On topological groups with a first-countable remainder, III

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Abstract

We prove a general theorem that allows us to conclude that under CH, the free topological group over a nontrivial convergent sequence $S$ has a first-countable remainder. It is also shown that any separable non-metrizable topological group with a first-countable remainder is Rajkov complete.

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1. Introduction

In a series of papers, Arhangel’skii studied topological spaces having a compactification with a first-countable remainder. Specific attention was paid to topological groups that belong to this class. For details, and references, see e.g., [1–4].

Recently, the authors continued this study in [7,8] and obtained among other things the following results: a topological group with a first-countable remainder has character at most $\omega_1$. 

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has weight at most $2^ω$, and is metrizable in case it is pre-compact. It was shown that moreover there exists a non-metrizable topological group with a first-countable remainder in ZFC, and a countable such space under CH (it was known by Arhangel’skii [2] that such a space cannot be countable under MA + ¬CH).

In this paper we continue these investigations. We prove a general theorem about spaces having a compactification with first-countable remainder that allows us to conclude that under CH, the free topological group over a nontrivial convergent sequence has a first-countable remainder. We also solve a problem in Juhász, van Mill and Weiss [14] in the negative. Spaces that have a strongly $ω$-bounded remainder play an important role in our investigations. We prove that a separable topological group with an $ω$-bounded remainder is Rajkov complete; this implies that any separable non-metrizable topological group with a first-countable remainder is Rajkov complete.

2. Preliminaries

Let $P$ be any topological property. It is natural to call a space $P$-bounded if every subset with property $P$ has compact closure. In particular, if $P \equiv \text{‘countable’}$ then we obtain the well-known class of $ω$-bounded spaces, and if $P \equiv \text{‘σ-compact’}$ then we obtain the class that is called strongly $ω$-bounded in Nyikos [16]. For other concepts that are in the same spirit, see e.g. Juhász, van Mill and Weiss [14].

A space $X$ has countable type if every compact subspace of $X$ is contained in a compact subspace of $X$ which has countable character in $X$. By a well-known result in Henriksen and Isbell [13], a space $X$ is of countable type if and only if the remainder in any (or in some) compactification of $X$ is Lindelöf.

If $X$ is a space, then $βX$ denotes its Čech–Stone compactification, and $X^* = βX \setminus X$. It follows by standard methods that a topological space $X$ has an $ω$-bounded remainder if and only if every remainder of it is $ω$-bounded. Similarly for strongly $ω$-bounded. These well-known facts are left as exercises to the reader. A point $p \in X^*$ is said to be a remote point of $X$ if $p \notin \overline{D}$, where $D$ is any nowhere dense subset of $X$ (here closure means closure in $βX$).

For all undefined notions, see Engelking [11]. For information on topological groups, see Arhangel’skii and Tkachenko [6].

3. The theorem

It is a well-known result by Franklin and Rajagopalan [12] that the space $X = ω \cup \{p\}$, where $p \in ω^*$ is a $P$-point, has a compactification $bX$ such that $bX \setminus X$ is $W(ω_1)$, the space of all countable ordinal numbers. Hence $bX \setminus X$ is first countable. We will generalize this result to Theorem 3.1 below, which is our main tool for constructing spaces with a first-countable remainder.

**Theorem 3.1 (CH).** Let $X$ be a strongly $ω$-bounded space of countable type with a compactification $bX$ such that $w(bX) \leq 2^ω$. Then there are a compact space $Z$ and a continuous surjection $f : bX \to Z$ having the following properties:

1. The restriction of $f$ to $Y = bX \setminus X$ is a homeomorphism from $Y$ onto $f(Y)$.
2. $f(Y) \cap f(X) = \emptyset$.
3. $Z$ is first-countable at all points of $f(X)$.

**Proof.** We may assume that $bX \subseteq [\frac{1}{2}, 1]^{ω_1}$. For every $α < ω_1$, let $π_α : [\frac{1}{2}, 1]^{ω_1} \to I$ be the projection onto the $α$-th coordinate, and let $p_α : X \to I$ be the restriction of $π_α$ to $X$. 
Let $\mathcal{E}$ be the family of all compact subspaces $F$ of $X$ such that $\chi(F, X) \leq \omega$. Observe that if $F \in \mathcal{E}$ then $\chi(F, bX) \leq \omega$ since $X$ is dense in $bX$. The next three statements are obvious from our assumptions.

(C1) $|\mathcal{E}| \leq \omega_1$.
(C2) $\bigcup \mathcal{E} = X$.
(C3) There exists an increasing $\omega_1$-sequence \{\(F_\alpha : \alpha < \omega_1\) of members of $\mathcal{E}$ such that 
\[
\bigcup_{\alpha < \omega_1} F_\alpha = X.
\]

For each $\alpha < \omega_1$ we fix a decreasing sequence \{\(W_{\alpha, n} : n < \omega\) of open neighborhoods of $F_\alpha$ in $bX$ which is a base at $F_\alpha$ in $bX$. Fix a continuous function $g_{\alpha, n} : bX \to \mathbb{I}$ satisfying the following conditions for $\alpha < \omega_1$ and $n < \omega$:

(s1) $g_{\alpha, n}(F_\alpha) = \{0\}$.
(s2) $g_{\alpha, n}(bX \setminus W_{\alpha, n}) = \{1\}$.

For every $\alpha < \omega_1$, put 
\[
\mathcal{H}_\alpha = \{p_\alpha \cdot g_{\alpha, n} : n < \omega\}
\]

and let 
\[
\mathcal{H} = \bigcup_{\alpha < \omega_1} \mathcal{H}_\alpha.
\]

Let $f : bX \to \mathbb{I}^\mathcal{H}$ denote the diagonal map of the family $\mathcal{H}$. That is, $f(p)_h = h(p)$ for every $p \in bX$ and $h \in \mathcal{H}$. We claim that $Z = f(bX)$ and $f$ are as required.

**Claim 1.** For every $\alpha < \omega_1$, $f(F_\alpha)$ is metrizable and \(f^{-1}(f(F_\alpha)) = F_\alpha\).

Pick $g \in \mathcal{H}_\beta$ for some $\beta > \alpha$. Then for some $n$, $g = p_\beta \cdot g_{\beta, n}$. But then $g(F_\alpha) = \{0\}$ since $g_{\beta, n}$ is identically 0 on $F_\alpha$. So we conclude that 
\[
f(F_\alpha) \subseteq \left\{ q \in \mathbb{I}^\mathcal{H} : q_h = 0 \text{ for all } h \in \bigcup_{\beta > \alpha} \mathcal{H}_\beta \right\}.
\]

Hence $f(F_\alpha)$ is metrizable since $\bigcup_{\beta \leq \alpha} \mathcal{H}_\beta$ is countable.

We will now show that $f^{-1}(f(F_\alpha)) = F_\alpha$. To prove this, pick an arbitrary $p \in bX \setminus F_\alpha$, and let $n < \omega$ be so large that $p \not\in W_{\alpha, n}$. Put $h = p_\alpha \cdot g_{\alpha, n}$. Then 
\[
f(p)_h = h(p) = p_\alpha \cdot g_{\alpha, n}(p) = p_\alpha(p) \geq \frac{1}{2},
\]

and 
\[
f(x)_h = h(x) = p_\alpha \cdot g_{\alpha, n}(x) = 0
\]

for every $x \in F_\alpha$. This completes the proof of the claim.

Hence for every $\alpha < \omega_1$ we have that $f(F_\alpha)$ is a metrizable closed $G_\delta$-subset of the compact space $Z$. Hence $Z$ is first-countable at all points of $f(X)$.

It also follows from the claim that $f(Y) \cap f(X) = \emptyset$.

We now claim that $f \upharpoonright Y$ is one-to-one. Indeed, if $y(0)$ and $y(1)$ are distinct elements of $Y$, then we may pick $\alpha < \omega_1$ such that $y(0)_\alpha \neq y(1)_\alpha$. Fix $n$ so that $y(0), y(1) \not\in W_{\alpha, n}$. Then 
\[
p_\alpha \cdot g_{\alpha, n}(y(0)) = p_\alpha(y(0)) = y(0)_\alpha
\]

and, similarly, $p_\alpha \cdot g_{\alpha, n}(y(1)) = y(1)_\alpha$, so we are done.
From this we get that for every \( y \in Y \) we have \( f^{-1}(f(y)) = \{y\} \). Hence \( f \) restricts to a homeomorphism on \( Y \). \( \square \)

Observe that in order to be in a position to apply Theorem 3.1, we need a nowhere locally compact space \( X \) which has a compactification \( bX \) such that \( w(bX) \leq 2^\omega \) while moreover \( bX \setminus X \) is strongly \( \omega \)-bounded and of countable type. Then Theorem 3.1 tells us that \( bX \setminus X \) can be replaced by a first-countable remainder.

**Corollary 3.2 (CH).** If \( Y \) is a nowhere locally compact Lindelöf space with a strongly \( \omega \)-bounded remainder and \( w(Y) \leq 2^\omega \), then \( Y \) has a compactification with a first-countable remainder. Even more is true: for every compactification of \( Y \), there exists a smaller compactification with a first-countable remainder.

**Proof.** Simply use the fact that \( \beta Y \setminus Y \) has countable type by the Henriksen and Isbell Theorem from [13]. \( \square \)

**Corollary 3.3 (CH).** Every strongly \( \omega \)-bounded space \( X \) of countable type such that \( w(X) \leq 2^\omega \) can be mapped onto a first-countable space \( R \) by a perfect mapping (then automatically \( R \) is strongly \( \omega \)-bounded and \( w(R) \leq 2^\omega \)).

We will now discuss the assumptions in Theorem 3.1. We will first show that CH is essential.

Let \( \text{Seq} \) denote the set of all finite sequences of elements from \( \omega \). Moreover, let \( p \) be a free ultrafilter on \( \omega \). Define a topology \( \mathcal{T} \) on \( \text{Seq} \) by the rule: \( V \subseteq \text{Seq} \) is open iff for every \( s \in V \), the set \( \{ n < \omega : s \upharpoonright n \in V \} \in p \). Here \( s \upharpoonright n \) denotes the concatenation of \( s \) by \( n \). It is easy to verify that \( \text{Seq} \) with this topology is Tychonoff, zero-dimensional, perfect and extremally disconnected. For details and variations, see Arhangel’skii and Franklin [5] and Dow, Gubbi and Szymański [10].

Let \( [\omega]^{<\omega} \) be the set of all finite subsets of \( \omega \). Clearly, the symmetric difference operator \( \triangle \) makes \( [\omega]^{<\omega} \) a Boolean group. For a free ultrafilter \( p \) on \( \omega \), define a topology \( \mathcal{T}_p \) on \( [\omega]^{<\omega} \) as follows:

\[
U \in \mathcal{T}_p \iff (\forall F \in U)(\{ n < \omega : F \triangle n \in U \} \in p).
\]

It is not difficult to see that \( \mathcal{T}_p \) is extremally disconnected.

This topology is due to Louveau [15] who proved that if \( p \) is selective, then \( \mathcal{T}_p \) is compatible with the group structure on \( [\omega]^{<\omega} \). The topological group thus obtained is denoted by \( L(p) \) and hence is an example of a non-discrete extremally disconnected topological group.

It was shown by Vaughan [23] that \( L(p) \) and \( \text{Seq} \) (for the same ultrafilter \( p \)) are homeomorphic.

**Theorem 3.4.** If \( p \) is a selective ultrafilter on \( \omega \) of character greater than \( \omega_1 \), then \( \text{Seq} \) is a countable non-discrete topological group whose Čech–Stone remainder is strongly \( \omega \)-bounded but none of its remainders is first-countable.

**Proof.** That \( \text{Seq}^* \) is strongly \( \omega \)-bounded was proved in [10, Remark 1]. It is easy to see that the character of \( \text{Seq} \) is greater than \( \omega_1 \). Just observe that the space \( \omega \cup \{p\} \) is a subspace of \( \text{Seq} \). Hence \( \text{Seq} \) does not have a first-countable remainder since any topological group with a first-countable remainder has character \( \omega_1 \) by Arhangel’skii and van Mill [7, Theorem 2.1]. \( \square \)

Hence in Theorem 3.1 CH is indeed essential since for example under \( \text{MA} + \neg \text{CH} \) there exist selective ultrafilters on \( \omega \) and they have character \( 2^\omega > \omega_1 \) [18].
The question of whether the assumption on strong $\omega$-boundedness is essential in Theorem 3.1 is very natural. We will answer it in the negative by using a powerful recent result of Dow [9]. For basic facts on Čech–Stone compactifications, see van Mill [22].

**Theorem 3.5** ($\oslash$). There are a $\sigma$-compact nowhere locally compact space $X$ and a compactification $bX$ of $X$ such that $bX \setminus X$ is $\omega$-bounded (and clearly of countable type) while moreover every compactification $cX$ of $X$ such that $cX \leq bX$ has the property that $cX \setminus X$ is not first-countable.

**Proof.** Put $K = \omega \times \omega^*$, and for every $n$, let $K_n = \{n\} \times \omega^*$. Dow [9] recently proved that under $\oslash$, $K$ has a remote point $p$ which is simultaneously a $P$-point of $K^*$. Put $S = \beta K \setminus \{p\}$. Since countable subsets of $K$ are nowhere dense, it clearly follows that $S$ is $\omega$-bounded. It is not strongly $\omega$-bounded though since it is not compact and contains a dense $\sigma$-compact subspace. We now put $X = (\beta K)^\omega \setminus S^\omega$ and $bX = (\beta K)^\omega$. We claim that $X$ and $bX$ are as required. Clearly, $bX \setminus X = S^\omega$ is $\omega$-bounded. We let $\Delta$ denote the diagonal in the product $(\beta K)^\omega$ and, by abuse of notation, identify it with $\beta K$. Hence we consider $p$ to also be a point of $X$. Assume that $cX$ is a compactification of $X$ such that $cX \leq bX$ and assume, striving for a contradiction, that $cX \setminus X$ is first-countable. Let $f : bX \to cX$ be a continuous surjection that restricts to the identity on $X$.

Fix $n$ for a while, and let $g_n = f \upharpoonright K_n : K_n \to cX \setminus X$. Since $f(K_n)$ is first-countable, the fibers of the map $g_n$ are all closed $G_\delta$-subsets of $K_n$. But every closed $G_\delta$-subset of $\omega^*$ has a dense interior. For every $s \in f(K_n)$, let $U^n_s$ denote the dense interior of $g^{-1}_n(s)$. Let $F_n$ be the complement in $K_n$ of the union of the disjoint family $\{U^n_s : s \in f(K_n)\}$. Then $F_n$ is a closed nowhere dense subset of $K_n$.

Since $p$ is a remote point of $K$, $p$ is not in the closure of the nowhere dense set $F = \bigcup_{n<\omega} F_n$. Hence there is a clopen set $C$ in $K$ which contains $p$ in its closure and is contained in the complement of $F$. The compact set $C_n = C \cap K_n$ is covered by $\{U^n_s : s \in f(K_n)\}$. Hence there is a finite subset $G_n$ of $f(K_n)$ such that $C_n \subseteq \bigcup_{s \in G_n} U^n_s$. Since $p$ is in the closure (in $cX$) of $f(C)$, we conclude that $p$ is in the closure of the countable subset $\bigcup_{n<\omega} G_n$ of $cX \setminus X$. Since $X$ is nowhere locally compact, the remainder $cX \setminus X$ is dense in $cX$. Since it is first-countable, this implies that $p$ has countable $\pi$-character in $X$. Since the restriction to $X$ of the projection $(\beta K)^\omega \to \beta K$ onto the first factor space is open, this shows that $p$ has countable $\pi$-character in $\beta K$, which is absurd. \qed

**Problem 3.6.** Is there in ZFC a nowhere locally compact Lindelöf space $X$ having no first-countable remainder while $X^*$ is $\omega$-bounded? What if $X$ in addition is a topological group?

4. **Applications to topological groups**

For topological groups with special properties, the property of having a first-countable remainder can be characterized as follows under CH:

**Theorem 4.1** (CH). Suppose that $G$ is a Lindelöf non-locally compact topological group with a strongly $\omega$-bounded remainder. Then the following conditions are equivalent:

(i) $G$ has a first-countable remainder.

(ii) The weight of $G$ equals $\omega_1$.

**Proof.** The implication (ii) $\Rightarrow$ (i) is a consequence of Corollary 3.2. For (i) $\Rightarrow$ (ii), we first use Arhangel’skii and van Mill [7] to conclude that the character of $G$ is $\omega_1$. But a Lindelöf
The first topological group with a first-countable remainder which is countable and not metrizable, was constructed by the authors under CH in [8]. This example is not a familiar topological group. The results in this paper allow us to conclude that many familiar topological groups have the same property.

**Corollary 4.2 (CH).** If $G$ is the free (Abelian) topological group over any infinite separable compactum, then $G$ has a first-countable remainder (clearly, $G$ is not metrizable).

**Proof.** It is known that $G$ is a $k_{\omega}$-space, see e.g. Ordman [17] and Arhangel’skii and Tkachenko [6, Theorem 7.4.1]. Moreover, by an unpublished result of van Douwen (see [14, Proposition 5.3]), the Čech–Stone remainder of any $k_{\omega}$-space is strongly $\omega$-bounded. (This was independently and unaware of van Douwen’s result also established in Arhangel’skii [2].) Hence $G^*$ is strongly $\omega$-bounded and of countable type since $G$ is Lindelöf being $\sigma$-compact. Since $G$ has weight $2^{\omega}$, being separable, we are done by Theorem 4.1.

We finish this section by answering the first part of Questions 6.4 and 6.5 in Juhász, van Mill and Weiss [14] in the negative.

**Corollary 4.3 (CH).** There is a first-countable strongly $\omega$-bounded space which has a dense hereditarily Lindelöf subspace and is neither ccc-bounded nor compact.

**Proof.** Let $Y$ be the remainder of the compactification $bG$ that was constructed in Corollary 4.2. It is clear that $Y$ is not compact, $G$ being nowhere locally compact. But $Y$ is strongly $\omega$-bounded and hence a Baire space. Since $Y$ has weight $2^{\omega}$, as $bG$ is separable, it follows that $Y$ has a dense Luzin (hence hereditarily Lindelöf) subspace by van Douwen, Tall and Weiss [21]. Since both $G$ and $Y$ are dense in $bG$, we conclude that $Y$ is ccc.

## 5. Rajkov completeness

A topological group $G$ is called **Rajkov complete** if all of its Cauchy filters (with respect to the two-sided uniformity) converge. It is known that a closed subgroup of a Rajkov complete topological group is Rajkov complete, that every Čech-complete topological group is Rajkov complete and that a metrizable group is Rajkov complete if and only if it is Čech-complete. It is also known that for every topological group $G$ there exists a unique (up to topological isomorphism) Rajkov complete topological group $\rho G$ containing (a topologically isomorphic copy of) $G$ as a dense subgroup. For this and more information about Rajkov completeness, see Arhangel’skii and Tkachenko [6, Sections 3.6 and 4.3].

Observe that every Rajkov complete subgroup $G$ of a topological group $H$ is closed in $H$. Let us call a topological group $G$ (a space $X$) **TOG-closed**, if for every topological group $H$ and every subgroup $A$ of $H$ which is homeomorphic to $G$ (homeomorphic to $X$) we have that $A$ is closed in $H$. This property can easily be characterized, as follows:

**Proposition 5.1.** A topological group $G$ is TOG-closed if and only if every topological group $H$ which is homeomorphic to $G$ is Rajkov complete.

**Proof.** Simply observe that if $G$ is homeomorphic to $H$ and $H$ is not Rajkov complete, then it is not closed in $\rho H$. □
Hence there are many such groups. For example, every Čech complete topological group is TOG-closed.

It is not true that a topological group is Rajkov complete if and only if it is TOG-closed: there are many examples of homeomorphic topological groups $G$ and $H$ such that $G$ is Rajkov complete, but $H$ is not. We will prove in Proposition 5.2 below that every topological group $G$ has the property that its free topological group $F(G)$ is homeomorphic to the product of $G$ and a nontrivial group $N$ (similarly for $A(G)$). Hence the topological group $A(\mathbb{Q})$ is homeomorphic to $\mathbb{Q} \times N$, for some topological group $N$. Here $\mathbb{Q}$ denotes the space of rational numbers. But $A(\mathbb{Q})$ is Rajkov complete (Arhangel'skii and Tkachenko [6, 7.9.7]), and the topological group $\mathbb{Q} \times N$ is not since $\mathbb{Q}$ is not Čech complete.

**Proposition 5.2.** Let $G$ be a topological group. Then its free topological group $F(G)$ is homeomorphic to $G \times N$, where $N$ is a nontrivial topological group (similarly for $A(G)$).

**Proof.** There is clearly a retraction $r: F(G) \to G$ which is also a homomorphism. Let $N$ denote its kernel. The function $f: F(G) \to G \times N$ defined by $f(p) = (r(p), p \cdot r(p)^{-1})$ is a homeomorphism. □

This suggests the following interesting problem.

**Problem 5.3.** Characterize the topological spaces $X$ for which $A(X)$ and $X \times A(X)$ are homeomorphic. Similarly for $F(X)$.

It is not true that for all spaces $X$ the product $X \times F(X)$ is homogeneous. For example, let $X = \beta\omega$. Indeed, the projection mapping from $X \times F(X)$ to $X$ is open and continuous. Since the cardinality of $X$ is greater than $2^\omega$, it follows from Theorem 4.1(a) of van Douwen’s paper [20] that no power of the space $X \times F(X)$ is homogeneous. Similarly for $X \times A(X)$. Of particular interest in Problem 5.3 is the case when $X$ is (compact) metrizable.

A topological group $G$ will be called Rajkov countably complete, if every countable subset of $G$ is contained in a Rajkov complete subgroup of $G$.

**Theorem 5.4.** Suppose that $G$ is a topological group with an $\omega$-bounded remainder. Then $G$ is Rajkov countably complete.

**Proof.** Fix a countable subset $A$ of $G$ and consider the Rajkov completion $\rho G$ of $G$. Let $\langle A \rangle$ be the countable subgroup of $G$ algebraically generated by $A$ in $G$. Fix any compactification $B$ of the space $\rho G$. Then $B$ is also a compactification of $G$, since $G$ is a dense subspace of $\rho G$. So we put $bG = B$ and $Y = bG \setminus G$. Let $H$ denote the closure of $\langle A \rangle$ in $\rho G$. Suppose that there exists an element $p \in H \setminus G$. Then $p\langle A \rangle$ is a countable dense subset of $H$ which is entirely contained in $Y$. But this disproves the fact that $Y$ is $\omega$-bounded. As a consequence, the Rajkov complete subgroup $H$ of $\rho G$ is contained in $G$ and hence we are done. □

Since a separable topological group is Rajkov countably complete if and only if it is Rajkov complete, the following corollary is obvious.

**Corollary 5.5.** Suppose that $G$ is a separable topological group with an $\omega$-bounded remainder. Then $G$ is TOG-closed. If moreover $G$ is countable, then every closed subgroup (subspace) of $G$ is TOG-closed.
Thus, we have a dichotomy:

**Theorem 5.6.** If $G$ is a separable topological group with a first-countable remainder, then either $G$ is metrizable, or $G$ is TOG-closed.

**Proof.** If $G$ is not metrizable, then $G$ has an $\omega$-bounded remainder by Arhangel’skii [2].

Not every countable Rajkov complete topological group has a first-countable remainder, as the following result shows.

**Example 5.7.** The topological group $A(\mathbb{Q})$ does not have a first-countable remainder.

**Proof.** Note that $\mathbb{Q}$ is closed in $G$. Assume, striving for a contradiction, that $G$ has a first-countable remainder $Y$ in some compactification $bG$ of $G$. Let $b\mathbb{Q}$ be the closure of $\mathbb{Q}$ in $bG$. Put $Z = b\mathbb{Q} \setminus \mathbb{Q}$. Clearly, $Z$ is a closed subspace of $Y$, since $\mathbb{Q}$ is closed in $G$. Since $\mathbb{Q}$ is not locally compact, $\mathbb{Q}$ is not open in $b\mathbb{Q}$. Since $\mathbb{Q}$ is first-countable, we conclude that the closure of some countable subset of $Z$ intersects $\mathbb{Q}$. Hence, neither $Z$, nor $Y$ is $\omega$-bounded. However, $Y$ is $\omega$-bounded, since $G$ is not metrizable and $Y$ is dense in $bG$ and first-countable [2]. This is a contradiction.

This is also true for the free topological group $F(\mathbb{Q})$ over $\mathbb{Q}$ since $F(\mathbb{Q})$ is Rajkov complete as well [19].

It follows from this and Corollary 4.2 that, under CH, $\mathbb{Q}$ does not embed in the free Abelian group $A(S)$ over a nontrivial convergent sequence $S$ as a closed subset. However, more is true. We claim that $\mathbb{Q}$ cannot be embedded in $A(S)$. Indeed, the ‘layers’ of $A(S)$ are countable compact spaces of finite Cantor–Bendixson height. Hence every compact subspace of $A(S)$ has finite Cantor–Bendixson height. But $\mathbb{Q}$ contains compacta of arbitrarily large (countable) Cantor–Bendixson height.

This last observation implies that every metrizable subspace of $A(S)$ is scattered. Indeed, if it were not scattered then it would contain a topological copy of $\mathbb{Q}$.

These results suggest the following problems.

**Problem 5.8.** Is every closed subgroup of a separable topological group with an $\omega$-bounded remainder TOG-closed?

**Problem 5.9.** Is there in ZFC a countable Rajkov complete topological group with an $\omega$-bounded remainder and no first-countable remainder?

The following problems are also quite interesting.

**Problem 5.10.** Is there a non-metrizable topological group with a first countable but not strongly $\omega$-bounded remainder?

**Problem 5.11.** Does there exist, under CH, a countable topological group with an $\omega$-bounded but not a strongly $\omega$-bounded remainder?

**References**


