a contradiction.

Problem 3.1. *Is there a (separable metrizable) space X such that $\omega + 1 < \operatorname{trt} X < \infty$?*

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References

[1] F. G. Arenas, V. A. Chatyrko, and M. L. Puertas, *Transfinite extension of Steinke's dimension*, Acta Math. Hungar. 88 (2000), 105–112.

[2] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Monograf. Mat. 58, PWN–Polish Sci. Publ., Warszawa, 1975.

[3] V. A. Chatyrko and Y. Hattori, *Scattered topological invariants*, preprint, 2011.
[4] J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland, Amsterdam, 2001.
[5] G. Steinke, *A new dimension by means of separating sets*, Arch. Math. (Basel) 40 (1983), 273–282.

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