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On topological groups with a first-countable remainder, III

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Abstract

We prove a general theorem that allows us to conclude that under CH, the free topological group over a nontrivial convergent sequence S has a first-countable remainder. It is also shown that any separable non-metrizable topological group with a first-countable remainder is Rajkov complete.

Keywords: Character; Remainder; Compactification; Topological group; First-countable; Metrizable; Rajkov complete; Continuum hypothesis

1. Introduction

In a series of papers, Arhangel'skii studied topological spaces having a compactification with a first-countable remainder. Specific attention was paid to topological groups that belong to this class. For details, and references, see e.g., [1–4].

Recently, the authors continued this study in [7,8] and obtained among other things the following results: a topological group with a first-countable remainder has character at most ω_1,

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has weight at most 2^ω, and is metrizable in case it is pre-compact. It was shown that moreover there exists a non-metrizable topological group with a first-countable remainder in ZFC, and a countable such space under CH (it was known by Arhangel'skii [2] that such a space cannot be countable under $\mathsf{MA} + \neg\mathsf{CH}$).

In this paper we continue these investigations. We prove a general theorem about spaces having a compactification with first-countable remainder that allows us to conclude that under CH, the free topological group over a nontrivial convergent sequence has a first-countable remainder. We also solve a problem in Juhász, van Mill and Weiss [14] in the negative. Spaces that have a strongly ω-bounded remainder play an important role in our investigations. We prove that a separable topological group with an ω-bounded remainder is Rajkov complete; this implies that any separable non-metrizable topological group with a first-countable remainder is Rajkov complete.

2. Preliminaries

Let $\mathscr{P}$ be any topological property. It is natural to call a space $\mathscr{P}$-bounded if every subset with property $\mathscr{P}$ has compact closure. In particular, if $P \equiv$ 'countable' then we obtain the well-known class of *ω-bounded spaces*, and if $\mathscr{P} \equiv$ 'σ-compact' then we obtain the class that is called *strongly ω-bounded* in Nyikos [16]. For other concepts that are in the same spirit, see e.g. Juhász, van Mill and Weiss [14].

A space X has *countable type* if every compact subspace of X is contained in a compact subspace of X which has countable character in X. By a well-known result in Henriksen and Isbell [13], a space X is of countable type if and only if the remainder in any (or in some) compactification of X is Lindelöf.

If X is a space, then βX denotes its Čech–Stone compactification, and $X^* = \beta X \setminus X$. It follows by standard methods that a topological space X has an ω-bounded remainder if and only if every remainder of it is ω-bounded. Similarly for strongly ω-bounded. These well-known facts are left as exercises to the reader. A point $p \in X^*$ is said to be a *remote point* of X if $p \notin \overline{D}$, where D is any nowhere dense subset of X (here closure means closure in βX).

For all undefined notions, see Engelking [11]. For information on topological groups, see Arhangel'skii and Tkachenko [6].

3. The theorem

It is a well-known result by Franklin and Rajagopalan [12] that the space $X = \omega \cup \{p\}$, where $p \in \omega^*$ is a P-point, has a compactification bX such that $bX \setminus X$ is $W(\omega_1)$, the space of all countable ordinal numbers. Hence $bX \setminus X$ is first countable. We will generalize this result to Theorem 3.1 below, which is our main tool for constructing spaces with a first-countable remainder.

Theorem 3.1 (CH). *Let X be a strongly ω-bounded space of countable type with a compactification bX such that $w(bX) \le 2^\omega$. Then there are a compact space Z and a continuous surjection $f: bX \to Z$ having the following properties:*

(1) *The restriction of f to $Y = bX \setminus X$ is a homeomorphism from Y onto $f(Y)$,*
(2) *$f(Y) \cap f(X) = \emptyset$,*
(3) *Z is first-countable at all points of $f(X)$.*

Proof. We may assume that $bX \subseteq [\frac{1}{2}, 1]^{\omega_1}$. For every $\alpha < \omega_1$, let $\pi_\alpha: \mathbb{I}^{\omega_1} \to \mathbb{I}$ be the projection onto the α-th coordinate, and let $p_\alpha: X \to \mathbb{I}$ be the restriction of π_α to X.

Let $\mathscr{E}$ be the family of all compact subspaces F of X such that $\chi(F, X) \leq \omega$. Observe that if $F \in \mathscr{E}$ then $\chi(F, bX) \leq \omega$ since X is dense in bX. The next three statements are obvious from our assumptions.

(C_1) $|\mathscr{E}| \leq \omega_1$.
(C_2) $\bigcup \mathscr{E} = X$.
(C_3) There exists an increasing ω_1-sequence $\{F_\alpha : \alpha < \omega_1\}$ of members of $\mathscr{E}$ such that $\bigcup_{\alpha<\omega_1} F_\alpha = X$.

For each $\alpha < \omega_1$ we fix a decreasing sequence $\{W_{\alpha,n} : n < \omega\}$ of open neighborhoods of F_α in bX which is a base at F_α in bX. Fix a continuous function $g_{\alpha,n}: bX \to \mathbb{I}$ satisfying the following conditions for $\alpha < \omega_1$ and $n < \omega$:

(s_1) $g_{\alpha,n}(F_\alpha) = \{0\}$.
(s_2) $g_{\alpha,n}(bX \setminus W_{\alpha,n}) = \{1\}$.

For every $\alpha < \omega_1$, put

$$\mathscr{H}_\alpha = \{p_\alpha \cdot g_{\alpha,n} : n < \omega\}$$

and let

$$\mathscr{H} = \bigcup_{\alpha<\omega_1} \mathscr{H}_\alpha.$$

Let $f: bX \to \mathbb{I}^{\mathscr{H}}$ denote the diagonal map of the family $\mathscr{H}$. That is, $f(p)_h = h(p)$ for every $p \in bX$ and $h \in \mathscr{H}$. We claim that $Z = f(bX)$ and f are as required.

Claim 1. *For every $\alpha < \omega_1$, $f(F_\alpha)$ is metrizable and $f^{-1}(f(F_\alpha)) = F_\alpha$.*

Pick $g \in \mathscr{H}_\beta$ for some $\beta > \alpha$. Then for some n, $g = p_\beta \cdot g_{\beta,n}$. But then $g(F_\alpha) = \{0\}$ since $g_{\beta,n}$ is identically 0 on F_α. So we conclude that

$$f(F_\alpha) \subseteq \left\{ q \in \mathbb{I}^{\mathscr{H}} : q_h = 0 \text{ for all } h \in \bigcup_{\beta>\alpha} \mathscr{H}_\beta \right\}.$$

Hence $f(F_\alpha)$ is metrizable since $\bigcup_{\beta\leq\alpha} \mathscr{H}_\beta$ is countable.

We will now show that $f^{-1}(f(F_\alpha)) = F_\alpha$. To prove this, pick an arbitrary $p \in bX \setminus F_\alpha$, and let $n < \omega$ be so large that $p \notin W_{\alpha,n}$. Put $h = p_\alpha \cdot g_{\alpha,n}$. Then

$$f(p)_h = h(p) = p_\alpha \cdot g_{\alpha,n}(p) = p_\alpha(p) \geq \frac{1}{2},$$

and

$$f(x)_h = h(x) = p_\alpha \cdot g_{\alpha,n}(x) = 0$$

for every $x \in F_\alpha$. This completes the proof of the claim.

Hence for every $\alpha < \omega_1$ we have that $f(F_\alpha)$ is a metrizable closed G_δ-subset of the compact space Z. Hence Z is first-countable at all points of $f(X)$.

It also follows from the claim that $f(Y) \cap f(X) = \emptyset$.

We now claim that $f \restriction Y$ is one-to-one. Indeed, if $y(0)$ and $y(1)$ are distinct elements of Y, then we may pick $\alpha < \omega_1$ such that $y(0)_\alpha \neq y(1)_\alpha$. Fix n so that $y(0), y(1) \notin W_{\alpha,n}$. Then

$$p_\alpha \cdot g_{\alpha,n}(y(0)) = p_\alpha(y(0)) = y(0)_\alpha$$

and, similarly, $p_\alpha \cdot g_{\alpha,n}(y(1)) = y(1)_\alpha$, so we are done.

From this we get that for every $y \in Y$ we have $f^{-1}(f(y)) = \{y\}$. Hence f restricts to a homeomorphism on Y. □

Observe that in order to be in a position to apply Theorem 3.1, we need a nowhere locally compact space X which has a compactification bX such that $w(bX) \leq 2^\omega$ while moreover $bX \setminus X$ is strongly ω-bounded and of countable type. Then Theorem 3.1 tells us that $bX \setminus X$ can be replaced by a first-countable remainder.

Corollary 3.2 (*CH*). *If Y is a nowhere locally compact Lindelöf space with a strongly ω-bounded remainder and $w(Y) \leq 2^\omega$, then Y has a compactification with a first-countable remainder. Even more is true: for every compactification of Y, there exists a smaller compactification with a first-countable remainder.*

Proof. Simply use the fact that $\beta Y \setminus Y$ has countable type by the Henriksen and Isbell Theorem from [13]. □

Corollary 3.3 (*CH*). *Every strongly ω-bounded space X of countable type such that $w(X) \leq 2^\omega$ can be mapped onto a first-countable space R by a perfect mapping (then automatically R is strongly ω-bounded and $w(R) \leq 2^\omega$).*

We will now discuss the assumptions in Theorem 3.1. We will first show that CH is essential.

Let Seq denote the set of all finite sequences of elements from ω. Moreover, let p be a free ultrafilter on ω. Define a topology $\mathscr{T}$ on Seq by the rule: $V \subseteq \text{Seq}$ is open iff for every $s \in V$, the set $\{n < \omega : s^\frown n \in V\} \in p$. Here $s^\frown n$ denotes the concatenation of s by n. It is easy to verify that Seq with this topology is Tychonoff, zero-dimensional, perfect and extremally disconnected. For details and variations, see Arhangel'skii and Franklin [5] and Dow, Gubbi and Szymański [10].

Let $[\omega]^{<\omega}$ be the set of all finite subsets of ω. Clearly, the symmetric difference operator $\triangle$ makes $[\omega]^{<\omega}$ a Boolean group. For a free ultrafilter p on ω, define a topology $\mathscr{T}_p$ on $[\omega]^{<\omega}$ as follows:

$$U \in \mathscr{T}_p \iff (\forall F \in U)(\{n < \omega : F \triangle n \in U\} \in p).$$

It is not difficult to see that $\mathscr{T}_p$ is extremally disconnected.

This topology is due to Louveau [15] who proved that if p is selective, then $\mathscr{T}_p$ is compatible with the group structure on $[\omega]^{<\omega}$. The topological group thus obtained is denoted by $L(p)$ and hence is an example of a non-discrete extremally disconnected topological group.

It was shown by Vaughan [23] that $L(p)$ and Seq (for the same ultrafilter p) are homeomorphic.

Theorem 3.4. *If p is a selective ultrafilter on ω of character greater than ω_1, then Seq is a countable non-discrete topological group whose Čech–Stone remainder is strongly ω-bounded but none of its remainders is first-countable.*

Proof. That Seq* is strongly ω-bounded was proved in [10, Remark 1]. It is easy to see that the character of Seq is greater than ω_1. Just observe that the space $\omega \cup \{p\}$ is a subspace of Seq. Hence Seq does not have a first-countable remainder since any topological group with a first-countable remainder has character ω_1 by Arhangel'skii and van Mill [7, Theorem 2.1]. □

Hence in Theorem 3.1 CH is indeed essential since for example under MA + ¬CH there exist selective ultrafilters on ω and they have character $2^\omega > \omega_1$ [18].

The question of whether the assumption on strong ω-boundedness is essential in Theorem 3.1 is very natural. We will answer it in the negative by using a powerful recent result of Dow [9]. For basic facts on Čech–Stone compactifications, see van Mill [22].

Theorem 3.5 ($\diamond$). *There are a σ-compact nowhere locally compact space X and a compactification bX of X such that $bX \setminus X$ is ω-bounded (and clearly of countable type) while moreover every compactification cX of X such that $cX \leq bX$ has the property that $cX \setminus X$ is not first-countable.*

Proof. Put $K = \omega \times \omega^*$, and for every n, let $K_n = \{n\} \times \omega^*$. Dow [9] recently proved that under $\diamond$, K has a remote point p which is simultaneously a P-point of K^*. Put $S = \beta K \setminus \{p\}$. Since countable subsets of K are nowhere dense, it clearly follows that S is ω-bounded. It is not strongly ω-bounded though since it is not compact and contains a dense σ-compact subspace. We now put $X = (\beta K)^\omega \setminus S^\omega$ and $bX = (\beta K)^\omega$. We claim that X and bX are as required. Clearly, $bX \setminus X = S^\omega$ is ω-bounded. We let $\triangle$ denote the diagonal in the product $(\beta K)^\omega$ and, by abuse of notation, identify it with βK. Hence we consider p to also be a point of X. Assume that cX is a compactification of X such that $cX \leq bX$ and assume, striving for a contradiction, that $cX \setminus X$ is first-countable. Let $f : bX \to cX$ be a continuous surjection that restricts to the identity on X.

Fix n for a while, and let $g_n = f \restriction K_n : K_n \to cX \setminus X$. Since $f(K_n)$ is first-countable, the fibers of the map g_n are all closed G_δ-subsets of K_n. But every closed G_δ-subset of ω^* has a dense interior. For every $s \in f(K_n)$, let U_s^n denote the dense interior of $g_n^{-1}(\{s\})$. Let F_n be the complement in K_n of the union of the disjoint family $\{U_s^n : s \in f(K_n)\}$. Then F_n is a closed nowhere dense subset of K_n.

Since p is a remote point of K, p is not in the closure of the nowhere dense set $F = \bigcup_{n<\omega} F_n$. Hence there is a clopen set C in K which contains p in its closure and is contained in the complement of F. The compact set $C_n = C \cap K_n$ is covered by $\{U_s^n : s \in f(K_n)\}$. Hence there is a finite subset G_n of $f(K_n)$ such that $C_n \subseteq \bigcup_{s \in G_n} U_s^n$. Since p is in the closure (in cX) of $f(C)$, we conclude that p is in the closure of the countable subset $\bigcup_{n<\omega} G_n$ of $cX \setminus X$. Since X is nowhere locally compact, the remainder $cX \setminus X$ is dense in cX. Since it is first-countable, this implies that p has countable π-character in X. Since the restriction to X of the projection $(\beta K)^\omega \to \beta K$ onto the first factor space is open, this shows that p has countable π-character in βK, which is absurd. $\square$

Problem 3.6. Is there in ZFC a nowhere locally compact Lindelöf space X having no first-countable remainder while X^* is ω-bounded? What if X in addition is a topological group?

4. Applications to topological groups

For topological groups with special properties, the property of having a first-countable remainder can be characterized as follows under CH:

Theorem 4.1 (*CH*). *Suppose that G is a Lindelöf non-locally compact topological group with a strongly ω-bounded remainder. Then the following conditions are equivalent:*

(i) *G has a first-countable remainder.*
(ii) *The weight of G equals ω_1.*

Proof. The implication (ii) $\Rightarrow$ (i) is a consequence of Corollary 3.2. For (i) $\Rightarrow$ (ii), we first use Arhangel'skii and van Mill [7] to conclude that the character of G is ω_1. But a Lindelöf topological group with character ω_1 clearly has weight ω_1. (Observe that G does not have countable weight since it has a strongly ω-bounded remainder.) □

The first topological group with a first-countable remainder which is countable and not metrizable, was constructed by the authors under CH in [8]. This example is not a familiar topological group. The results in this paper allow us to conclude that many familiar topological groups have the same property.

Corollary 4.2 (*CH*). *If G is the free (Abelian) topological group over any infinite separable compactum, then G has a first-countable remainder (clearly, G is not metrizable).*

Proof. It is known that G is a k_ω-space, see e.g. Ordman [17] and Arhangel'skii and Tkachenko [6, Theorem 7.4.1]. Moreover, by an unpublished result of van Douwen (see [14, Proposition 5.3]), the Čech–Stone remainder of any k_ω-space is strongly ω-bounded. (This was independently and unaware of van Douwen's result also established in Arhangel'skii [2].) Hence G^* is strongly ω-bounded and of countable type since G is Lindelöf being σ-compact. Since G has weight 2^ω, being separable, we are done by Theorem 4.1. □

We finish this section by answering the first part of Questions 6.4 and 6.5 in Juhász, van Mill and Weiss [14] in the negative.

Corollary 4.3 (*CH*). *There is a first-countable strongly ω-bounded space which has a dense hereditarily Lindelöf subspace and is neither ccc-bounded nor compact.*

Proof. Let Y be the remainder of the compactification bG that was constructed in Corollary 4.2. It is clear that Y is not compact, G being nowhere locally compact. But Y is strongly ω-bounded and hence a Baire space. Since Y has weight 2^ω, as bG is separable, it follows that Y has a dense Luzin (hence hereditarily Lindelöf) subspace by van Douwen, Tall and Weiss [21]. Since both G and Y are dense in bG, we conclude that Y is *ccc*. □

5. Rajkov completeness

A topological group G is called *Rajkov complete* if all of its Cauchy filters (with respect to the two-sided uniformity) converge. It is known that a closed subgroup of a Rajkov complete topological group is Rajkov complete, that every Čech-complete topological group is Rajkov complete and that a metrizable group is Rajkov complete if and only if it is Čech-complete. It is also known that for every topological group G there exists a unique (up to topological isomorphism) Rajkov complete topological group ρG containing (a topologically isomorphic copy of) G as a dense subgroup. For this and more information about Rajkov completeness, see Arhangel'skii and Tkachenko [6, Sections 3.6 and 4.3].

Observe that every Rajkov complete subgroup G of a topological group H is closed in H. Let us call a topological group G (a space X) *TOG-closed*, if for every topological group H and every subgroup A of H which is homeomorphic to G (homeomorphic to X) we have that A is closed in H. This property can easily be characterized, as follows:

Proposition 5.1. *A topological group G is TOG-closed if and only if every topological group H which is homeomorphic to G is Rajkov complete.*

Proof. Simply observe that if G is homeomorphic to H and H is not Rajkov complete, then it is not closed in ρH. □

Hence there are many such groups. For example, every Čech complete topological group is TOG-closed.

It is not true that a topological group is Rajkov complete if and only if it is TOG-closed: there are many examples of homeomorphic topological groups G and H such that G is Rajkov complete, but H is not. We will prove in Proposition 5.2 below that every topological group G has the property that its free topological group $F(G)$ is homeomorphic to the product of G and a nontrivial group N (similarly for $A(G)$). Hence the topological group $A(\mathbb{Q})$ is homeomorphic to $\mathbb{Q} \times N$, for some topological group N. Here $\mathbb{Q}$ denotes the space of rational numbers. But $A(\mathbb{Q})$ is Rajkov complete (Arhangel'skii and Tkachenko [6, 7.9.7]), and the topological group $\mathbb{Q} \times N$ is not since $\mathbb{Q}$ is not Čech complete.

Proposition 5.2. *Let G be a topological group. Then its free topological group $F(G)$ is homeomorphic to $G \times N$, where N is a nontrivial topological group (similarly for $A(G)$).*

Proof. There is clearly a retraction $r: F(G) \to G$ which is also a homomorphism. Let N denotes its kernel. The function $f: F(G) \to G \times N$ defined by

$$f(p) = (r(p), p \cdot r(p)^{-1})$$

is a homeomorphism. $\square$

This suggests the following interesting problem.

Problem 5.3. Characterize the topological spaces X for which $A(X)$ and $X \times A(X)$ are homeomorphic. Similarly for $F(X)$.

It is not true that for all spaces X the product $X \times F(X)$ is homogeneous. For example, let $X = \beta\omega$. Indeed, the projection mapping from $X \times F(X)$ to X is open and continuous. Since the cardinality of X is greater than 2^{ω}, it follows from Theorem 4.1(a) of van Douwen's paper [20] that no power of the space $X \times F(X)$ is homogeneous. Similarly for $X \times A(X)$. Of particular interest in Problem 5.3 is the case when X is (compact) metrizable.

A topological group G will be called *Rajkov countably complete*, if every countable subset of G is contained in a Rajkov complete subgroup of G.

Theorem 5.4. *Suppose that G is a topological group with an ω-bounded remainder. Then G is Rajkov countably complete.*

Proof. Fix a countable subset A of G and consider the Rajkov completion ρG of G. Let $\langle\langle A\rangle\rangle$ be the countable subgroup of G algebraically generated by A in G. Fix any compactification B of the space ρG. Then B is also a compactification of G, since G is a dense subspace of ρG. So we put $bG = B$ and $Y = bG \setminus G$. Let H denote the closure of $\langle\langle A\rangle\rangle$ in ρG. Suppose that there exists an element $p \in H \setminus G$. Then $p\langle\langle A\rangle\rangle$ is a countable dense subset of H which is entirely contained in Y. But this disproves the fact that Y is ω-bounded. As a consequence, the Rajkov complete subgroup H of ρG is contained in G and hence we are done. $\square$

Since a separable topological group is Rajkov countably complete if and only if it is Rajkov complete, the following corollary is obvious.

Corollary 5.5. *Suppose that G is a separable topological group with an ω-bounded remainder. Then G is TOG-closed. If moreover G is countable, then every closed subgroup (subspace) of G is TOG-closed.*

Thus, we have a dichotomy:

Theorem 5.6. *If G is a separable topological group with a first-countable remainder, then either G is metrizable, or G is TOG-closed.*

Proof. If G is not metrizable, then G has an ω-bounded remainder by Arhangel'skii [2]. □

Not every countable Rajkov complete topological group has a first-countable remainder, as the following result shows.

Example 5.7. The topological group $A(\mathbb{Q})$ does not have a first-countable remainder.

Proof. Note that $\mathbb{Q}$ is closed in G. Assume, striving for a contradiction, that G has a first-countable remainder Y in some compactification bG of G. Let $b\mathbb{Q}$ be the closure of $\mathbb{Q}$ in bG. Put $Z = b\mathbb{Q} \setminus \mathbb{Q}$. Clearly, Z is a closed subspace of Y, since $\mathbb{Q}$ is closed in G. Since $\mathbb{Q}$ is not locally compact, $\mathbb{Q}$ is not open in $b\mathbb{Q}$. Since $\mathbb{Q}$ is first-countable, we conclude that the closure of some countable subset of Z intersects $\mathbb{Q}$. Hence, neither Z, nor Y is ω-bounded. However, Y is ω-bounded, since G is not metrizable and Y is dense in bG and first-countable [2]. This is a contradiction. □

This is also true for the free topological group $F(\mathbb{Q})$ over $\mathbb{Q}$ since $F(\mathbb{Q})$ is Rajkov complete as well [19].

It follows from this and Corollary 4.2 that, under CH, $\mathbb{Q}$ does not embed in the free Abelian group $A(S)$ over a nontrivial convergent sequence S as a closed subset. However, more is true. We claim that $\mathbb{Q}$ cannot be embedded in $A(S)$. Indeed, the 'layers' of $A(S)$ are countable compact spaces of finite Cantor–Bendixson height. Hence every compact subspace of $A(S)$ has finite Cantor–Bendixson height. But $\mathbb{Q}$ contains compacta of arbitrarily large (countable) Cantor–Bendixson height.

This last observation implies that every metrizable subspace of $A(S)$ is scattered. Indeed, if it were not scattered then it would contain a topological copy of $\mathbb{Q}$.

These results suggest the following problems.

Problem 5.8. Is every closed subgroup of a separable topological group with an ω-bounded remainder TOG-closed?

Problem 5.9. Is there in ZFC a countable Rajkov complete topological group with an ω-bounded remainder and no first-countable remainder?

The following problems are also quite interesting.

Problem 5.10. Is there a non-metrizable topological group with a first countable but not strongly ω-bounded remainder?

Problem 5.11. Does there exist, under CH, a countable topological group with an ω-bounded but not a strongly ω-bounded remainder?

References

[1] A.V. Arhangel'skii, Remainders in compactifications and generalized metrizability properties, Topology Appl. 150 (2005) 79–90.
[2] A.V. Arhangel'skii, First-countability, tightness, and other cardinal invariants in remainders of topological groups, Topology Appl. 154 (2007) 2950–2961.

[3] A.V. Arhangel'skii, More on remainders close to metrizable spaces, Topology Appl. 154 (2007) 1084–1088.
[4] A.V. Arhangel'skii, Two types of remainders of topological groups, Comment. Math. Univ. Carolin. 47 (2008) 119–126.
[5] A.V. Arhangel'skii, S.P. Franklin, Ordinal invariants for topological spaces, Michigan Math. J. 15 (1968) 313–320; A.V. Arhangel'skii, S.P. Franklin, Ordinal invariants for topological spaces, Michigan Math. J. 15 (1968) 506. Addendum.
[6] A.V. Arhangel'skii, M.G. Tkachenko, Topological Groups and Related Structures, in: Atlantis Studies in Mathematics, vol. 1, Atlantis Press, World Scientific, Paris, 2008.
[7] A.V. Arhangel'skii, J. van Mill, On topological groups with a first-countable remainder, Topology Proc. 42 (2013) 157–163.
[8] A.V. Arhangel'skii, J. van Mill, On topological groups with a first-countable remainder, II, 2012.
[9] A. Dow, A remote P-point from $\diamond$. Preprint.
[10] A. Dow, A.V. Gubbi, A. Szymański, Rigid stone spaces in ZFC, Proc. Amer. Math. Soc. 102 (1988) 745–748.
[11] R. Engelking, General Topology, second ed., Heldermann-Verlag, Berlin, 1989.
[12] S.P. Franklin, M. Rajagopalan, Some examples in topology, Trans. Amer. Math. Soc. 155 (1971) 305–314.
[13] M. Henriksen, J.R. Isbell, Some properties of compactifications, Duke Math. J. 25 (1957) 83–105.
[14] I. Juhász, J. van Mill, W. Weiss, Variations on ω-boundedness, Israel J. Math. 194 (2013) 745–766.
[15] A. Louveau, Sur un article de S. Sirota, Bull. Sci. Math. (2) 96 (1972) 3–7.
[16] P. Nyikos, First countable, countably compact, noncompact spaces, in: E. Pearl (Ed.), Open Problems in Topology II, North-Holland Publishing Co., Amsterdam, 2007, pp. 217–224.
[17] E.T. Ordman, Free k-groups and free topological groups, Gen. Top. Appl. 5 (1975) 205–219.
[18] M.E. Rudin, Lectures on Set Theoretic Topology, in: Regional Conference Series in Mathematics, vol. 23, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, RI, 1975.
[19] V.V. Uspenskiy, Free topological groups of metrizable spaces, Izv. Akad. Nauk SSSR Ser. Mat. 54 (6) (1990) 1295–1319.
[20] E.K. van Douwen, Nonhomogeneity of products of preimages and π-weight, Proc. Amer. Math. Soc. 69 (1978) 183–192.
[21] E.K. van Douwen, F.D. Tall, W.A.R. Weiss, Nonmetrizable hereditarily Lindelöf spaces with point countable bases from CH, Proc. Amer. Math. Soc. 64 (1977) 139–145.
[22] J. van Mill, An introduction to $\beta\omega$, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland Publishing Co., Amsterdam, 1984, pp. 503–567.
[23] J.E. Vaughan, Two spaces homeomorphic tp Seq(p), Comment. Math. Univ. Carolin. 42 (2001) 209–218.