Houston Journal of Mathematics
Volume 40, No. 3, 2014

HOMOGENEITY AND GENERALIZATIONS OF 2-POINT SETS

WILL BRIAN, JAN VAN MILL, AND ROLF SUABEDISSEN

Communicated by Yasunao Hattori

Abstract. We prove the existence of homogeneous κ-point sets in the plane for every finite $\kappa \geq 3$. We also show that for every zero-dimensional subset A of the real line there is a subset X of the plane such that every line intersects X in a topological copy of A.

1. Introduction

A two-point set is a subset X of the plane which meets every line in precisely 2 points. Since the first proof of the existence of two-point sets in [8], these rather strange geometric objects have received considerable interest.

Of course the notion has been generalized to κ-point sets (subsets of the plane meeting every line in precisely κ many points) and a wide variety of κ-point sets with some extra topological or geometric properties have been constructed for various values of κ. Typically, if one can obtain a two-point set satisfying some property $\mathcal{P}$, it is possible to construct κ-point sets satisfying property $\mathcal{P}$ (at least for finite κ). Curiously, [4] (for $\kappa = 2$) and [2] (for infinite $\kappa < \mathfrak{c}$) have constructed κ-point sets which are multiplicative subgroups of $\mathbb{C} \setminus \{0\}$ and in particular homogeneous. However, neither of these approaches generalizes directly to give a homogeneous κ-point set for $3 \leq \kappa < \aleph_0$.

In the first part of the paper we give a proof that homogeneous κ-point sets exist for finite $\kappa \geq 3$. This relies on a lemma showing that for zero-dimensional, first-countable topological spaces the notions of homogeneity and almost homogeneity coincide. We note, however, that although the κ-point sets so constructed are homogenous as topological spaces, they are not homogeneously embedded in the plane.

2010 *Mathematics Subject Classification.* 54G99,54H15.
Key words and phrases. two-point set, homogeneity, slice set.

Noting that the generalization to κ-point sets (for infinite values of κ) is rather coarse, we give a much finer more topological generalization in the second part of the paper: the idea is that if $\mathcal{P}$ is a topological property, we say that $X \subseteq \mathbb{R}^2$ is a $\mathcal{P}$-slice set if and only if for every line L the subspace $X \cap L$ satisfies $\mathcal{P}$. Of particular interest to us is the case when $\mathcal{P}$ is simply 'homeomorphic to A' for some fixed subset A of $\mathbb{R}$. In this case we simply talk about 'A-slice sets'. We show that if $1 \neq |A| < \mathfrak{c}$ or $1 \neq |\mathbb{R} \setminus A| < \mathfrak{c}$, then there is an A-slice set. We also observe that no $[0,1]$-slice set exists and give some further results concerning slice sets. In the final section we prove that, for any zero-dimensional subset A of $\mathbb{R}$, there is an A-slice set.

2. Notation

We use the following notation, common in work on κ-point sets: We use both $\mathbb{R}^2$ and $\mathbb{C}$ to denote the plane. The set of lines in the plane is denoted by $\mathcal{L}$ and usually well-ordered as $\{L_\alpha : \alpha < \mathfrak{c}\}$. If $A \subseteq \mathbb{R}^2$, the set of lines spanned by points of A is denoted by $\langle A \rangle = \{L \in \mathcal{L} : |L \cap A| \geq 2\}$. If A is infinite then $|\langle A \rangle| = |A|$. By a partial κ-point set we mean a subset X of the plane such that X meets every line in at most κ many points.

If G is a group acting on $\mathbb{C}$ and maps lines to lines then there is a natural induced action of G on $\mathcal{L}$. Typically we will not distinguish between these and no misunderstanding should arise. If G is a multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ then the action of G on $\mathbb{C}$ will always be given by multiplication $(g, z) \mapsto gz$.

3. Homogeneous n-point sets

Theorem 3.1 ([9]). *If X is a zero-dimensional, first-countable topological space which is almost homogeneous (i.e. for any $x, y \in X$ and any open $U \ni x$, $V \ni y$ there are clopen W, Z with $x \in W \subseteq U$ and $y \in Z \subseteq V$ such that W and Z are homeomorphic), then X is homogeneous.*

Theorem 3.2. *Suppose G is a countable, dense, partial two-point multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$, $n \geq 3$. If the natural action of G on the lines in $\mathbb{C}$ is faithful (i.e. for every line L and every $g \in G \setminus \{1\}$ we have $gL \neq L$) then there is a zero-dimensional n-point subset X of $\mathbb{C}$ such X is invariant under G, i.e. $GX = X$.*

For those familiar with the construction of two points sets, we give a sketch proof before embarking on the formal construction:

Sketch. The standard construction will be applied with the following modifications:

- for each n we will cover $\mathbb{C}$ by countably many closed disks of size $1/n$. Writing C for the union of their boundaries, we note that C intersects each line in at most countably many points and that GC does so as well, as each $g \in G$ maps circles to circles and G is countable. We will ensure that $X \cap GC = \emptyset$ so that X is zero-dimensional. This excludes only a small number of points on each line.
- when adding a point on a line L we will of course add Gx to X. We will choose x such that Gx is disjoint from any line in a different G-coset with at least 2 points on it already. Since G and hence Gx is a partial two-point set, this will ensure that the new X will still be a partial n-point set. As there are only a small number of lines with at least 2 points on it and G is countable, this excludes only a small number of points on the given line. Note that 2 may be replaced by $n-1$.
- when adding a point on a line L with $n-1$ points already on that line, then by the faithfulness of the action (and countability of G) there are only countably many $x \in L$ with $Gx \cap L \neq \{x\}$. We will not add one of these small number of points to X.

□

Proof. For each $n \in \mathbb{N}$, use Lindeloefness of $\mathbb{C}$ to find $\{x^n_m \in \mathbb{C} \colon m \in \omega\}$ be such that $\bigcup_m B_{1/n}(x^n_m)$ covers $\mathbb{C}$. Let

$$C = \{x \in \mathbb{C} \colon \exists n \in \mathbb{N}, m \in \omega \ \ |x - x^n_m| = 1/n\}$$

be the union of the bounding circles of the $B_{1/n}(x^n_m)$. Since C is a union of countably many circles, it meets every line in at most countably many points. Since G is countable and maps lines to lines we have that GC meets every line L in at most countably many points Z_L.

We will now construct an n-point way modifying the familiar inductive construction.

Let $\{L_\alpha \colon \alpha < \mathfrak{c}\}$ be an enumeration of the lines of $\mathbb{C}$. We will construct sets $X_\alpha \subset \mathbb{C}$ and write $T_\alpha = \bigcup_{\beta \leq \alpha} X_\beta$ such that for each $\alpha < \mathfrak{c}$:

(1) $|X_\alpha| \leq \aleph_0$;
(2) $X_\alpha \subset \mathbb{C} \setminus GC$;
(3) $GX_\alpha = X_\alpha$, i.e. X_α is invariant under G;
(4) T_α is a partial n-point set;
(5) $|T_\alpha \cap L_\alpha| = n$.

Once we have achieved this, we let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha = \bigcup_{\alpha<\mathfrak{c}} T_\alpha$. Clearly X is an n-point set invariant under G. Also $X \subseteq \mathbb{C} \setminus C$ so is zero-dimensional.

So, suppose we have constructed X_β (and T_β) for $\beta < \alpha$ satisfying the above properties. Set $T'_\alpha = \bigcup_{\beta<\alpha} X_\beta$ which has cardinality $< \mathfrak{c}$. Note that T'_α and hence $\langle T'_\alpha \rangle$ is invariant under G since all $X_\beta, \beta < \alpha$ are. There are two cases to consider:

If $k = |T'_\alpha \cap L_\alpha| = n$ we set $X_\alpha = \emptyset$ and note that X_α (and $T_\alpha = T'_\alpha \cup X_\alpha$) satisfies all the inductive conditions.

Assume otherwise, i.e. $k < n$. We will show how to obtain a countable $X'_\alpha \subset \mathbb{C} \setminus GC$ invariant under G such that $T'_\alpha \cup X'_\alpha$ is a partial n-point set and $|T'_\alpha \cup X'_\alpha \cap L_\alpha| > k$. Iterating this construction (with $T'_\alpha \cup X'_\alpha$ in place of T'_α) finitely often (up to n times) and taking the union of the obtained X'_α will clearly produce a set X_α as required.

For $L \in \langle T'_\alpha \rangle \setminus GL_\alpha$ we set

$$F_L = \bigcup_{g \in G} g^{-1} L \cap L_\alpha.$$

Note that since $g \in G$ maps lines to lines and $L \neq gL_\alpha$ we have $\left|g^{-1}L \cap L_\alpha\right| \leq 1$ so that $|F_L| \leq |G| = \aleph_0$. So the set

$$F = \bigcup_{L \in \langle T'_\alpha \rangle \setminus GL_\alpha} F_L$$

has cardinality $\aleph_0 . |T'_\alpha| < \mathfrak{c}$.

Next note that for $g \in G, g \neq 1$ there is at most one $x \in L_\alpha$ with $gx \in L_\alpha$: if $x, y \in L_\alpha$ were distinct with $gx, gy \in L_\alpha$ then, since g maps lines to lines, g would map L_α to itself. But by assumption G acts faithfully on lines, so we must have $g = 1$, a contradiction. We thus see that

$$S = \{x \in L_\alpha \colon \exists g \in G \setminus \{1\} \ \ gx \in L_\alpha\}$$

has cardinality $\leq |G| = \aleph_0$.

Finally, as noted above $GC \cap L_\alpha$ is countable and that $T'_\alpha \cap L_\alpha$ is finite.

We can therefore find $x \in L_\alpha \setminus (F \cup S \cup GC \cup T'_\alpha)$ and claim that $X'_\alpha = Gx$ is as desired. Clearly X'_α is countable and $T'_\alpha \cup X'_\alpha$ meets L_α in at least the additional point x compared to T'_α. It remains to show that $T'_\alpha \cup X'_\alpha$ is a partial n-point set.

To this end, assume not and let $L \in \mathcal{L}$ witness this fact. Note that T'_α is a partial n-point set and that G and hence $Gx = X'_\alpha$ is a partial 2-point set. Hence T'_α must meet L in at least $n-1$ points and since $n \geq 3$, we must have $L \in \langle T'_\alpha \rangle$. If $L \notin GL_\alpha$ then $x \notin F_L$ and thus for every $g \in G$, $gx \notin L$. But as $X'_\alpha = Gx$

we then must have $X'_\alpha \cap L = \emptyset$, implying that T'_α is not a partial n-point set, a contradiction. Thus there is $g \in G$ such that $gL_\alpha = L$. As T'_α and X'_α are G-invariant this implies that L_α meets $T'_\alpha \cup X'_\alpha$ in at least $n+1$ points. Since by assumption T'_α meets L_α in at most $n-1$ points, we must have that there is $h \in G$ with $x \neq hx \in L_\alpha$. But then $h \neq 1$ so that $x \in S$, a contradiction again. Hence $T'_\alpha \cup X'_\alpha$ is indeed a partial n-point set. □

We remark that the above proof is not subtle in its exclusion of points from L_α. We note for example that it is sufficient to define

$$F = \bigcup \{F_L \colon L \in \mathcal{L} \setminus GL_\alpha, |L \cap T'_\alpha| \geq n-1\}.$$

This might be exploited when one wishes to construct homogeneous n-point sets with additional properties (or in fact homogeneous A-slice sets).

Corollary 3.3. *Under the same assumptions as in 3.2 there is a homogeneous n-point subset of $\mathbb{C}$.*

Proof. Taking the n-point set from Theorem 3.2 we will show that it satisfies the conditions of Theorem 3.1. Clearly X is first-countable and zero-dimensional. Now suppose that $x, y \in X$ and $\epsilon > 0$. Without loss of generality $\epsilon < |y|/2$. Note that if $\delta < \epsilon$ and $g \in G$ satisfies $|gx - y| < \delta$ then $0 < m = \frac{|y|}{2|x|} < |g| < \frac{3|y|}{2|x|} = M = 3m$.

Since G is dense in $\mathbb{C}$ we have that Gx is dense in $\mathbb{C}$ so there is $g \in G$ with $gx \in B_{\epsilon m/4M}(y) \subseteq B_{\epsilon/2}(y)$ so that $g^{-1}y \in B_{\epsilon/4M}(x)$. Since X is Lindeloef it is strongly zero-dimensional and hence we can find an X-clopen W with $\overline{B_{\epsilon/4M}(x)} \subseteq W \subseteq B_{\epsilon/2M}(x)$ so that $gW \subseteq B_{\epsilon/2}(gx)$. We then have $y \in gW \subseteq B_\epsilon(y)$. By Theorem 3.1 X is homogeneous. □

From [4] we will use the following lemma to construct the partial two-point group required in the above results.

Lemma 3.4. *Let X be a partial two-point set such that $|X| < \mathfrak{c}$, let $L = \{re^{i\theta_0} \colon r \in \mathbb{R}\}, \theta_0 \notin \pi\mathbb{Q}$ such that $X \cap L = \emptyset$. Then there are fewer than $\mathfrak{c}$ many $g \in L$ such that $\bigcup_{n \in \mathbb{Z}} g^n(X)$ is not a partial two-point set.*

Lemma 3.5. *There is a countable, dense, partial two-point multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ such that the action of G on the lines in $\mathbb{C}$ is faithful.*

Proof. Let $\{B_n \colon n \in \omega\}$ be a countable basis of $\mathbb{C}$. By induction on n we will construct $g_n \in \mathbb{C} \setminus \{0\}$ and write G_n for the smallest multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ containing $\{g_m \colon m \leq n\}$. In general, if $A \subset \mathbb{C} \setminus \{0\}$ we will write $[A]$ for the group generated by A, i.e. the smallest multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ containing A. We will construct the g_n such that

(1) $g_n \in B_n$;
(2) G_n is a partial two-point set;
(3) G_n acts faithfully on lines.

Note that unless $g \in \mathbb{C}\setminus\{0\}$ has $\arg(g) = q\pi$ for some $q \in \mathbb{Q}$ we have that for every line $L \in \mathcal{L}$ $gL \neq L$. Thus to satisfy 3 it is sufficient that $1, \arg(g_0), \ldots, \arg(g_n)$ are linearly independent over $\pi\mathbb{Q}$.

We define $g_0 = 1$. Suppose we have obtained g_k and G_k for $k \leq n$ satisfying the above conditions. Let

$$\mathcal{L}_F = \{L \in \mathcal{L} \colon 0 \in L, \exists g \in L\ [G_k \cup \{g\}] \text{ does not act faithfully on lines}\}.$$

By the comment above $\mathcal{L}_F$ is a countable set. We can thus find a line $L = \left\{re^{i\theta_0} \colon r \in \mathbb{R}\right\}$ such that $L \notin \mathcal{L}_F$, $L \cap G_n = \emptyset$, $L \cap B_n \neq \emptyset$ (and $\theta_0 \notin \pi\mathbb{Q}$ which would follow anyway from $L \notin \mathcal{L}_F$). By Lemma 3.4 and the fact that $|L \cap B_n \setminus \{0\}| = \mathfrak{c}$ as well as $|G_k| < \mathfrak{c}$, we can find $g_{n+1} \in L \cap B_n$ such that $[G_k \cup \{g_{n+1}\}] = \bigcup_{n \in \mathbb{Z}} g^n(G_k)$ is a partial two-point set, as required.

Finally, let $G = [\{g_n \colon n \in \omega\}] = \bigcup_{n\in\omega} G_n$ and observe that G is as required. □

Corollary 3.6. *There are homogeneous n-point sets for $3 \leq n < \aleph_0$.*

As remarked in the introduction, it is known that for $\kappa = 2$ and $\aleph_0 \leq \kappa < \mathfrak{c}$ there are κ-point sets which are homogeneously embedded in the plane (which are in fact multiplicative subgroups of $\mathbb{C}$). It is easy to see that for $3 \leq \kappa < \aleph_0$ there is no multiplicative subgroup of $\mathbb{C}$ which is a κ-point set. However, the following is open:

Question 3.7. *For $3 \leq \kappa < \aleph_0$, are there κ-point sets which are homogeneously embedded in the plane?*

4. Slice sets

Definition 1. Let A be a subset of $\mathbb{R}$. $X \subseteq \mathbb{R}^2$ is an **A-slice set** (or a slice set for A) if, for every line L in $\mathbb{R}^2$, the intersection $X \cap L$ is homeomorphic to A. More generally, if $\mathcal{P}$ is a topological property then we say that $X \subseteq \mathbb{R}$ is a $\mathcal{P}$-slice set whenever $X \cap L$ has property $\mathcal{P}$ for every line L.

Now that we are interested in the topological structure of $X \cap L$, we can't simply add points in an inductive construction. The following lemma is the key to solve this problem.

Lemma 4.1. *Suppose that A and B are subsets of $\mathbb{R}$ with $2 \leq |A| < \mathfrak{c}$ and $|B| < \mathfrak{c}$, and suppose that $x_1, x_2 \in \mathbb{R} \setminus B$. There is a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f[A] \cap B = \emptyset$ and $x_1, x_2 \in f[A]$. Moreover, f can be taken to be C^∞.*

Proof. Assume that $x_1, x_2 \in A$ (if not, we can dilate and translate $\mathbb{R}$ so that this becomes true). Consider the three intervals $(-\infty, x_1)$, (x_1, x_2), and (x_2, ∞). We will find three C^∞ automorphisms of $\mathbb{R}$, each of which is the identity off of one of these intervals, and, on the interval where it is not the identity, maps points of A to $\mathbb{R} \setminus B$.

There is a C^∞ bump function ψ on $\mathbb{R}$ with the following properties:

- $\psi(x) = 0$ for all $x \notin (x_1, x_2)$.
- $\psi(x) > 0$ for all $x \in (x_1, x_2)$.
- There is a positive constant h_0 such that, for $0 < h < h_0$, $\left|\frac{d(h\psi)}{dx}\right| < 1$ at every point in $\mathbb{R}$.

If $0 < h < h_0$ then the map $\phi_h(x) = x + h\psi(x)$ is a C^∞ automorphism of $\mathbb{R}$. We claim that there is a constant h_1 such that $0 < h_1 < h_0$ and, for all $x \in A \cap (x_1, x_2)$, $\phi_{h_1}(x) \notin B$. Suppose that this is not the case. Then, for every $h \in (0, h_0)$, there is (at least one) pair $(a_h, b_h) \in A \times B$ such that $a_h \in (x_1, x_2)$ and $\phi_h(a_h) = b_h$. If $h < h'$ and $a_h = a_{h'}$ then, since $\psi(a_h) > 0$,

$$b_h = a_h + h\psi(a_h) < a_h + h'\psi(a_h) = b_{h'}$$

It follows that $(a_h, b_h) \neq (a_{h'}, b_{h'})$ whenever $h \neq h'$. This is impossible since $|A \times B| < \mathfrak{c}$. Thus some such h_1 exists. $f_1 = \phi_{h_1}$ is a C^∞ automorphism of $\mathbb{R}$ which is the identity on $\mathbb{R} \setminus (x_1, x_2)$ and which maps all $x \in A \cap (x_1, x_2)$ into $\mathbb{R} \setminus B$.

Similarly, there is a C^∞ automorphism f_2 of $\mathbb{R}$ which is the identity on $\mathbb{R} \setminus (-\infty, x_1)$ and which maps all $x \in A \cap (-\infty, x_1)$ into $\mathbb{R} \setminus B$, and there is a C^∞ automorphism f_3 of $\mathbb{R}$ which is the identity on $\mathbb{R} \setminus (x_2, \infty)$ and which maps all $x \in A \cap (x_2, \infty)$ into $\mathbb{R} \setminus B$. Set $f = f_3 \circ f_2 \circ f_1$. □

Using this lemma, we can do 'the usual' inductive reconstruction, being careful never to put more than two points onto a line L before we are at the appropriate stage (when $L_\alpha = L$) in the recursion.

Theorem 4.2. *If $A \subseteq \mathbb{R}$ and $2 \leq |A| < \mathfrak{c}$, then there is a slice set for A.*

Proof. Let $\langle L_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of all lines in $\mathbb{R}^2$. As above, we build X by transfinite recursion. Let $X^0 = \emptyset$. Let $\alpha < \mathfrak{c}$ and assume that we have constructed $\langle X^\beta : \beta < \alpha \rangle$ such that

- For $\gamma < \beta < \alpha$, $X^\beta \cap L_\gamma$ is homeomorphic to A
- For $\gamma \geq \beta < \alpha$, $|X^\beta \cap L_\gamma| \leq 2$
- If $\gamma < \beta < \alpha$ then $X^\gamma \subseteq X^\beta$

If α is a limit ordinal, take $X^\alpha = \bigcup_{\beta<\alpha} X^\beta$. If $\alpha = \beta + 1$ then, by assumption, $X^\beta \cap L_\beta$ contains at most two points, say x_1 and x_2. Let

$$B = \{x \in L_\beta : x \notin X^\beta \text{ but } x \in L_\gamma \text{ for some } \gamma < \beta\}$$

$$B' = \{x \in L_\beta : \text{ for some } \gamma > \beta,\ \left|L_\gamma \cap X^\beta\right| = 2 \text{ and } x \in L_\gamma\}$$

It is straightforward to show that $|B| < \mathfrak{c}$ and $|B'| < \mathfrak{c}$. By Lemma 4.1, there is a subset Y of L_β which is homeomorphic to A, which includes both x_1 and x_2, and which is disjoint from $B \cup B'$. Setting $X^\alpha = X^\beta \cup Y$, it is clear that X^α satisfies the inductive hypotheses, so this completes the induction. $X = \bigcup_{\alpha<\mathfrak{c}} X^\alpha$ is the desired slice set. □

Corollary 4.3. *If $A \subseteq \mathbb{R}$ and $2 \leq |\mathbb{R} \setminus A| < \mathfrak{c}$, then there is a slice set for A.*

Proof. By Theorem 4.2 there is a slice set X for $\mathbb{R} \setminus A$. Furthermore, because of our use of Lemma 4.1 in the induction step, X has the additional property that for every line L in $\mathbb{R}^2$ there is a homeomorphism $\mathbb{R} \to L$ which restricts to a homeomorphism from $\mathbb{R} \setminus A$ onto $X \cap L$. Taking complements, it follows that for every line L in $\mathbb{R}^2$ there is a homeomorphism $\mathbb{R} \to L$ which restricts to a homeomorphism from A onto $L \setminus (X \cap L) = L \cap (\mathbb{R}^2 \setminus X)$. Thus $\mathbb{R}^2 \setminus X$ is a slice set for A. □

Using the techniques of section 3 in the case that $3 \leq |A| < \mathfrak{c}$, we may take our A-slice sets to be homogeneous.

Corollary 4.4. *If $A \subseteq \mathbb{R}$ and $3 \leq |A| < \mathfrak{c}$, then there is a homogeneous subset of $\mathbb{R}^2$ which is a slice set for A.*

Proof. The proof follows closely the proof of Theorem 3.2. The only extra tool that is needed is a modification of Lemma 4.1: Suppose that A and B are subsets of $\mathbb{R}$ with $3 \leq |A| < \mathfrak{c}$ and $|B| < \mathfrak{c}$, and suppose that $x_1, x_2, x_3 \in \mathbb{R} \setminus B$; then there is a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $\phi[A] \cap B = \emptyset$ and $x_1, x_2, x_3 \in \phi[A]$. The proof of this lemma is similar to the original proof of Lemma 4.1. □

Corollary 4.5. *It is consistent with ZFC that there is a homogeneous subset X of $\mathbb{R}^2$ such that, for every line L, $X \cap L$ is rigid.*

Proof. It is shown in [1] that there is a generic extension in which there is a rigid subset of $\mathbb{R}$ of cardinality less than $\mathfrak{c}$. Applying Corollary 4.4, we obtain the desired result. □

Unlike Theorem 4.2, Corollary 4.4 does not extend via complementation to the case $3 \leq |\mathbb{R} \setminus A| < \mathfrak{c}$, and it remains unknown whether homogeneous A-slice sets exist for such A.

It is obvious that there are slice sets for $\emptyset$ and $\mathbb{R}$ and that there is not a slice set for a singleton. This observation, together with Theorem 4.2 and Corollary 4.3, nearly answers the question of the existence of slice sets for small and co-small subsets of $\mathbb{R}$ (the one case which remains unsolved is $|\mathbb{R} \setminus A| = 1$, i.e., a subset of the plane which meets every line in exactly two open intervals). The next natural question to ask is: for which subsets A of $\mathbb{R}$ with $|A| = |\mathbb{R} \setminus A| = \mathfrak{c}$ do A-slice sets exist? A general characterization has not been found, but in the next section we will show that any totally disconnected subset of $\mathbb{R}$ has a slice set. The following theorem summarizes a few results for various subsets of $\mathbb{R}$ which are not totally disconnected:

Theorem 4.6.
(i) There is no slice set for $[0,1]$ *or for* $[0,1)$.
(ii) There are slice sets for a countable sum of closed intervals and for a countable sum of open intervals.
(iii) If $A \subseteq \mathbb{R}$ *is such that* $A = -A$ *and, for any* $r \in \mathbb{R}$, A *is homeomorphic to the image of* A *in the quotient space* $\mathbb{R}/[-r,r]$, *then there is a slice set for* A. *Note that this property is not topological, so it is sufficient for* A *to be homeomorphic to such a space.*

Proof.
(i) Suppose that X is a slice set for $[0,1]$ or for $[0,1)$. Since either of $[0,1]$ or $[0,1)$ is connected, X is convex. For each line L in $\mathbb{R}^2$, there is an open ray in L which does not belong to X, i.e., some $p \in L$ such that every point of L on one side of p does not belong to X; without loss of generality, we may take the origin to be in X and, using polar coordinates, take the open ray $\{(r,\pi) \colon r > 0\}$ to be a subset of $\mathbb{R}^2 \setminus X$. Let $\theta_1 \leq \pi$ be the smallest and $\theta_2 \geq \pi$ the largest values for which $\mathbb{R}^2 \setminus X$ contains the open wedge

$$W = \{(r,\theta) \colon r > 0, \theta_1 < \theta < \theta_2\}$$

If $\theta_2 - \theta_1 < \pi$, consider the open ray $R = \left\{(r, \frac{\theta_1+\theta_2}{2}) \colon r > 0\right\}$; even in the degenerate case $\theta_1 = \theta_2 = \pi$, we have $R \subseteq \mathbb{R}^2 \setminus X$. Let

$$A = \left\{(r,\theta) \colon r > 0, \frac{\theta_1+\theta_2}{2} - \frac{\pi}{2} < \theta < \frac{\theta_1+\theta_2}{2}\right\}$$
$$B = \left\{(r,\theta) \colon r > 0, \frac{\theta_1+\theta_2}{2} < \theta < \frac{\theta_1+\theta_2}{2} + \frac{\pi}{2}\right\}$$

These are the two quadrants on either side of R. It must be that either $A \subseteq \mathbb{R}^2 \setminus X$ or $B \subseteq \mathbb{R}^2 \setminus X$; otherwise, by the convexity of X, we can find a point of R which is in X. This contradicts either the minimality of θ_1 or the maximality of θ_2; thus we have $\theta_2 - \theta_1 \geq \pi$. However, if $\theta_2 - \theta_1 \geq \pi$, then there is a line in $\mathbb{R}^2$ which is completely contained in $\mathbb{R}^2 \setminus X$, contradicting the assumption that X is a slice set for a nonempty set.
(ii) Consider the hexagonal honeycomb packing of circles of radius 1 in the plane. Keeping the centers of the circles fixed, shrink the radius of each circle by some constant $\frac{2-\sqrt{3}}{2} < c < 1$. Now remove the interiors of the circles. The set which remains meets every line in a countable sum of closed intervals. The complement of this set meets every line in a countable sum of open intervals.
(iii) For each $r > 0$, let C_r denote the circle of radius r centered at the origin and let $C_0 = \{(0,0)\}$. Take $X = \bigcup_{a \in A \cap [0,\infty)} C_a$. $\square$

Many open questions remain concerning slice sets for $A \subseteq \mathbb{R}$, $|A| = |\mathbb{R} \setminus A| = \mathfrak{c}$. For instance, it is unknown whether there is a slice set for $[0,1] \times \{0,1\}$, $[0,1] \times n$, or, more generally, whether there is a subset of $\mathbb{R}^2$ which meets every line in a finite union of closed intervals. Similarly, it is unknown whether there is a slice set for $(0,1) \times \{0,1\}$ (Corollary 4.3 covers the case of larger sums of open intervals).

Alternatively, we can ask for a subset of the plane which meets every line in a unique way:

Lemma 4.7. *The number of distinct homeomorphism classes of countable subsets of* $\mathbb{R}$ *is* $\mathfrak{c}$.

Proof. Every countable subset of $\mathbb{R}$ can be embedded in $\mathbb{Q}$, so the number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at most $|\mathcal{P}(\mathbb{Q})| = \mathfrak{c}$.

Let $X \subseteq \mathbb{R}$. Let P be the largest dense-in-itself subset of X and let $S = X \setminus P$ be the scattered part of X. We define the **scattered signature** $H(X)$ of X as follows. $H(X)$ is a set of ordinals, and $\alpha \in H(X)$ if and only if there is some $p \in P$ such that p has Cantor-Bendixson rank α in $S \cup \{p\}$.

Let $A = \{\alpha_n\}_{n \in \mathbb{N}}$ be a countable subset of ω_1. We show that there is a countable subset of $\mathbb{R}$ with scattered signature A. On the interval $[n + \frac{1}{4}, n + \frac{1}{2}]$, embed $\omega^{\alpha_n} + 1$, making sure that the point ω^{α_n} maps to the point $n + \frac{1}{2}$. Include the points $\mathbb{Q} \cap [n + \frac{1}{2}, n + \frac{3}{4}]$ and call the resulting set X. It is a routine exercise to show that $H(X) = A$.

As there are $\mathfrak{c}$-many countable subsets of ω_1, this proves that the number of distinct homeomorphism classes of countable subsets of $\mathbb{R}$ is at least $\mathfrak{c}$. $\square$

Theorem 4.8. *There is a subset of the plane whose intersection with each line has unique homeomorphism type, i.e., no two such intersections are homeomorphic.*

PROOF. Let $\langle A_\alpha : \alpha < \mathfrak{c}\rangle$ be a sequence of countable subsets of $\mathbb{R}$ such that if $\alpha \neq \beta$ then A_α is not homeomorphic to A_β.

We now construct the desired set X by transfinite induction. The construction is exactly the same as in Theorem 4.2 except that, at the successor step $\alpha + 1$, we use Lemma 4.1 to guarantee that $L_\alpha \cap X$ is homeomorphic to A_α. $\square$

5. Zero-dimensional subsets of $\mathbb{R}$

In this section we will use algebraically independent Cantor subsets of $\mathbb{R}$ to construct a Cantor-slice set. We then use this to show that for any zero-dimensional subset A of $\mathbb{R}$ an A-slice set exists. The use of algebraic independence is interesting for the following reason:

When one wants to construct a Cantor-slice set, the fundamental problem with an inductive construction is that there are Cantor sets $C \subseteq \mathbb{R}$ such that $C - C$ contains an interval. If one wants to carry out an inductive construction, then for each $\alpha \geq 3$ one has $\mathfrak{c}$ many lines already containing two points. But since C is compact the moment one has chosen to include infinitely many points on a particular line L, one must have its closure in the eventual slice-set, which may of course cause problems. So a simple counting argument will not work for the construction of Cantor-slice sets.

To get around this problem, the first author replaced the notion of 'smallness' as '$< \mathfrak{c}$ many' by 'null set'. However, as the inductive construction may be longer than $\aleph_1$ (depending on whether or not CH holds) one needs to ensure that the ideal of null subsets of $\mathbb{R}$ is $< \mathfrak{c}$-complete, e.g. by Martin's Axiom. Choosing the Cantor sets carefully then yields a construction of a Cantor-slice set which is consistent relative to ZFC.

By again replacing the notion of 'smallness' by $\mathbb{R}$ having a $\mathfrak{c}$-transcendence degree over the Cantor set, we were finally able to achieve a ZFC-construction of a Cantor-slice set. We note that this is reminiscent of the improvement of the construction of a 2-point set contained in the union of countably many concentric circles from a consistency result (in this case [5] needed CH) to a ZFC-result by [11] which also used algebraic independence in an essential way.

We briefly review some of the necessary terminology from algebra and only consider subfields of $\mathbb{C}$. If A is a subset of $\mathbb{C}$ the field generated by A is the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q} \cup A$ (which can be obtained by intersecting all subfields of $\mathbb{C}$ containing $\mathbb{Q} \cup A$). A subfield F of $\mathbb{C}$ is algebraically closed if for every polynomial p with coefficients in F and every $z \in \mathbb{C}$ with $p(z) = 0$ we have $z \in F$. The algebraic closure of A is the smallest algebraically closed subfield containing $\mathbb{Q} \cup A$ (again obtained by taking intersections over all algebraically closed subfields containing $\mathbb{Q} \cup A$). We say that A is algebraically independent if and only if no $x \in A$ belongs to the algebraic closure of $A \setminus \{x\}$. The transcendence degree of some field F over some field F' is $\min\{|B| : B \subseteq F \text{ and } F \text{ is contained in the algebraic closure of } F' \cup B\}$. We relativize these notions to $\mathbb{R}$ in the obvious way (i.e. replacing $\mathbb{C}$ by $\mathbb{R}$ in the above constructions). For more details, we refer the reader to [10].

For the construction, let us first note that there are algebraically independent Cantor subsets of $\mathbb{R}$ (see for example [6], Lemma 3.9). By partitioning such a subset into two Cantor subsets (for example) we have that there is a Cantor subset C of $\mathbb{R}$ such that the transcendence degree of $\mathbb{R}$ over the (relative to $\mathbb{R}$) algebraic closure of C is $\mathfrak{c}$ ([10] Theorem VIII.1.1).

Theorem 5.1. *Suppose C is a subset of $\mathbb{R}$ such that the transcendence degree of $\mathbb{R}$ over the (relative to $\mathbb{R}$) algebraic closure of C is $\mathfrak{c}$ and such that the union of two copies of C in $\mathbb{R}$ is homeomorphic to C. Then there is a subset X of the plane such that for every line L the set $L \cap X$ is homeomorphic to C. In particular there is a Cantor-slice set.*

Proof. As always, we will well order the lines in the plane as $\langle L_\alpha \colon \alpha < \mathfrak{c}\rangle$ and we will parametrize a line L_α as

$$L_\alpha(t) = r_\alpha e^{i\theta_\alpha} + te^{i(\theta_\alpha + \frac{\pi}{2})} = r_\alpha(\cos\theta_\alpha, \sin\theta_\alpha) + t(-\sin\theta_\alpha, \cos\theta_\alpha)$$

for $r_\alpha \in \mathbb{R}_0^+$ and $\theta_\alpha \in [0, \pi)$ such that $r_\alpha e^{i\theta_\alpha}$ is the closest point of L to 0. We identify L_α with $\mathbb{R}$ by the parametrization given above, so that if we talk about a real number t (resp. a set A of real numbers) 'interpreted as a an element (resp. subset) of L_α' we mean the point $L_\alpha(t) \in \mathbb{R}^2$ (resp. the set $\{L_\alpha(a) \colon a \in A\} \subseteq L_\alpha \subset \mathbb{R}^2$) whereas if we consider t (resp. A) 'interpreted as an element (resp. subset) of $\mathbb{R}$' we mean the actual real number (resp. set of real numbers). Let us note that the two coordinates of $L_\alpha(t)$ are given by an algebraic equation in $\{r_\alpha, \cos\theta_\alpha, \sin\theta_\alpha, t\}$. For each α, we will write $D_\alpha = \{r_\alpha, \cos\theta_\alpha, \sin\theta_\alpha\}$.

Fix some $i_0 \in C$.

We will construct sequences $a_\alpha, b_\alpha, t_\alpha \in \mathbb{R}, \alpha < \mathfrak{c}$ by transfinite induction. Having constructed $a_\beta, b_\beta, t_\beta$ for $\beta < \alpha$ we will write $C_\beta = (a_\beta + t_\beta(C - i_0)) \cup (b_\beta + t_\beta(C - i_0))$ (interpreted as a subset of L_β) and $P_\alpha = \bigcup_{\beta<\alpha} C_\beta$ (interpreted as a subset of $\mathbb{R}^2$). The conditions of our transfinite induction are:

(1) $P_\alpha \cap L_\alpha \subseteq \{L_\alpha(a_\alpha), L_\alpha(b_\alpha)\}$
(2) If $L_\alpha(a_\alpha) \notin \bigcup_{\beta<\alpha} L_\beta$ then $L_\alpha(a_\alpha) \notin \langle P_\alpha \rangle$ and similarly for b_α.
(3) t_α is not in the algebraic closure of

$$C \cup \bigcup_{\beta \le \alpha} D_\beta \cup \{t_\beta : \beta < \alpha\} \cup \{a_\beta, b_\beta : \beta \le \alpha\}.$$

(4) For $\gamma > \alpha$ we have $|P_{\alpha+1} \cap L_\gamma| \le 2$.

So suppose we are at stage α in our construction (if $\alpha = 0$ then of course $P_\alpha = \emptyset$, $\bigcup_{\beta<\alpha} L_\beta = \emptyset$ etc):

The key fact is that the (relative to $\mathbb{R}$) algebraic closure of $\langle P_\alpha \rangle \cap L_\alpha$ in $\mathbb{R}$ (after identifying L_α with $\mathbb{R}$) is still small, in the sense that $\mathbb{R}$ has transcendence degree $\mathfrak{c}$ over it. For suppose $t \in L_\alpha$ (identified with $\mathbb{R}$) is in $\langle P_\alpha \rangle$. Then there are $\beta_i < \alpha, i = 1, 2$ with $t_i \in C_{\beta_i}$ (interpreted as a subset of $\mathbb{R}$) such that t, t_1, t_2 are collinear (this time interpreting t,t_1, and t_2 as points on L_α, L_{β_1} and L_{β_2} respectively). But collinearity of these points can be expressed as an algebraic equation in their coordinates and hence as an algebraic equation over $D_\alpha \cup D_{\beta_1} \cup D_{\beta_2} \cup \{t, t_1, t_2\}$ (where again t, t_1, t_2 are interpreted as real numbers). But each $t_i \in C_{\beta_i}$ so is given by an algebraic equation over $C \cup \{t_{\beta_i}, i_0, a_{\beta_i}, b_{\beta_i}\}$. Hence $\{t \in L_\alpha : t \in \langle P_\alpha \rangle\}$ is contained in the (relative to $\mathbb{R}$) algebraic closure A_α of $C \cup \{t_\beta, a_\beta, b_\beta : \beta < \alpha\} \cup \bigcup_{\beta \le \alpha} D_\beta$. Since $\mathbb{R}$ has transcendence degree $\mathfrak{c}$ over C and $\{t_\beta, a_\beta, b_\beta : \beta < \alpha\} \cup \bigcup_{\beta \le \alpha} D_\beta$ has size $< \mathfrak{c}$, this means that $\mathbb{R}$ has transcendence degree $\mathfrak{c}$ over A_α. In particular $\mathbb{R} \setminus A_\alpha \neq \emptyset$ and if F is any finite subset of $\mathbb{R}$ then $\mathbb{R}$ still has transcendence degree $\mathfrak{c}$ over the algebraic closure of $A_\alpha \cup F$.

Next note that P_α will meet L_α in at most two points: if α is a successor then this follows from condition 4 from the previous step of the induction; if on the other hand α is a limit and P_α would meet L_α in three points, then there is $\beta < \alpha$ such that $P_{\beta+1}$ already meets L_α in three points, a contradiction to condition 4 at inductive stage β again.

If the intersection $P_\alpha \cap L_\alpha$ has exactly two points, we choose a_α, b_α so that $L_\alpha(a_\alpha), L_\alpha(b_\alpha)$ are precisely those two points. If it only meets L_α in one point, then we choose $a_\alpha = b_\alpha$ to be that point. Finally, if the intersection is empty, then we choose $a_\alpha = b_\alpha$ outside A_α.

Lastly, since $\mathbb{R}$ had transcendence degree $\mathfrak{c}$ over A_α, it will have transcendence degree $\mathfrak{c}$ over the algebraic closure of $A_\alpha \cup \{a_\alpha, b_\alpha\}$ so we can choose t_α such that condition 3 is satisfied. Note that by this choice and an argument as in the first paragraph, this means that the set C_α will meet $\langle P_\alpha \rangle$ in at most those points in which it already meets P_α. Hence the fourth inductive condition continues to be satisfied.

The construction of X is now clear: simply take $X = \bigcup_{\alpha<\mathfrak{c}} C_\alpha$. □

Corollary 5.2. *If $A \subseteq \mathbb{R}$ is zero-dimensional then there is a slice set for A.*

Proof. A can be embedded in a Cantor set in such a way that any two pre-assigned points are in the image of A. We can follow the construction of the previous theorem (Theorem 5), but instead of choosing a Cantor set C_α at stage α of the construction, we embed A into C_α, such that $L_\alpha(a_\alpha)$ and $L_\alpha(b_\alpha)$ both belong to the image of A under the embedding. □

References

[1] W. R. Brian, "Topics in general and set-theoretic topology," doctoral thesis, University of Oxford (2012).
[2] B. Chad, *Doctoral Thesis*, University of Oxford, (2011).
[3] B. Chad and R. Suabedissen, *Symmetries of two-point sets*, Topology Appl. 155 (2008), 1213-1220.
[4] B. Chad and R. Suabedissen, *On affine groups admitting invariant two-point sets*, Topology Appl. 156 (2009), 2209-2213
[5] B. Chad, R. W. Knight and R. Suabedissen, *Set-theoretic constructions of 2-point sets*, Fund. Math. 203 (2009) 179-189
[6] J. J. Dijkstra, K. Kunen and J. van Mill, *Hausdorff measures and two point set extensions*, Fund. Math. 157 (1998) 44-60
[7] Kunen, K. *Set Theory.* Studies in Logic and the Foundations of Mathematics, vol. 102. (1980), Elsevier, Amsterdam, The Netherlands.
[8] S. Mazurkiewicz, *On a certain planar set that has precisely two points in common with each straight line*, C. R. Acad. Sci. Lettres Varsovie 7 (1914), 382-384 (in Polish).
[9] J. van Mill, *Homogeneous subsets of the real line*, Comp. Math. 45 (1982), 3-13.
[10] S. Lang, *Algebra*, 3rd ed., Addison-Wesley, Reading, 1993
[11] J. H. Schmerl, *Some 2-point sets*, Fund. Math. 208 (2010), 87-91

Received February 1, 2012
Revised version received May 8, 2012

William R. Brian, Department of Mathematics, Tulane University
E-mail address: wbrian.math@gmail.com

Jan van Mill Korteweg-de Vries Institute for Mathematics, University of Amsterdam
E-mail address: j.vanMill@uva.nl

Rolf Suabedissen, Department of Mathematics, University of Oxford
E-mail address: suabedis@maths.ox.ac.uk