ISRAEL JOURNAL OF MATHEMATICS **194** (2013), 745–766
DOI: 10.1007/s11856-012-0062-8

VARIATIONS ON ω-BOUNDEDNESS

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ABSTRACT

Let $\mathscr{P}$ be a property (or, equivalently, a class) of topological spaces. A space X is called **$\mathscr{P}$-bounded** if every subspace of X with (or in) $\mathscr{P}$ has compact closure. Thus, countable-bounded has been known as ω-bounded and (σ-compact)-bounded as strongly ω-bounded.

In this paper we present a systematic study of the interrelations of these two known "boundedness" concepts with $\mathscr{P}$-boundedness where $\mathscr{P}$ is one of the further countability properties weakly Lindelöf, Lindelöf, hereditarily Lindelöf, and *ccc*.

* The first author was supported by OTKA grants no. 68262 and 83726.
** The second and third author are pleased to thank the Alfréd Rényi Institute of Mathematics for generous hospitality.
Received May 5, 2011 and in revised form September 28, 2011

1. Introduction

All spaces under discussion are Tychonoff.

Let $\mathscr{P}$ be any topological property. It is natural to call a space X **$\mathscr{P}$-bounded** if every subset of X with property $\mathscr{P}$ has compact closure in X. In particular, if $\mathscr{P} \equiv$ "countable" then we obtain the well-known class of ω-bounded spaces, and if $\mathscr{P} \equiv$ "σ-compact" then we obtain the class that is called **strongly ω-bounded** in Nyikos [23].

Following this pattern, we may now consider $\mathscr{P}$-bounded spaces for each "countability property" $\mathscr{P}$, by which we mean any property that all countable spaces possess. (Warning: first and second countability are not such properties!) By definition, $\mathscr{P}$-bounded spaces of this sort are always ω-bounded, hence in what follows we shall call (σ-compact)-bounded spaces σC-bounded rather than strongly ω-bounded.

We now list five further countability properties (and their abbreviations) that will also be studied in this paper:

Lindelöf-bounded — abbreviated as L-bounded;

(hereditarily Lindelöf)-bounded — abbreviated as HL-bounded;

(weakly Lindelöf)-bounded — abbreviated as wL-bounded;

ccc-bounded;

(countable spread)-bounded — abbreviated as CS-bounded.

2. Equivalence of certain boundedness notions

It is obvious that if $\mathscr{P} \Rightarrow \mathscr{Q}$ then $\mathscr{Q}$-bounded $\Rightarrow$ $\mathscr{P}$-bounded. The aim of this section is to present some additional implications between boundedness properties that are less obvious. These results then yield equivalences between certain boundedness properties, sometimes under extra topological and/or set-theoretical assumptions.

The first such equivalence is absolutely trivial:

$$\text{separable-bounded} \equiv \omega\text{-bounded}.$$

This explains why separable-bounded was not included in the above list.

The next result is less obvious.

Theorem 2.1: *Any HL-bounded space is CS-bounded. Consequently,*

$$\textit{HL-bounded} \equiv \textit{CS-bounded}.$$

Proof. What we have to show is that if a space X is HL-bounded then it is also CS-bounded. For this we may assume that X has countable spread and our task is to prove that it is compact.

If X is separable, then it is compact by our assumptions. Otherwise it is not separable and therefore contains a dense *left separated* subspace of type κ, where $\kappa > \omega$. That is, a subspace S that can be written as $\{s_\alpha : \alpha < \kappa\}$ such that $s_\alpha \notin \overline{\{s_\beta : \beta < \alpha\}}$ for every $\alpha < \kappa$. We claim that every right-separated subset of S is countable, hence S is hereditarily Lindelöf (see Juhász [14, 2.6]). Indeed, otherwise we had an uncountable subset $T \subset S$ that is both left and right separated, implying that T contains a discrete subspace of size ω_1 (see Juhász [14, the proof of 2.7]). However, this contradicts that X has countable spread. So we are done because X is HL-bounded, moreover S is hereditarily Lindelöf and dense in X. ∎

Hence, in what follows, we shall only deal with HL-boundedness and forget about CS-boundedness.

Our next (easy) equivalence is valid only for locally compact spaces.

Theorem 2.2: *If X is locally compact and σC-bounded then X is wL-bounded. Consequently, in the class of locally compact spaces we have*

$$\sigma C\text{-bounded} \equiv L\text{-bounded} \equiv wL\text{-bounded}.$$

Proof. We need to prove that any weakly Lindelöf subspace S of the σC-bounded locally compact space X has compact closure. For this, simply take a cover $\mathscr{U}$ of S by open sets having compact closure in X and choose a countable subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $\bigcup \mathscr{V}$ is dense in S. Then

$$\overline{S} \subset \overline{\bigcup \mathscr{V}} \subset \overline{\bigcup \{\overline{V} : V \in \mathscr{V}\}} = T\,,$$

and T is compact by our assumption, hence so is $\overline{S}$ as well. ∎

Let us recall, before giving our next equivalence result, that an L-space is a hereditarily Lindelöf space that is not hereditarily separable.

Theorem 2.3: *Let X be an ω-bounded space that contains no L-subspace. Then X is HL-bounded. Consequently,* $\mathsf{MA}_{\aleph_1}$ *implies*

$$\omega\text{-bounded} \equiv HL\text{-bounded}$$

in the class of first countable spaces.

Proof. Assume that S is a hereditarily Lindelöf subspace of X. Then S is (hereditarily) separable because it cannot be an L-space, hence $\overline{S}$ is compact, as X is ω-bounded. The rest now follows because $\mathsf{MA}_{\aleph_1}$ implies that there are no first countable L-spaces, as was shown in Szentmiklóssy [26]. ∎

We can strengthen the second part of Theorem 2.3, provided that we add local compactness to first countability.

THEOREM 2.4: $\mathsf{MA}_{\aleph_1}$ *implies*

$$\omega\text{-bounded} \equiv \text{ccc-bounded}$$

in the class of first countable and locally compact spaces.

Proof. As it is well-known, $\mathsf{MA}_{\aleph_1}$ implies that every locally compact, first-countable, and ccc space is separable. (More generally, any first-countable and ccc space in which every closed subspace is Martin-ω_1-complete is separable; see Fremlin [10, 43M and 43N] and Juhász [15].) But this trivially implies our claim. ∎

The following result yields another equivalent of the HL-boundedness property for first countable spaces, but this time under the continuum hypothesis (CH) that, of course, contradicts $\mathsf{MA}_{\aleph_1}$. So, the reader should be aware that Theorems 2.3 and 2.5 cannot be applied simultaneously.

THEOREM 2.5 (CH): *If X is HL-bounded and $\chi(X) \leq \omega_1$, then X is ccc-bounded. In other words,*

$$\text{HL-bounded} \equiv \text{ccc-bounded}$$

for spaces of character at most ω_1.

Proof. Since the closure of a ccc subspace is again ccc, it suffices to show that if $A \subset X$ is closed and ccc then A is compact. But we also have $\pi\chi(A) \leq \chi(A) \leq \omega_1$, hence by Šapirovskii's celebrated theorem (see, e.g., Juhász [16, 2.37]) and by CH we get

$$w(A) \leq \varrho(A) \leq \pi\chi(A)^{c(A)} \leq \omega_1^{\omega} = \omega_1\,.$$

Now, observe that A is a Baire space, being ω-bounded. Consequently, we may apply van Douwen, Tall and Weiss [5, Theorem 1] to conclude that A contains a dense Luzin, and hence hereditarily Lindelöf, subspace. But then A is compact because X is HL-bounded. ∎

3. Some results from βX

The aim of this section is to present some results that will turn out to play a crucial role in the construction of our examples separating various boundedness properties. The first group of these results culminates in Theorem 3.4 below which was inspired by Franklin and Rajagopalan [9] and Walter Rudin [24].

The long segment $\mathbb{L}$ is obtained from ω_1+1 by adding arcs that join successive countable ordinals. More precisely, we define, for each countable ordinal α and each real number $r \in (0,1)$, an element $\alpha+r$, such that $\alpha < \alpha+r < \alpha+s < \alpha+1$ whenever $0 < r < s < 1$. The long segment $\mathbb{L}$ is the union of the set of all such elements $\alpha+r$ and ω_1+1, and is endowed with the natural order topology that is clearly compact.

THEOREM 3.1: *Let K be a compact space without isolated points and $P \subset K$ be a closed P-set of character ω_1 in K. Then there is a continuous surjection $f\colon K \to \mathbb{L}$ such that $f^{-1}(\{\omega_1\}) = P$.*

Proof. First, using that P is a closed P-set of character ω_1, we may easily construct an increasing ω_1-sequence $\{V_\alpha : \alpha < \omega_1\}$ of nonempty open subsets of K such that

(1) $\overline{V}_\beta \subset V_\alpha$ for all $\beta < \alpha$ (here '$\subset$' means proper inclusion),
(2) $\overline{V}_\alpha \setminus V_\alpha \neq \emptyset$ for every α,
(3) $\bigcup_{\alpha<\omega_1} V_\alpha = K \setminus P$.

We will use this ω_1-sequence to define, step-by-step, our continuous surjection $f\colon K \to \mathbb{L}$ as follows. The set $\overline{V}_0$ is mapped to 0 by f, and $\overline{V}_1 \setminus V_1$ is mapped to 1. Let U be a nonempty open subset of V_1 whose closure misses $\overline{V}_0 \cup (\overline{V}_1 \setminus V_1)$. Then $\overline{U}$ is a compact space without isolated points, hence can be mapped onto $\mathbb{I}$ by f. Now we use the Tietze Extension Theorem to extend f from $\overline{V}_0 \cup \overline{U} \cup (\overline{V}_1 \setminus V_1)$ to a map $f\colon \overline{V_1} \to [0,1]$. Clearly, this map is a continuous surjection. In this way we continue up to stage ω. At this stage we may further continuously extend f to $\overline{\bigcup_{n<\omega} V_n}$ by sending $\overline{\bigcup_{n<\omega} V_n} \setminus \bigcup_{n<\omega} V_n$ to the point ω. Then the set $\overline{V}_\omega \setminus V_\omega$ is mapped to the point $\omega+1$, and we may again use the above procedure to extend our partially defined map to a continuous surjection of $\overline{V}_\omega$ onto $[0,\omega+1]$. In this manner the construction can be carried out all the way up to ω_1 to define the continuous surjection $f : K \setminus P \to \mathbb{L} \setminus \{\omega_1\}$. We finish by sending P to the point ω_1. The map f obtained in this way is clearly the required continuous surjection $f\colon K \to \mathbb{L}$. ∎

It should be clear that if K and P are as above but K is also zero-dimensional, then a similar (but simpler) construction yields a continuous surjection $f\colon K \to \omega_1 + 1$ such that $f^{-1}(\{\omega_1\}) = P$.

Our next result will be used to produce situations in which the previous theorem can be applied.

LEMMA 3.2: *Let X be a noncompact, locally compact, and σ-compact space and $\mathscr{A}$ be a family of at most ω_1 nowhere dense subsets of X. Then there is a nonempty closed P-set P of character ω_1 in the Čech–Stone remainder $X^* = \beta X \setminus X$ satisfying $P \cap \overline{A} = \emptyset$ for each $A \in \mathscr{A}$ (where closure is taken in βX).*

Proof. Observe that for every $A \in \mathscr{A}$ we have that $\overline{A} \cap X^*$ is nowhere dense in X^*. Also, X^* has the property that every nonempty G_δ-subset has infinite interior, in particular, X^* has no isolated points. This can be found in Gillman and Jerison [12] (see also [21, Theorem 1.2.5]). Enumerate $\mathscr{A}$ as $\{A_\alpha : \alpha < \omega_1\}$. Now we use the technique in Walter Rudin's proof in [24] of the existence of P-points in ω^* to obtain a strictly decreasing sequence $\{U_\alpha : \alpha < \omega_1\}$ of nonempty closed G_δ-subsets of X^* such that $U_0 = X^*$, moreover $\overline{U_{\alpha+1}} \subset U_\alpha$ and $U_\alpha \cap \overline{A}_\alpha = \emptyset$ for every $\alpha < \omega_1$. Put $P = \bigcap_{\alpha<\omega_1} U_\alpha$ and observe that, as X^* is compact, P is a nonempty P-set of character ω_1 in X^*. ∎

The following result, which is an immediate consequence of Lemma 3.2, is well-known and is implicit for example in the proof of Theorem 2.3 in Fine and Gillman [7]. For completeness' sake, we will present its proof.

COROLLARY 3.3: *Let $\mathscr{A}$ be a family of ω_1 nowhere dense subsets of a nonpseudocompact space X. Then $\beta X \neq \bigcup_{A\in\mathscr{A}} \overline{A}$ (where closure is taken in βX).*

Proof. Since X is not pseudocompact, there is a nonempty closed G_δ-subset S of βX that misses X, [12, Problem 6I.1]. Put $Y = \beta X \setminus S$. Then Y is locally compact, noncompact, and σ-compact, moreover $\beta Y = \beta X$, [12, Theorem 6.7]. As X is dense in Y, $\overline{A} \cap Y$ is nowhere dense in Y for every member $A \in \mathscr{A}$. Hence we are done by applying Lemma 3.2 to Y and the family $\{\overline{A} \cap Y : A \in \mathscr{A}\}$. ∎

In the proof of the following result we will make use of Magill's Theorem from [20]. This theorem states that if αX is any compactification of a locally compact space X and $f\colon \alpha X \setminus X \to Y$ is a continuous surjection, then

$$\big\{\{x\} : x \in X\big\} \cup \{f^{-1}(y) : y \in Y\}$$

forms an upper semicontinuous decomposition of αX. Consequently, the corresponding quotient space of αX is a compactification γX of X whose remainder is (homeomorphic to) Y, so (if we assume that X and Y are disjoint) $\mathrm{id}_X \cup f$ is a continuous map from αX onto γX which restricts to f on $\alpha X \setminus X$.

Theorem 3.4: *Let $\mathscr{A}$ be a family of ω_1 nowhere dense subsets of a noncompact, locally compact, and σ-compact space X. Then there is a compactification γX of X having the following properties:*

(1) *$\gamma X \setminus X$ is the long segment $\mathbb{L}$,*

(2) *$\omega_1 \notin \bigcup_{A\in\mathscr{A}} \overline{A}$ (here closure is taken in γX).*

Proof. By Lemma 3.2 there is a closed P-set P in $X^* = \beta X \setminus X$ of character ω_1 such that

$$P \cap \bigcup\{\overline{A}^{\beta X} : A \in \mathscr{A}\} = \emptyset.$$

Let $f\colon X^* \to \mathbb{L}$ be the continuous surjection we get from applying Theorem 3.1 to $K = X^*$ and this P-set P. Then, by Magill's Theorem which we just quoted, there is a compactification γX of X whose remainder is $\mathbb{L}$, moreover $g = \mathrm{id}_X \cup f$ is a continuous map from βX onto γX. But then (1) holds trivially, moreover (2) holds because $g^{-1}(\{\omega_1\}) = P$ and $g^{-1}(\overline{A}) = \overline{A}^{\beta X}$ for any $A \subset X$. ∎

In view of our remark made after the proof of Theorem 3.1, if X (and hence βX) is zero-dimensional, then we can replace $\mathbb{L}$ with $\omega_1 + 1$ in Theorem 3.4.

We call a space X a **k_ω-space** if it can be written as $X = \bigcup_{n<\omega} X_n$, where each X_n is compact, $X_n \subset X_{n+1}$ for $n < \omega$, and a set $A \subset X$ is closed if and only if $A \cap X_n$ is closed for every n. That is, a k_ω-space has the weak topology determined by a(n increasing) sequence of compact subspaces that cover it. Alternatively, it is easy to see that k_ω-spaces are exactly the quotients of countable topological sums of compact spaces.

The following proposition is due to van Douwen and was explained by him to the second author of the present paper around 1980. As far as we know, it was neither published by van Douwen nor discovered independently by somebody else. So we present its proof here. See Dow, Gubbi and Szymański [6] for related results.

Proposition 3.5 (van Douwen): *If X is any k_ω-space then its Čech–Stone remainder X^* is σC-bounded.*

Proof. Write X as $\bigcup_{n<\omega} X_n$, where the sequence of X_n's witnesses that X is a k_ω-space. Let $\{S_n : n < \omega\}$ be a sequence of compact subsets of X^*. For every n there exists an open neighborhood U_n of X_n in βX such that $\overline{U}_n \cap S_n = \emptyset$ (all closures are taken in βX).

Now take an arbitrary point $p \in X$ and assume, without loss of generality, that $p \in X_0$. Let V_0 be an open neighborhood of p in βX such that $\overline{V}_0 \subset U_0$. Since $\overline{V}_0 \cap X_0 \subset X_0 \subset X_1 \subset U_1$, there is in βX an open neighborhood V_1 of the compact set $\overline{V}_0 \cap X_0$ such that $\overline{V}_1 \subset U_0 \cap U_1$. Continuing in this way inductively, we may define a sequence $V_0, V_1, \ldots$ such that V_n is an open neighborhood of $\overline{V}_{n-1} \cap X_{n-1}$ in βX satisfying $\overline{V}_n \subset U_0 \cap U_1 \cap \cdots \cap U_n$. Now put

$$W = (V_0 \cap X_0) \cup \big(V_1 \cap (X_1 \setminus X_0)\big) \cup \cdots \cup \big(V_n \cap (X_n \setminus X_{n-1})\big) \cup \cdots .$$

We may show by an easy induction that for each $n < \omega$ we have

$$X_n \setminus W = (X_0 \setminus V_0) \cup (X_1 \setminus V_1) \cup \cdots \cup (X_n \setminus V_n),$$

which is compact and hence closed in X_n. Consequently, $W \cap X_n$ is open in X_n for every n, hence W is an open neighborhood of p in X. Let $\widehat{W}$ be an open subset of βX such that $\widehat{W} \cap X = W$. Observe that $\widehat{W}$ is included in the closure of W, hence it suffices to prove that $\overline{W}$ misses $\bigcup_{n<\omega} S_n$. But for every $n < \omega$ we have by construction that

$$W \setminus X_n \subset U_n \subset \overline{U}_n \subset \beta X \setminus S_n,$$

hence $\overline{W} \cap S_n = \emptyset$ as X_n is compact, and this completes the proof. ∎

There are countable k_ω-spaces that have no isolated points and hence are nowhere locally compact, for example the well-known space S_ω from Arhangel′skiĭ and Franklin [1]. A simple description of S_ω is as follows: The points are the finite sequences of natural numbers (i.e. the underlying set is $\omega^{<\omega}$) and a set G is open in S_ω iff $s \in G$ implies $s^\frown n \in G$ for all but finitely many $n \in \omega$. To see that S_ω is a k_ω-space, take any ω-type enumeration $\{s_i : i < \omega\}$ of $\omega^{<\omega}$ and put $X_n = \bigcup_{i\le n}\{s_i\} \cup \{s_i^\frown n : n < \omega\}$.

The Čech–Stone remainder $S_\omega^* = \beta S_\omega \setminus S_\omega$ is σC-bounded by Proposition 3.5. S_ω^* is also ccc, in fact its topology is even σ-centered as S_ω is countable and nowhere locally compact, hence S_ω^* is dense in βS_ω. Consequently, S_ω^* is not ccc-bounded, and our next result will imply that it is not L-bounded, either.

THEOREM 3.6: *Assume that the space X admits a continuous surjection $f\colon X \to Y$ onto a separable metrizable space Y such that $f^{-1}(K)$ is nowhere dense in X for every compact subset K of Y. Then the Čech–Stone remainder X^* of X has a dense Lindelöf subspace.*

Proof. Let γY be any metrizable compactification of Y, and consider the Stone extension $\beta f\colon \beta X \to \gamma Y$. Clearly, βf is also surjective. Let us put $S = (\beta f)^{-1}(Y)$ and $T = \beta X \setminus S$. Note that $T \subset X^*$.

CLAIM 1: *T is dense in βX and hence in X^*.*

Assume otherwise. Then there is a nonempty open subset U of βX whose compact closure $\overline{U}$ is contained in S, hence $K = \beta f(\overline{U})$ is a compact subset of Y. However, this would imply $\emptyset \neq U \cap X \subset f^{-1}(K)$, contradicting our assumption.

CLAIM 2: *T is Lindelöf.*

This is clear since $T = (\beta f)^{-1}(\gamma Y \setminus Y)$ and the perfect preimage of a Lindelöf space is Lindelöf. ∎

COROLLARY 3.7: *Assume that $f\colon X \to Y$ is a continuous surjection where Y is separable metric, every compact subspace of Y is scattered, and the fibers $f^{-1}(y)$ are nowhere dense in X for all $y \in Y$. Then X^* has a dense Lindelöf subspace. In particular, this holds if X is a countable space with no isolated points.*

Proof. To see that the conditions of Theorem 3.6 are satisfied, let $K \subset Y$ be compact and assume that $f^{-1}(K)$ contains a nonempty open subset of X, say U. Then the closure of $f(U)$ in Y is compact, hence it has an isolated point, say y. Let V be open in Y such that $V \cap \overline{f(U)} = \{y\}$. Then $f^{-1}(V) \cap U$ is a nonempty open subset of $f^{-1}(y)$, which contradicts our assumption on f.

If X is a countable space with no isolated points, then there is a continuous bijection $f : X \to \mathbb{Q}$. Indeed, let $\mathscr{B}$ be a countable point separating a collection of clopen subsets of X; then $\mathscr{B}$ generates a second countable topology on X which is evidently weaker than the original topology. With this new topology, X is homeomorphic to the space $\mathbb{Q}$ of rational numbers. ∎

In particular, we may conclude from this that S^*_ω has a dense Lindelöf subspace.

Our next result, also based on Theorem 3.6, says that the Čech–Stone remainders of certain k_ω-spaces have dense Lindelöf subspaces. However, it is not clear if it can be applied to S^*_ω.

THEOREM 3.8: *Let $X = \bigcup_{n<\omega} X_n$, and assume that*

(1) *each X_n is a compact nowhere dense G_δ-subset of X_{n+1},*

(2) *X has the weak topology determined by the sequence $\{X_n : n < \omega\}$.*

Then X^* has a dense Lindelöf subspace.

Proof. For every $n < \omega$, let $\lambda_n : X_{n+1} \to \mathbb{I}$ be a continuous map such that $\lambda_n^{-1}(0) = X_n$. We define by recursion for every $n < \omega$ a compact metrizable space Y_n and a continuous surjection $\alpha_n : X_n \to Y_n$, as follows. For $n = 0$ we put $Y_0 = \{0\}$ and $\alpha_0 : X_0 \to Y_0$ the constant function with value 0. At stage $n+1$, we think of Y_n as a subspace of the Hilbert cube Q. The map $\alpha_n : X_n \to Y_n \hookrightarrow Q$ can be extended to a continuous function $\bar{\alpha}_n : X_{n+1} \to Q$. Define $\alpha_{n+1} : X_{n+1} \to Q \times \mathbb{I}$ by

$$\alpha_{n+1}(x) = \langle \bar{\alpha}_n(x), \lambda_n(x) \rangle.$$

Put $Y_{n+1} = \alpha_{n+1}(X_{n+1})$. Observe that if $x \in X_n$, then

$$\alpha_{n+1}(x) = \langle \bar{\alpha}_n(x), \lambda_{n+1}(x) \rangle = \langle \alpha_n(x), 0 \rangle \in X_n \times \{0\}.$$

Moreover, if $x \in X_{n+1} \setminus X_n$, then $\lambda_{n+1}(x) > 0$, and hence

$$\alpha_{n+1}(x) = \langle \bar{\alpha}_n(x), \lambda_{n+1}(x) \rangle \notin X_n \times \{0\}.$$

Hence we may think of Y_n as being a subspace of Y_{n+1}, so that α_{n+1} has the following properties:

(1) $\alpha_{n+1} \restriction X_n = \alpha_n$,
(2) $\alpha_{n+1}(X_{n+1} \setminus X_n) = Y_{n+1} \setminus Y_n$.

Put $Y = \bigcup_{n<\omega} Y_n$, and $\alpha = \bigcup_{n<\omega} \alpha_n : X \to Y$. We endow Y with the weak topology determined by the sequence $\{Y_n : n < \omega\}$. Then α is continuous and for every $n < \omega$ we have

(3) $\alpha^{-1}(Y_n) = X_n$.

For every $n < \omega$, fix an embedding $i_n\colon Y_n \to Q$. Since Y is σ-compact, and hence normal, this embedding can be extended to a continuous map $\bar\imath_n\colon Y \to Q$. Let $\xi\colon Y \to Q^\omega$ be the diagonal mapping obtained from the sequence $\{\bar\imath_n : n<\omega\}$. Then $\xi\colon Y \to \xi(Y) = Z$ is a continuous bijection with a separable metric range.

We aim to apply Theorem 3.6 to the map $\eta = \xi \circ \alpha\colon X \to Z$. To this end, take an arbitrary compact subset K in Z and, striving for a contradiction, assume that $\eta^{-1}(K)$ contains a nonempty open subset of X, say U. Let L be the closure of $\eta(U)$. Then L is compact and $\eta(U)$ is dense in L. Since $L = \bigcup_{n<\omega} \eta(X_n) \cap L$, by the Baire Category Theorem there exists $N < \omega$ such that $\eta(X_N) \cap L$ has nonempty interior in L. Let V be a nonempty open subset of L such that $V \subset \eta(X_N) \cap L$. Then $V \cap \eta(U)$ is a nonempty open subset of $\eta(U)$, hence $W = (\eta\restriction U)^{-1}(V \cap \eta(U))$ is a nonempty open subset of U and hence of X. However, by (3) we have

$$W \subset \eta^{-1}(V) \subset \eta^{-1}(\eta(X_N)) = X_N,$$

contradicting that X_N has empty interior already in X_{N+1}.

Hence the desired result indeed follows from Theorem 3.6. ∎

Remark 3.9: The assumption in Theorem 3.8 that every X_n is a G_δ-subset of X_{n+1} is essential for getting a dense Lindelöf subspace of X^*. In fact, if each X_n happens to be a nowhere dense P-set in X_{n+1}, then the argument used in the proof of Theorem 3.5 shows that every X_n is a P-set in X and hence in βX. But then X^* is even wL-bounded. Indeed, if $A \subset X^*$ is weakly Lindelöf, then for each $n < \omega$ there is a σ-compact subset F_n of $\beta X \setminus X_n$ such that $\overline{A} = \overline{A \cap F_n}$. But we have $\overline{F_n} \cap X_n = \emptyset$, for X_n is a P-set in βX, hence $\overline{A} \cap X_n = \emptyset$.

4. Examples for separating various boundedness properties

Nyikos [23, Problem 2] asked whether there is a first countable, ω-bounded space that is not σC-bounded. He also observed that without first countability the problem has an affirmative answer: If p is a weak P-point in $\omega^* = \beta\omega \setminus \omega$ that is not a P-point, then $\omega^* \setminus \{p\}$ is locally compact, ω-bounded but not σC-bounded. That such points exist in ω^* is highly nontrivial and was shown by Kunen [18].

Spaces of the form $\omega^* \setminus \{p\}$ are never Fréchet. We point out that there are easy examples of Fréchet spaces that are ω-bounded but not σC-bounded. In fact, it is well-known that the subspace $\Sigma = \{x \in 2^{\omega_1} : |\{\alpha < \omega_1 : x_\alpha = 1\}| \le \omega\}$

of the Cantor cube 2^{ω_1} is both ω-bounded and Fréchet. Since Σ contains the dense σ-compact subset $\sigma = \{x \in 2^{\omega_1} : |\{\alpha < \omega_1 : x_\alpha = 1\}| < \omega\}$, it is not σC-bounded. As Σ is ccc, it is also not ccc-bounded. However, unlike Nyikos' example, it is not locally compact.

Nyikos' Problem 2 from [23] was solved by Aurichi [2]. His example is also locally compact. Our first example is an easier construction of such a space, which is even ccc-bounded. Aurichi's space is ccc-bounded if one applies a special Aronszajn tree in his construction, but will not be ccc-bounded for a Suslin tree.

Our example will be just a particular instance of a general construction of ω-bounded but not σC-bounded spaces that is based on Theorem 3.4. First, however, we introduce some new terminology.

A space X will be called ω_1-**short** if it can be written in the form

$$X = \bigcup_{\alpha<\omega_1} A_\alpha,$$

where $\{A_\alpha : \alpha < \omega_1\}$ is an increasing sequence of nowhere dense closed subsets of X. Now, if X is both ω_1-short and noncompact, locally compact, and σ-compact, then we may apply Theorem 3.4 to obtain a compactification γX of X with remainder $\mathbb{L}$ (or $\omega_1 + 1$ if X is zero-dimensional) such that the point $\omega_1 \notin \overline{A_\alpha}$ for all $\alpha < \omega_1$. Since every countable subset of X is included in some A_α, moreover ω_1 is a P-point in $\mathbb{L}$, it immediately follows that the open subspace $M(X) = \gamma X \setminus \{\omega_1\}$ of γX is ω-bounded. Moreover, $M(X)$ is not σC-bounded because X is σ-compact and dense in it. The following result summarizes this and says some more.

Theorem 4.1: *Let X be an ω_1-short, noncompact, locally compact, and σ-compact space. Then $M(X)$ is normal, locally compact, ω-bounded but not σC-bounded. Moreover, if X is first countable then so is $M(X)$.*

Proof. First, to see that $M(X)$ is normal, note that our assumptions on X imply that we may write X as $X = \bigcup_{n<\omega} U_n$ where U_n is open with compact closure in X and $\overline{U_n} \subset U_{n+1}$ for each $n < \omega$. Now, let H and K be two disjoint closed sets in $M(X)$. Then we may assume that $K \cap \mathbb{L}$ is bounded in $\mathbb{L}$ and hence compact. So, there exist disjoint open sets V_0 and W_0 in $M(X)$ such that $H \subset V_0$ and $K \cap \mathbb{L} \subset W_0$.

We claim that $K_1 = K \setminus W_0 \subset U_n$ for some $n < \omega$. Indeed, otherwise we could choose a countably infinite subset $A \subset K_1$ such that $A \cap U_n$ is finite for each $n < \omega$, and then every limit point x of A in $M(X)$ has to belong to $\mathbb{L}$ and hence to $K \cap \mathbb{L}$. Note that there is such a limit point because $M(X)$ is ω-bounded. This, however, is impossible because $A \subset K_1$ implies $x \in K_1$, as K_1 is closed.

But then K_1 is compact because each $\overline{U_n}$ is, consequently there exist disjoint open sets V_1 and W_1 in $M(X)$ such that $H \subset V_1$ and $K_1 \subset W_1$. Now, it is obvious that $V_0 \cap V_1$ and $W_0 \cup W_1$ are disjoint open sets containing H and K, respectively.

The second part follows because, as X is σ-compact, $\mathbb{L}$ is a G_δ in γX, and hence so is every point of $\mathbb{L} \setminus \{\omega_1\}$. Consequently, every point of $\mathbb{L} \setminus \{\omega_1\} = M(X) \setminus X$ has countable character already in γX. ∎

If X is also zero-dimensional and $\mathbb{L}$ is replaced with $\omega_1 + 1$ in the above, we shall write $M_0(X)$ instead of $M(X)$. Clearly, everything we proved for $M(X)$ is also valid for $M_0(X)$.

The following simple result gives a useful sufficient condition for a space to be ω_1-short.

Theorem 4.2: *Every countably tight and nowhere separable space X of density ω_1 is ω_1-short.*

Proof. Let $\{x_\alpha : \alpha < \omega_1\}$ be dense in X. It is obvious that the sequence $\{A_\alpha : \alpha < \omega_1\}$, where A_α is the closure of the set $\{x_\beta : \beta < \alpha\}$ in X, witnesses that X is ω_1-short. ∎

We can now present our above promised example.

Theorem 4.3: *There is a zero-dimensional, normal, first countable, and locally compact space that is ccc-bounded but not σC-bounded.*

Proof. Let $A(\mathbb{C})$ be the Alexandroff duplicate of the Cantor set $\mathbb{C}$. Remove all but ω_1 isolated points from $A(\mathbb{C})$. The subspace Y of $A(\mathbb{C})$ that we get in this way is a first countable zero-dimensional compact space which is not ccc and has density ω_1. Hence $Z = Y^\omega$ is a zero-dimensional first countable compact space which is nowhere ccc and has density ω_1. Finally, let us put $X = \omega \times Z$. Then X is a zero-dimensional, noncompact, locally compact, and σ-compact space which is also ω_1-short by Theorem 4.2.

Now, it is easy to check that Y, and hence Z, and hence X, have the property that every ccc subspace in them is actually separable. Clearly, every ordinal space has this property as well. Consequently, $M_0(X)$ which we know is ω-bounded, is also ccc-bounded, and so has all the properties required in our theorem. ∎

Next we shall consider results that will help us produce examples that are not ccc-bounded.

THEOREM 4.4: *Fix a cardinal number κ. Then there is a locally compact, noncompact, ccc space in which all Lindelöf subspaces of size at most κ have compact closure.*

Proof. It is known (see Fremlin [11, 523La]) that there is a cardinal λ such that all subsets of the Cantor cube 2^λ of size at most 2^{2^κ} have μ-measure 0, where μ is the standard product measure on 2^λ. We let $\mathscr{U}$ be the collection of all clopen subsets U of the space $X = \omega \times 2^\lambda$ for which

$$\sum_{n<\omega} \mu\big(U \cap (\{n\} \times 2^\lambda)\big) < \infty.$$

(Here we actually use the natural transfer of the measure μ to $\omega \times 2^\lambda$, that we also denote by μ.)

The family $\mathscr{V}$ of all complements of the elements of $\mathscr{U}$ clearly has the finite intersection property, hence $P = \bigcap\{\overline{V} : V \in \mathscr{V}\}$ is nonempty, where closures are taken in βX. It is clear that $\bigcap \mathscr{V} = \emptyset$, hence $P \subset X^*$, and we claim that P is a P-set in X^*.

To see this, consider any compact set $S \subset X^* \setminus P$ and a positive real number ε. There clearly exists $U \in \mathscr{U}$ for which $S \subset \overline{U}$. Since

$$\sum_{i<\omega} \mu\big(U \cap (\{i\} \times 2^\lambda)\big) < \infty,$$

there exists $N < \omega$ such that

$$\sum_{i \geq N} \mu\big(U \cap (\{i\} \times 2^\lambda)\big) < \varepsilon\,.$$

Now, let $H = \bigcup_{n<\omega} S_n \subset X^*$, with each S_n compact, be any σ-compact set in X^*. According to the above, for each $n < \omega$ we may then choose a set

$U_n \in \mathscr{U}$ with $S_n \subset \overline{U_n}$ and a natural number $N_n \geq n$ such that

$$\sum_{i \geq N_n} \mu\big(U_n \cap (\{i\} \times 2^\lambda)\big) < 2^{-n}.$$

Now, if we set $W_n = U_n \setminus \bigcup_{i<N_n}(\{i\} \times 2^\lambda)$, then $S_n \subset \overline{W_n}$, moreover $W = \bigcup_{n<\omega} W_n$ is clopen in X and $\mu(W) < 1$. Hence we conclude that $\overline{W}$ is a clopen set in βX that contains H and misses P.

Now put $Y = \beta X \setminus P$. Then Y is locally compact and not ccc-bounded since it contains the dense ccc subset X. To prove the rest, let $L \subset Y$ be a Lindelöf subspace of size at most κ and M be the closure of $L \cap X$ in X. Then $|M| \leq 2^{2^\kappa}$, so $\mu(M) = 0$. Therefore, M is contained in a clopen subset of X of finite measure, since μ is a Radon measure. We then conclude that P misses $\overline{L \cap X}$. But P also misses the closure of $L \cap X^*$ in βX, since $L \cap X^* \subset Y$ is Lindelöf and P is a P-set in X^*. ∎

Remark 4.5: The idea to use measures to get filters avoiding null sets is quite old and goes back to Eberlein; see Fine and Gillman [8].

Corollary 4.6: *There is a locally compact space that is HL-bounded but not ccc-bounded.*

Proof. By de Groot's Theorem, every hereditarily Lindelöf space has size at most $\mathfrak{c}$; see Juhász [14, 2.5]. Hence we are done by applying the previous theorem for $\kappa = \mathfrak{c}$. ∎

After separating HL-bounded from ccc-bounded, the next result separates it from ω-bounded.

Theorem 4.7:

(1) *There is an ω-bounded space which is not HL-bounded.*
(2) *There is a noncompact ω-bounded space with a dense first countable and ccc subspace.*

Proof. For (1), we make use of the celebrated result of Moore [22] that there is an L-space X. We may assume that X is left-separated in type ω_1 and nowhere separable. By removing if necessary the left-most point of X and hence a nowhere dense G_δ-subset of X, we may additionally assume that X is not pseudocompact.

Thus we may apply Corollary 3.3 to X and the family $\mathscr{A}$ of all countable initial segments of the ω_1-type left-separating well-order of X to conclude that

$$Z = \bigcup\{\overline{A} : A \in [X]^{\leq\omega}\} \neq \beta X$$

(closures are taken in βX). Since the noncompact space Z is clearly ω-bounded and the hereditarily Lindelöf subspace X is dense in it, we are done.

For (2), we consider the Pixley–Roy hyperspace $F[X]$ of an ω_1-dense subspace of $\mathbb{R}$ of size ω_1. It is well-known that $F[X]$ is first countable, ccc, nowhere separable, and has density ω_1. By Theorem 4.2, $F[X]$ is ω_1-short, witnessed by the closures of some countable sets $\{A_\alpha : \alpha < \omega_1\}$. Also, $F[X]$ is not pseudocompact. Indeed, let $\{U_n : n < \omega\}$ be any disjoint collection of nonempty open subsets of X. For each n, let F_n be a subset of U_n of size n. Then $\{\,[F_n, U_n] : n < \omega\}$ is an infinite discrete collection of clopen subsets of $F[X]$. Corollary 3.3 may thus be applied to conclude that the space

$$Z = \bigcup\{\overline{A_\alpha} : \alpha < \omega_1\} \neq \beta F[X]$$

is as required. Note, however, that Z — unlike its dense subspace $F[X]$ — is not first countable. ∎

The following theorem summarizes what we know of the boundedness properties of the space S_ω^* that was considered in the previous section.

THEOREM 4.8: *The space S_ω^* has the following properties:*

(1) *S_ω^* is σC-bounded but not compact,*
(2) *the topology of S_ω^* is σ-centered, hence S_ω^* is not ccc-bounded,*
(3) *S_ω^* has a dense Lindelöf subspace, hence it is not L-bounded,*
(4) *if CH holds then S_ω^* has a dense hereditarily Lindelöf subspace, hence it is not even HL-bounded.*

Proof. In view of our remarks made after Proposition 3.5 and of Corollary 3.8, we only have to show (4). To this end, observe first that S_ω^* is a Baire space. Indeed, this follows both because S_ω^* is Čech-complete and because it is countably compact. We also have $w(S_\omega^*) \leq w(\beta S_\omega) = \mathfrak{c}$. Thus, if CH holds then we may apply van Douwen, Tall and Weiss [5, Theorem 1] to conclude that S_ω^* contains a dense Luzin subspace. But all Luzin spaces are hereditarily Lindelöf, so we are done. ∎

By Kunen [17], there are no uncountable Luzin spaces under $\mathsf{MA}_{\aleph_1}$, hence the method we used to prove Theorem 4.8(4) cannot work to get a ZFC example of a σC-bounded but not HL-bounded space. However, our next result implies that under CH we have an even locally compact and first countable space which is ω-bounded but not HL-bounded.

THEOREM 4.9: *If there is a compact L-space, then there exists a normal, locally compact, and first countable space which is ω-bounded but not HL-bounded.*

Proof. Let K be any compact L-space; we may assume without loss of generality that K is nowhere separable. Moreover, K is automatically first countable and has density ω_1, see [16, 3.13]. Thus, by Theorem 4.2, K is ω_1-short, and hence so is the hereditarily Lindelöf space $X = \omega \times K$. But then Theorem 4.1 can be applied to conclude that $M(X)$ is as required. ∎

Of course, a Suslin segment is a compact L-space, moreover Kunen constructed a compact L-space from CH in [19]. Consequently, both of these set-theoretic assumptions imply the conclusion of Theorem 4.9.

In addition to the non-compact space $M(X)$ constructed in Theorem 4.9, the "restored" compactum $\gamma X = M(X) \cup \{\omega_1\}$ is also noteworthy. Indeed, it is easy to check that γX is pseudoradial, moreover the point $\omega_1 \in \overline{X}$ is not in the closure of any discrete subset of X because those are all countable. Consequently we have the following result.

COROLLARY 4.10: *If there is a compact L-space, then there exists a pseudoradial compactum that is not discretely determined.*

In particular, both the existence of a Suslin line and CH imply the existence of such a compactum. The Suslin line case was proved in [2, 7.5] and the CH case in [4, Theorem 7]. The question if such a compactum exists in ZFC remains open.

Compact L-spaces are ccc, first countable, and non-separable, hence the following result is a partial strengthening of Theorem 4.9.

THEOREM 4.11: *If there is a ccc, first countable, and nonseparable compactum, then there is a normal, locally compact, first countable space which is ω-bounded but not ccc-bounded.*

Proof. Let X be any first countable compact ccc nonseparable space. We first apply Juhász [14, 2.25 and 2.26] to conclude that there is a closed ccc subset

Y of X which has density ω_1. We may assume without any loss of generality that Y is also nowhere separable and hence ω_1-short. So Theorem 4.1 can be applied to conclude that $M(\omega \times Y)$ is as required. ∎

Of course, if the compactum X above is zero-dimensional, then we could end up with the zero-dimensional space $M_0(\omega \times Y)$ instead of $M(\omega \times Y)$. This leads us to the following natural question that we could not answer.

Question 4.12: If there is a ccc, first countable, and nonseparable compactum, is there one which is also zero-dimensional?

The following corollary uses a set-theoretic assumption under which Theorem 4.11 applies but we do not know if a compact L-space exists, hence Theorem 4.9 may not apply. This makes use of Martin's Axiom for countable posets, abbreviated as MAC. It is well-known that MAC is equivalent with the equality $\mathfrak{m} = \mathfrak{c}$, where $\mathfrak{m}$ denotes the additivity of the meager ideal.

Corollary 4.13: *Assume* MAC *and* $\mathrm{cf}(\mathfrak{c}) = \omega_1$. *Then there is a normal, locally compact, first countable space which is ω-bounded but not ccc-bounded.*

Proof. Bell [3] used MAC to construct a compact ccc nonseparable space every point of which has character less than $\mathrm{cf}(\mathfrak{c})$. So if the cofinality of the continuum is ω_1 then his space is also first countable, hence Theorem 4.11 may be applied to it. ∎

Comparing Theorems 2.4 and 4.11 we see that the question whether a locally compact, first countable, and ω-bounded spaces is ccc-bounded, is undecidable in ZFC. We could not answer, however, the following related question.

Question 4.14: Does $\mathsf{MA}_{\aleph_1}$ imply that every first countable and ω-bounded space is ccc-bounded? Or equivalently, does $\mathsf{MA}_{\aleph_1}$ imply that every first countable, ccc, and ω-bounded space is separable?

Remark 4.15: It is worth noting however, that we can not weaken ω-bounded to countably compact in this question. Indeed, Soukup [25, Theorem 2.1] proved that if $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$, then every first countable space of cardinality less than $\mathfrak{c}$ can be densely embedded in a first countable and countably compact space. Let Z be any first countable, ccc, and nonseparable space of cardinality ω_1, for instance we may have $Z = F[X]$, the Pixley–Roy hyperspace that we considered in the proof of Theorem 4.7 (2). Note that MA+¬CH implies both $\mathsf{MA}_{\aleph_1}$ and

$\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$, so by Soukup's Theorem it also implies that Z can be densely embedded in a first countable and countably compact space Y. Then Y is ccc since Z is, and it is not separable because it is first countable and Z is not separable.

5. A graphic survey of what we know

In this section we present four diagrams that aim to give an overview of what we know about the interrelations of our various boundedness concepts in the classes of all (Tychonoff) spaces, all locally compact spaces, all first countable spaces, and all locally compact and first countable spaces, respectively.

On our diagrams, as usual, double arrows represent (mostly trivial) implications. The blunt tipped, dotted lines indicate negations of implications. The label attached to such a line tells the reader where the example establishing the appropriate non-implication can be found in the text. A label marked by a star indicates a result that needs additional set theoretical assumptions. Finally, question marks as labels indicate open problems for further research.

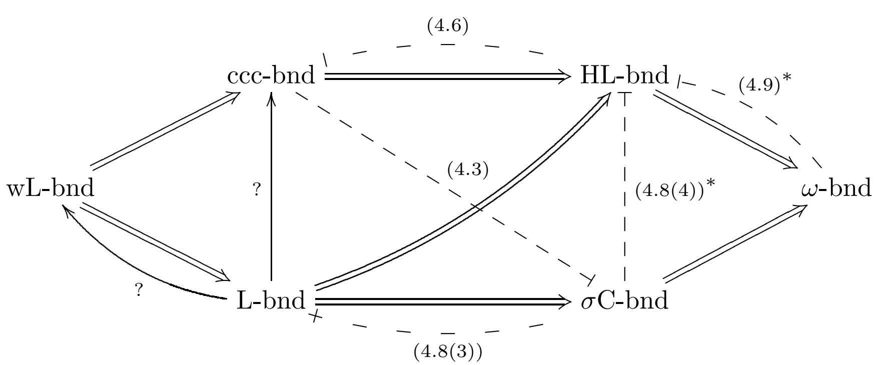

Diagram 1: All spaces

By Theorem 2.2, the diagram for the class of locally compact spaces is much simpler than Diagram 1.

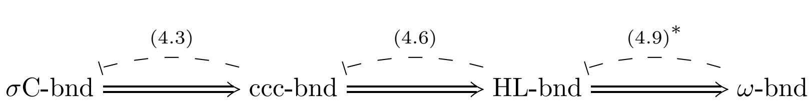

Diagram 2: All locally compact spaces

$$
\begin{tikzcd}
\sigma\text{C-bnd} \arrow[r, Rightarrow] & \text{ccc-bnd} \arrow[l, dashed, bend right, "(4.3)"', -|] \arrow[r, Rightarrow] & \text{HL-bnd} \arrow[l, bend right, "?"'] \arrow[r, Rightarrow] & \omega\text{-bnd} \arrow[l, dashed, bend right, "(4.9)^*"', -|] \arrow[ll, Rightarrow, bend left, "(2.4)^*"]
\end{tikzcd}
$$

Diagram 3: All locally compact and first countable spaces

By Theorem 2.4, diagram 3 has only two nodes under $\mathsf{MA}_{\aleph_1}$ but, by Theorem 4.9, at least three nodes if there is a compact L-space.

$$
\begin{tikzcd}
 & \text{ccc-bnd} \arrow[rr, Rightarrow] \arrow[ddrr, dashed, "(4.3)", -|] & & \text{HL-bnd} \arrow[ll, Rightarrow, bend right, "(2.5)^*"'] \arrow[dr, Rightarrow] & \\
\text{wL-bnd} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & & \omega\text{-bnd} \arrow[ul, Rightarrow, bend right, "(2.3)^*"'] \\
 & \text{L-bnd} \arrow[uu, "?"] \arrow[ul, bend left, "?"] \arrow[uurr, Rightarrow] \arrow[rr, Rightarrow] & & \sigma\text{C-bnd} \arrow[uu, "?"'] \arrow[ll, bend left, "?"] \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Diagram 4: All first countable spaces

Here we have more questions than answers.

6. Questions: what we do not know

Finally, we list in this section the problems that we tried to settle but could not. Most of these can be read off our diagrams.

In the light of the fact that our examples are normal and evidently not $\mathscr{N}$-bounded, where $\mathscr{N}$ means being normal, one is tempted to ask whether there is a first countable $\mathscr{N}$-bounded space which is not compact. But this question makes no sense and first countability is totally irrelevant since every noncompact space contains a discrete subspace whose closure is not compact (Tkachuk [27]).

Question 6.1: If a space is both L-bounded and ccc-bounded, is it also wL-bounded?

Question 6.2 (Diagram 1): Is every L-bounded space ccc-bounded? Is every L-bounded space wL-bounded?

Question 6.3 (Diagram 3): Is every locally compact, first countable, and HL-bounded space ccc-bounded?

Question 6.4 (Diagram 4): Are ω-bounded first countable spaces HL-bounded? Are σC-bounded first countable spaces L-bounded? Are L-bounded first countable spaces ccc-bounded? Are L-bounded first countable spaces wL-bounded?

Question 6.5: Is every first countable, ccc, σC-bounded space with a dense (hereditarily) Lindelöf subspace compact?

Question 6.6: If there is a compact, first countable, and ccc but nonseparable space, is there one which is also zero-dimensional?

Question 6.7: Are first countable ω-bounded spaces ccc-bounded under $\mathsf{MA}_{\aleph_1}$?

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