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On Countable Dense and n-homogeneity

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Abstract. We prove that a connected, countable dense homogeneous space is n-homogeneous for every n, and strongly 2-homogeneous provided it is locally connected. We also present an example of a connected and countable dense homogeneous space which is not strongly 2-homogeneous. This answers in the negative Problem 136 of Watson in the Open Problems in Topology Book.

1 Introduction

Unless otherwise stated, all spaces under discussion are Tychonoff.

Recall that a separable space X is *countable dense homogeneous* (CDH) if, given any two countable dense subsets D and E of X, there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$. The concept of CDH-ness obviously does not make sense for spaces that are not separable, therefore separability is included in the definition.

Bennett [1] proved that a connected first countable CDH-space is homogeneous. It was asked in Problem 136 of Watson [13] in the Open Problems in Topology Book whether every connected CDH-space is strongly 2-homogeneous. Observe that the real line $\mathbb{R}$ is an example of a space that is CDH but not strongly 3-homogeneous. We show that every connected CDH-space is n-homogeneous for every n, and strongly 2-homogeneous provided it is locally connected. Moreover, we construct an example of a connected Lindelöf CDH-space that is not strongly 2-homogeneous. This answers Watson's problem in the negative.

2 Preliminaries

Notation We use 'countable' for 'at most countable'. For a set X and $n \in \mathbb{N}$, $[X]^{<n}$ and $[X]^n$ denote $\{A \subseteq X : |A| < n\}$ and $\{A \subseteq X : |A| = n\}$, respectively. In addition, $[X]^{<\omega}$ abbreviates the collection of all finite subsets of X.

Homogeneity notions If X is a space, then $\mathcal{H}(X)$ denotes the group of homeomorphisms of X. If G is a subgroup of $\mathcal{H}(X)$ and if Y is a subspace of X, then by G^Y we denote the subgroup $\{g \in G : g(Y) = Y\}$ of G. Moreover, for every $F \subseteq X$ we let G_F denote the subgroup $\{ g \in G : (\forall\, x \in F)\big(g(x) = x\big) \}$. Hence G^Y_F denotes the subgroup of G consisting of all the elements of G that keep Y invariant and restrict to the identity on F. We do not requite that Y and F are related.

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If Y is a subspace of X, G is a subgroup of $\mathcal{H}(X)^Y$ and $y \in Y$, then by $\tau_G(y)$ we denote the *G-orbit* of y, *i.e.*, $\tau_G(y) = \{g(y) : g \in G\}$. Observe that $\tau_G(y)$ is a subset of Y.

Now we define the central concepts in this paper.

Definition 2.1 Let X be a separable space. We say that

1. X *ω-absorbing*, abbreviated ωA, if it has the following absorption property: for every countable dense subset D of X and every $x \in X$, there is a homeomorphism $f\colon X \to X$ such that $f(D \cup \{x\}) \subseteq D$;
2. X is *weakly countable dense homogeneous*, abbreviated wCDH, provided that for all finite $F \subseteq X$ and $D, E \subseteq X \setminus F$ countable and dense in X, then there is a homeomorphism $f\colon X \to X$ that restricts to the identity on F, while $f(D) \subseteq E$;
3. X is *countable dense homogeneous*, abbreviated CDH, provided that for all countable dense subsets D and E of X there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$;
4. X is *n-homogeneous*, where $n \geq 1$, if for all n-point subsets F and G of X, there is a homeomorphism $f\colon X \to X$ such that $f(F) = G$;
5. X is *strongly n-homogeneous*, where $n \geq 1$, if, given any two n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ of distinct points of X, there exists a homeomorphism g of X such that $g(x_i) = y_i$ for every $i \leq n$.

The concept of a wCDH-space may seem to be the less intuitive one. In [8, Proposition 3.1] (see also Proposition 4.1 below) it was shown that every CDH-space is wCDH. That is our basic tool in this paper. Hence it is clear that

$$\text{CDH} \implies \text{wCDH} \implies \omega\text{A}.$$

It is not known whether these implications are strict.

In contrast to CDH, it is easy to check that the properties wCDH and ωA are open hereditary for a restricted class of open subspaces and that is precisely why our proof works.

Let the group G act on the space X. We say that G *makes* X CDH provided that for all countable dense subsets D and E of X there is an element $g \in G$ such that $gD = E$. So, informally speaking, G witnesses the fact that X is CDH. In the considerations to come, X is quite often a subspace of a space Y, and G is a subgroup of $\mathcal{H}(Y)^X$. We use similar terminology for the concepts wCDH, ωA, n homogeneous, and strongly n-homogeneous.

Connectivity Let X be a connected space. A *segment* of X is a component of $X \setminus \{p\}$ for some $p \in X$.

3 Bennett's Theorem

For later use, we will prove a slightly stronger result than just the homogeneity that we are after in this section.

Theorem 3.1 *Suppose that X is a separable subspace of Y, and the subgroup G of $\mathcal{H}(Y)^X$ makes X ω-absorbing. Then for every $x \in X$, $\tau_G(x)$ is clopen in X.*

Proof Let $D \subseteq X$ be countable and dense. Fix an $x \in X$, and put $E = \tau_G(x) \cap D$.

Claim 1 $\overline{E} \subseteq \tau_G(x)$.

Proof Pick an arbitrary $p \in \overline{E}$, and put $D_0 = E \cup (D \setminus \overline{E})$. Then D_0 is clearly dense, and $\tau_G(x) \cap (D \setminus \overline{E}) = \varnothing$. There is by assumption an element $f_0 \in G$ such that $f_0(D_0 \cup \{p\}) \subseteq D_0$. Since $f_0\big(\tau_G(x)\big) = \tau_G(x)$ we get $f_0(E) \subseteq E$, hence $f_0(\overline{E}) \subseteq \overline{E}$. This means that $f_0(p) \in \overline{E} \cap D_0 = E \subseteq \tau_G(x)$, and so $p \in \tau_G(x)$. ■

Claim 2 $\overline{E}$ is not nowhere dense in X.

Proof Striving for a contradiction, assume that $\overline{E}$ is nowhere dense in X. Then $D_1 = D \setminus \overline{E}$ is dense. Observe that $D_1 \cap \tau_G(x) = \varnothing$. There is by assumption an element $f_1 \in G$ such that $f_1(D_1 \cup \{x\}) \subseteq D_1$. Hence $f_1(x) \in D_1 \cap \tau_G(x) = \varnothing$, which is a contradiction. ■

Let U be the interior of $\overline{E}$.

Claim 3 $\tau_G(x)$ is open.

Proof Let $D_2 = (D \cap U) \cup (D \setminus \overline{E})$ and observe that it is dense. Moreover, $(D \setminus \overline{E}) \cap \tau_G(x) = \varnothing$. Fix $p \in \tau_G(x)$. There is an element $f_2 \in G$ such that $f_2(D_2 \cup \{p\}) \subseteq D_2$. Hence $f_2(p) \in D_2 \cap \tau_G(x) = D \cap U$. We conclude that $p \in f_2^{-1}(U) \subseteq \tau_G(x)$. Hence $\tau_G(x)$ is a neighborhood of p. ■

Since the collection $\{\tau_G(x) : x \in X\}$ partitions X, it follows from the previous claim that every $\tau_G(x)$ is clopen. ■

Corollary 3.2 *Every connected ω*A*-space is homogeneous.*

It consequently follows that every connected CDH-space is homogeneous, a result which is due to Fitzpatrick and Lauer [4].

Corollary 3.3 *Let X be a space without isolated points. Assume that the group G makes X* wCDH. *Then for every finite subset $F \subseteq X$, every G_F-invariant subset of $X \setminus F$ is open.*

Proof Observe that G_F makes $X \setminus F$ ω-absorbing. Hence we are done by Theorem 3.1. ■

This leads us to the following result, which generalizes van Mill [8, Theorem 1.2] where the same result was proved for separable metrizable spaces.

Theorem 3.4 *If the group G makes the infinite space X* wCDH *and no set of size $n-1$ separates X, then G makes X strongly n-homogeneous.*

Proof All we need to show is that for every subset F of size $n-1$, the group G_F acts transitively on $X \setminus F$. By Corollary 3.3 every orbit $G_F x$ for $x \in X \setminus F$ is open. Since orbits are disjoint, they are clopen. So we are done by connectivity. ■

4 Tools

Our basic tool in this paper is the following result [8, Proposition 3.1] of which we include the simple proof for the sake of completeness.

Proposition 4.1 *Let the group G make X CDH. If $F \subseteq X$ is finite and $D, E \subseteq X \setminus F$ are countable and dense in X, then there is an element $f \in G_F$ such that $f(D) \subseteq E$.*

Proof Let h_0 be an arbitrary element in G. Suppose $\{h_\beta : \beta < \alpha\} \subseteq G$ have been constructed for some $\alpha < \omega_1$. Now by CDH, pick $h_\alpha \in G$ such that

$$(\dagger) \qquad h_\alpha(F \cup E) = \bigcup_{\beta<\alpha} h_\beta(D).$$

For $1 \leq \alpha < \omega_1$, let T_α be a nonempty finite subset of $[1, \alpha)$ such that $h_\alpha(F) \subseteq \bigcup_{\beta\in T_\alpha} h_\beta(D)$. By the Pressing Down Lemma, for the function $T \colon [1, \omega_1) \to [\omega_1]^{<\omega}$ defined by $T(\alpha) = T_\alpha$, the fiber $B = T^{-1}(A)$ is uncountable for some $A \in [\omega_1]^{<\omega}$. Then $h_\alpha(F) \subseteq \bigcup_{\beta\in A} h_\beta(D)$ for every $\alpha \in B$. Since $\bigcup_{\beta\in A} h_\beta(D)$ is countable, and B is uncountable, we may consequently assume without loss of generality that $h_\alpha \restriction F = h_\beta \restriction F$ for all $\alpha, \beta \in B$. Hence if $\alpha, \beta \in B$ are such that $\beta < \alpha$, then $h_\alpha \restriction F = h_\beta \restriction F$ and by $(\dagger)$, $(h_\alpha^{-1} \circ h_\beta)(D) \subseteq E$. ■

In [11], Ungar proved that for a locally compact separable metrizable space in which no finite set separates, countable dense homogeneity is equivalent to (strong) n-homogeneity for every n. He claimed that for such spaces, every dense open set is CDH. However, his argument is incomplete and whether it is true is still open. For Polish spaces it is not true: there is by van Mill [7] an example of a connected Polish CDH-space with a dense rigid connected open subset. Proposition 4.1 allows us to prove in Proposition 4.2 below that certain open subspaces of wCDH-spaces are wCDH. Although it seems to be a rather weak result, it is precisely what we will need later on.

Proposition 4.2 *Let the group G make X wCDH. If U is connected, and $F = \overline{U} \setminus U$ is finite, then G_F^U makes U wCDH.*

Proof Let $A \subseteq U$ be finite, and pick two countable dense subsets D and E in $U \setminus A$. By connectivity, we may assume that $A \neq \varnothing$. Since X is separable, we may select a countable dense subset B of the open set $X \setminus \overline{U}$. There is an element $f \in G_{A\cup F}$ such that $f(B \cup D) \subseteq B \cup E$. We claim that $f(U) = U$, and hence that $f(D) \subseteq E$. First observe that $f(U)$ does not intersect F since f is a homeomorphism and restricts to the identity on F. Hence if $f(U)$ would intersect $X \setminus U$, then it would be contained in $U \cup (X \setminus \overline{U})$ and since $A \neq \varnothing$ it would intersect both U and $X \setminus \overline{U}$; but this would contradict the connectivity of $f(U)$. Similarly for f^{-1}. Hence indeed $f(U) = U$, and we are done. ■

We will now continue by proving two results about segments in wCDH-spaces.

Proposition 4.3 *Let X be nontrivial, connected and wCDH. Then every segment of X is open.*

Proof Let C be an arbitrary component of $X\setminus\{p\}$ for certain $p\in X$. Then $T=X\setminus C$ is connected, by Kok [6, Lemma 9, p. 10]. Hence if C is a singleton, then $X\setminus\{p\}$ is connected, by homogeneity (Corollary 3.2). So we may assume without loss of generality that every component of $X\setminus\{p\}$ is nontrivial.

Let C be an arbitrary component of $X\setminus\{p\}$. Moreover, let U be the possibly empty interior of C, and take an arbitrary element $x\in C$. By the above we may pick an element $y\in C\setminus\{x\}$. Let D be any countable dense subset of X. Then

$$D_0=\big(D\cap(U\setminus\{y\})\big)\cup\big(D\setminus(\{p\}\cup C)\big)$$

is dense, as well as $D_1=D_0\cup\{x\}$. By Proposition 4.1, there is a homeomorphism $f\colon X\to X$ that restricts to the identity on $\{p,y\}$, while moreover $f(D_1)\subseteq D_0$. Clearly, $f(C)=C$, hence $f(x)\in U$. But this implies that $x\in f^{-1}(U)\subseteq C$. Hence C is open. ■

Proposition 4.4 *Let X be nontrivial, connected and* wCDH. *If $p\in X$, then $X\setminus\{p\}$ has at most two components.*

Proof Striving for a contradiction, assume that we can split $X\setminus\{p\}$ into three pairwise disjoint nonempty open sets, say U, V and W. Pick arbitrary elements $u\in U$, $v\in V$ and $w\in W$. Let $D\subseteq X\setminus\{p,u,v,w\}$ be countable and dense. There is a homeomorphism $f\colon X\to X$ such that f restricts to the identity on $\{u,v,w\}$ and $f(D\cup\{p\})\subseteq D$. We may assume without loss of generality that $f(p)\in V$ and $f^{-1}(p)\in V\cup W$. Let C_u be the component of $X\setminus\{p\}$ containing u. Then C_u is an open subset of U by Proposition 4.3 and hence $p\in\overline{C_u}$ by connectivity of X. Hence $f(p)\in\overline{f(C_u)}$ and so $V\cap f(C_u)\neq\varnothing$. Since $p\notin f(C_u)$ and $u\in f(C_u)$ we conclude that $f(C_u)$ is a connected set that intersects both U and V but does not contain p. But this is impossible. ■

This leads to the following.

Proposition 4.5 *Let X be nontrivial, connected and* wCDH. *Then for every finite subset $F\subseteq X$, every component of $X\setminus F$ is open.*

Proof If $|F|=1$, then there is nothing to prove by Proposition 4.3. So assume the result is true for all finite subsets of size $n\geq 1$ of all nontrivial connected wCDH-spaces, and assume that F has size $n{+}1$. Pick an arbitrary $p\in F$. Let U be an arbitrary component of $X\setminus\{p\}$. Then U is open by Proposition 4.3 and wCDH by Proposition 4.2. Hence every component of $U\setminus F$ is open by our inductive hypothesis. And all these components are components of $X\setminus F$. Let C be a component of $X\setminus F$ that intersects U. Then $C\subseteq U$ since $p\notin C$. Hence C is a component of $U\setminus F$. Since U was arbitrary, we consequently conclude that all components of $X\setminus F$ are open. ■

5 Proof of the Main Result

The aim of this section is to prove the following result.

Theorem 5.1 *Let X be a nontrivial connected space. If the group G makes X* wCDH, *then G makes X n-homogeneous for every n.*

For the remaining part of this section, let X be a nontrivial connected space and let the group G make X wCDH. That G makes X 1-homogeneous was proved in Theorem 3.4. So assume that for $n \geq 1$, every group H that makes the nontrivial connected space Y wCDH, makes Y n-homogeneous. Our task is to show that G makes X $(n+1)$-homogeneous.

By Proposition 4.4 and Corollary 3.2 there are two cases to consider. The first case is that $X \setminus \{p\}$ is connected for some (equivalently: for all) $p \in X$. A moment's reflection shows that all we need to prove is that for a given $p \in X$, the group $G_{\{p\}}$ makes $X \setminus \{p\}$ n-homogeneous. But this is clear from Proposition 4.2 and our inductive hypothesis. So we may assume that $X \setminus \{p\}$ has exactly two components for some (equivalently: every) $p \in X$. Hence if X were locally connected, then by Ward [12] we would get that X is homeomorphic to the real line $\mathbb{R}$ and one could then try to use its properties to complete the proof. However, it is unknown whether a Polish connected CDH-space is locally connected (Fitzpatrick and Zhou [5, Problem 386]), hence this approach does not seem to work.

Lemma 5.2 *Let $U \subseteq X$ be nonempty, open and connected, and assume that $F = \overline{U} \setminus U$ is finite. Then the group G_F^U makes U wCDH, and if $p \in U$, then $U \setminus \{p\}$ has precisely two components. Moreover, every segment in U is open in X.*

Proof Assume that $V = U \setminus \{p\}$ is connected. Since $\overline{U} = \overline{V} \cup \overline{\{p\}} = \overline{V} \cup \{p\}$, it follows that $F \subseteq \overline{V}$. Now let W be any component of $X \setminus F$ that misses U. Then W is open by Proposition 4.5, hence $\overline{W} \cap F \neq \varnothing$ by connectivity. This clearly implies that $X \setminus \{p\}$ is connected, which is a contradiction. Hence $U \setminus \{p\}$ is disconnected, and so we are done by Propositions 4.3 and 4.2. ■

It will be convenient to introduce the following notation. If $p \in X$, then C_p^0 and C_p^1 denote the components of $X \setminus \{p\}$. Observe that they are open subsets of X.

The following result is the first step in the proof of Ward's Theorem A in [12]. Its proof is repeated here for the sake of completeness. If $i \in \{0, 1\}$, then $\bar{\imath}$ denotes $1+i$ (mod 2).

Lemma 5.3 *If x and y are any two distinct points in X, and assume that $x \in C_y^j$ and $y \in C_x^i$ for certain $i, j \in \{0, 1\}$. Then $C_x^{\bar{\imath}} \cap C_y^{\bar{\jmath}} = \varnothing$.*

Proof It is clear that the connectivity of X gives us that $\overline{C_x^i} = C_x^i \cup \{x\}$ for $i \in \{0, 1\}$. Suppose that $C_x^{\bar{\imath}} \cap C_y^{\bar{\jmath}} \neq \varnothing$. Then

$$X \setminus \{y\} = C_y^0 \cup C_y^1 = C_y^0 \cup C_y^1 \cup \overline{C_x^{\bar{\imath}}}$$

is connected, which is a contradiction. ■

This leads us to the following.

Proposition 5.4 *If x, y and z are any three distinct points of X, there is one of them which separates the other two.*

Proof By Lemma 5.3 we may assume without loss of generality that $x \in C_y^0$, $y \in C_x^1$ and $C_x^0 \cap C_1^y = \varnothing$. Consider the set C_x^1 which contains C_y^1. We claim that C_y^1 is a component of $X \setminus \{x, y\}$. For assume that C is a connected set in $X \setminus \{x, y\}$ that contains C_y^1. If it intersects $X \setminus C_y^0$ then $X \setminus \{y\} = C_y^0 \cup C$ is connected which contradicts our assumptions. Hence C_y^1 is a component of $X \setminus \{x, y\}$ and hence also of $C_x^1 \setminus \{y\}$. There is another component of $C_x^1 \setminus \{y\}$ which consequently has to be equal to $E = C_y^0 \setminus (C_x^0 \cup \{x\}) = C_x^1 \setminus (C_y^1 \cup \{y\})$ (Lemma 5.2). It follows similarly that C_x^0 is a component of $X \setminus \{x, y\}$ as well as C_y^0. So we conclude that the components of $X \setminus \{x, y\}$ are C_x^0, E and C_y^1. Hence there are three cases to consider.

Case 1 $z \in C_x^0$. Then x separates z and y.

Case 2 $z \in C_y^1$. Then y separates x and z.

Case 3 $z \in E$. Let A and B be the components of $E \setminus \{z\}$ (Lemma 5.2). Suppose first that $\overline{A} \cap \{x, y\} = \varnothing$. We claim that $\{x, y\} \subseteq \overline{B}$. For assume that, *e.g.*, $x \notin \overline{B}$. Then the partition

$$C_x^0 \cup \{x\}, \quad E \cup \{y\} \cup C_y^1$$

contradicts the connectivity of X. Pick arbitrary points $u \in C_x^0$, $v \in C_y^1$ and $w \in A$, and let $D \subseteq X \setminus \{u, v, w, x, y, z\}$ be a countable dense set. There is by assumption a homeomorphism $f\colon X \to X$ such that f restricts to the identity on $\{u, v, w, y, z\}$ and $f(D \cup \{x\}) \subseteq D$. It is clear that $f(A) = A$ and $f(C_y^1) = C_y^1$. Hence $f(x) \in C_x^0 \cup E$. Assume first that $p = f(x) \in E$. Observe that $f(C_x^0 \cup \{x\})$ is a connected set that is contained in $C_x^0 \cup \{x\} \cup E$ and meets both C_x^0 and E. As a consequence, it contains x. Hence we conclude that $f(E)$ is a connected subset of $C_x^0 \cup \{x\} \cup E$ that contains z, and does not contain x. Hence we conclude that $f(E) \subseteq E$. Hence the components of $E \setminus \{p\}$ are $K = f(C_x^0) \cap E$ and $L = f(E)$.

Observe that the boundary of K consists of two elements, namely, x and p. Moreover, the boundary of $f(E)$ is the set $\{p, y\}$. By Lemma 5.2 and Theorem 3.4, there is a homeomorphism $g\colon E \cup \{x, y\} \to E \cup \{x, y\}$ which restricts to the identity on $\{x, y\}$ and has the property that $g(z) = p$. But the only boundary point of A is z, hence the only boundary point of $g(A)$ is p, which clearly is a contradiction since we know what the components are of $E \setminus \{p\}$. If $p \in C_x^0$, then $q = f^{-1}(x) \in E$ and we can reach a contradiction by a similar reasoning.

Hence we conclude that $\overline{A} \cap \{x, y\} \neq \varnothing$, and similarly for B. Suppose that, *e.g.*, $\{x, y\} \subseteq \overline{A}$. Since $\overline{B} \cap \{x, y\} \neq \varnothing$, this means that $X \setminus \{z\}$ is connected, which is a contradiction. Hence we may assume without loss of generality that $\overline{A} \cap \{x, y\} = \{x\}$, and $\overline{B} \cap \{x, y\} = \{y\}$. But then both $C_x^0 \cup \{x\} \cup A$ and $B \cup \{y\} \cup C_y^1$ are connected, *i.e.*, z separates x and y. ■

It follows now by Ward [12] that X admits a weaker orderable topology, and so by separability, X with this topology can be identified with a subinterval J of $\mathbb{R}$ (see also Kok [6, Theorem 3, p. 16]). Moreover, X and J have the same connected sets by Kok [6, Theorem 3, p. 5]. Hence, for example by homogeneity, J does not have endpoints, and so we may assume without loss of generality that $J = \mathbb{R}$.

Now let A and B be subsets of X of size $n+1$. Let a_0 and b_0 be the minima of A and B. Take a point $s < \min\{a_0, b_0\}$. There is by Lemma 5.2 (take $U = (s, \infty)$) a homeomorphism $f\colon X \to X$ such that $f(s) = s$ and $f(a_0) = b_0$. Then f is clearly order preserving. Hence

$$f(A \setminus \{a_0\}) \cup (B \setminus \{b_0\})$$

is contained in the interval $[b_0, \rightarrow)$. By our inductive assumption and Lemma 5.2, there is a homeomorphism $g\colon X \to X$ such that $g(b_0) = b_0$ and $g\big(f(A \setminus \{a_0\})\big) = B \setminus \{b_0\}$. Hence if $h = g \circ f$, then h is a homeomorphism of X such that $h(A) = B$.

Corollary 5.5 *Every nontrivial connected and locally connected* wCDH-*space is strongly* 2*-homogeneous.*

Proof Let X be as in the corollary. If $X \setminus \{p\}$ is connected for some $p \in X$, then we are done by Theorem 3.4. If not, then by Ward [12] we arrive at the conclusion that X is homeomorphic to $\mathbb{R}$, and hence there is nothing to prove. ■

6 The Example

In this section we will present an example of a connected CDH-space that is not strongly 2-homogeneous. Our space is of course Tychonoff, but not metrizable. It may be possible to construct a separable metrizable space with similar properties using the methods of Saltsman [9], [10]. However, his methods need the Continuum Hypothesis, while our result requires no additional set theoretic assumptions.

As usual, λ denotes Lebesgue measure on $\mathbb{R}$. For every $x \in \mathbb{R}$ we will define a certain collection of subsets $\mathcal{F}_x$ of $(\leftarrow, x)$, as follows: $F \in \mathcal{F}_x$ iff F is closed in $(\leftarrow, x)$, and

$$\sum_{n=0}^{\infty} 2^n \lambda([x-2^{-n}, x-2^{-n-1}] \cap F) < \infty.$$

Observe that $\mathcal{F}_x$ is closed under finite unions and contains all closed subsets of $(\leftarrow, x)$ of measure 0. Topologize $\mathbb{R}$ as follows: a basic neighborhood of $x \in \mathbb{R}$ has the form $U \setminus F$, where U is an open subset of $\mathbb{R}$ containing x, and $F \in \mathcal{F}_x$. Let $\mathcal{B}(x)$ denote all sets of this form. We will prove that $\mathcal{B}(x)$ is a neighborhood system for x, and the space with the topology τ generated by these neighborhood systems will be denoted by X. Clearly, τ is stronger than the euclidean topology on $\mathbb{R}$.

To this end, observe that $\mathcal{B}(x)$ is closed under finite intersections since $\mathcal{F}_x$ is closed under finite unions. Moreover, let $U \setminus F \in \mathcal{B}(x)$, where $U \subseteq \mathbb{R}$ is open and $F \in \mathcal{F}_x$. Assume that $y \in U \setminus F$. We will prove that there is an element $B \in \mathcal{B}_y$ such that $B \subseteq U \setminus F$. If $y = x$, then there is nothing to prove. Moreover, if $y \neq x$, then y is in the euclidean interior of $U \setminus F$, which is open in X. Hence again there is nothing to prove.

Lemma 6.1 *X is regular and Lindelöf.*

Proof Let $x \in X$, $U \subseteq \mathbb{R}$ open such that $x \in U$, and $F \in \mathcal{F}_x$. There is an open neighborhood A of F in $(\leftarrow, x)$ such that the closure G of A in $(\leftarrow, x)$ belongs to $\mathcal{F}_x$. Let V be an open neighborhood of x in $\mathbb{R}$ such that $\overline{V} \subseteq U$, and consider $W = V \setminus G$.

We claim that the closure of W in X is contained in $U \setminus F$. To check this, let p be an arbitrary element of that closure. Clearly, $p \in \overline{V} \subseteq U$. Assume that $p \in F$. Then A is a neighborhood of p in $\mathbb{R}$ and hence in X which misses W which is a contradiction. Hence $p \in U \setminus F$. Hence X is regular.

To see that X is Lindelöf, it suffices to observe that the topology on X is weaker than the Sorgenfrey topology on $\mathbb{R}$, which is Lindelöf [3, 3.8.14]. ■

We conclude from this that X is normal, and hence Tychonoff [3, 3.8.2].

Lemma 6.2 *X is* CDH.

Proof Let D and E be any two countable dense subsets of X. Then D and E are countable dense subsets of $\mathbb{R}$, and hence by Zamora Avilés [14] (see also [2]), there is a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ having the following properties:

(1) $f(D) = E$,

(2) for all distinct $x, y \in \mathbb{R}$, $1/2 \leq \dfrac{|f(x) - f(y)|}{|x - y|} \leq 2$.

This implies that for every measurable subset S of $\mathbb{R}$ we have that

$$1/2\lambda(S) \leq \lambda\big(f(S)\big) \leq 2\lambda(S).$$

Hence $f\colon X \to X$ is a homeomorphism as well since it maps for every $x \in X$ every element of $\mathcal{F}_x$ onto an element of $\mathcal{F}_{f(x)}$, *etc.* ■

Lemma 6.3 *X is connected but not strongly* 2*-homogeneous.*

Proof The proof that X is connected follows the same pattern as the standard proof that $\mathbb{R}$ is connected. Assume that X can be written as $U \cup V$, where U and V are nonempty disjoint open sets. Pick arbitrary $a \in U$ and $b \in V$. We may assume without loss of generality that $a < b$; let $c = \inf(V \cap [a, b])$. Assume that $c \in V$. Then $a < c$ and since V is open and any open neighborhood of c contains points that are strictly smaller than c and belong to $[a, b]$, this contradicts $c = \inf(V \cap [a, b])$. So $c \in U$, and hence $c < b$. Since U is open, there are $x, y \in \mathbb{R}$ and $F \in \mathcal{F}_c$ such that $x < c < y < b$ and $(x, y) \setminus F \subseteq U$. Hence $(c, y) \cap V = \varnothing$, and this again contradicts $c = \inf(V \cap [a, b])$.

Since the identity $X \to \mathbb{R}$ is a bijection, this implies that X and $\mathbb{R}$ have the same connected sets by Kok [6, Theorem 3, p. 5].

It will be convenient for every $x \in \mathbb{R}$ to denote $(\leftarrow, x)$ and $(x, \rightarrow)$ by L_x and R_x, respectively.

Take $p, q \in \mathbb{R}$ such that $p < q$. We claim that there does not exist a homeomorphism $f\colon X \to X$ such that $f(p) = q$ and $f(q) = p$. Striving for a contradiction, assume that such a homeomorphism f exists. Since X and $\mathbb{R}$ have the same connected sets, a moments reflection shows that $f(R_q \cup \{q\}) = L_p \cup \{p\}$. There is a sequence $(q_n)_n$ in R_q such that $q_n \to q$. Hence $\big(f(q_n)\big)_n$ is a sequence in L_p such that $f(q_n) \to p$. But this is clearly impossible since no sequence in L_p converges to p, being of measure 0. ■

Question 6.4 Is there a separable and metrizable connected space X which is CDH but not strongly 2-homogeneous?

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