Topological Homogeneity

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1 Some preliminaries: notation and terminology

All topological spaces considered in this article are assumed to be Hausdorff. In terminology and notation we follow rather closely Engelking's book [65]. Some of the definitions we recall below. Notice that in the definitions of cardinal invariants, such as the weight, the density, the Souslin number, the tightness and others, we use only infinite cardinal numbers.

The *tightness* $t(X)$ of a space X is the least cardinal number κ with the property that if $A \subset X$ and $x \in \overline{A}$, then there is some set $B \in [A]^{\leq\kappa}$ such that $x \in \overline{B}$.

A π-*base* of a space X at a point $x \in X$ is a family η of non-empty open subsets of X such that every open neighbourhood of x contains some member of η.

We denote by $\pi\chi(x, X)$ the smallest infinite cardinal number τ such that X has a π-base η at x such that $|\eta| \leq \tau$.

We denote by $\chi(x, X)$ the smallest infinite cardinal number τ such that X has a base η at x such that $|\eta| \leq \tau$.

A *pseudobase* of a space X at a point $x \in X$ is a family η of open neighbourhoods of x in X such that $\bigcap \eta = \{x\}$. We denote by $\psi(x, X)$ the smallest infinite cardinal number τ such that X has a pseudobase η at x such that $|\eta| \leq \tau$.

The cardinal numbers $\pi\chi(x, X)$, $\chi(x, X)$, and $\psi(x, X)$ are called the *π-character, the character, and the pseudocharacter*, respectively, of X at x. Taking the suprema of these functions, we obtain the definitions of $\psi(X)$,

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$\pi(X)$, and $\chi(X)$ which we call the *pseudocharacter*, *π-character*, and *character* of X respectively. By $\psi w(X)$, $\pi w(X)$ and $w(X)$ we denote the *pseudoweight*, *π-weight* and *weight*. By $nw(X)$ we denote the *network-weight*, by $l(X)$ we denote the *Lindelöf-degree* of X. The Souslin number of X is denoted by $c(X)$, and its density by $d(X)$.

Recall also that a *base of a space X at points of a subset F* of X is a family of open subsets of X that contains a base of X at any point of F. We say that the *weight of F in X* does not exceed τ, and write $w(F, X) \le \tau$, if there exists a base $\mathscr{B}$ of X at points of F such that $|\mathscr{B}| \le \tau$.

We say that the *G_κ-density of a space X at a point x* does not exceed κ if there exists a closed G_κ-subset H of X and a set $S \in [X]^{\le\kappa}$ such that $x \in H \subseteq \overline{S}$. We say that the G_κ-density of X does not exceed κ, if the G_κ-density does not exceed κ at all points x in X.

If X is a topological space, μ is an infinite cardinal number, and $A \subseteq \mu$, then by π_A we denote the projection from X^μ onto X^A. By π we denote π_0, the projection on the first coordinate. Whenever $x \in X$, by $\mathrm{constant}(x)$ we denote the element of X^μ which is equal to x on all coordinates. If $x \in X^\mu$ and $A \subset \mu$, then by x_A we denote the point $\pi_A(x)$. If $Y \subset X^\mu$, then $Y_A = \{y_A : y \in Y\}$.

We say that $x \in X$ is a G_κ-point if $\{x\}$ is a G_κ-subset of X, i.e. iff $\psi(x, X) \le \kappa$. Recall that if X is compact, then x is a G_κ-point iff the character of x in X does not exceed κ.

We use D as abbreviation for the space $\{0, 1\}$.

A space X is *Polish* it it is separable and completely metrizable.

If A is a subset of a group G, then $\langle\langle A\rangle\rangle$ denotes the subgroup of G generated by A.

2 The direction of the article

Our main interest in this survey article is directed at the concept of homogeneity, at homogeneous spaces. A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism h of X such that $h(x) = y$.

There are well developed *homeomorphism theories* for (separable metrizable) manifold-like spaces, both finite- and infinite-dimensional. Such theories play a crucial role in the following fundamental characterization theorems:

1. Toruńczyk [162, 163]: Hilbert cube and Hilbert space manifolds,
2. Edwards and Quinn [51]: n-manifolds,
3. Bestvina [36]: Menger manifolds,
4. Ageev [2, 3, 4], Levin [105], Nagórko [136]: Nöbeling manifolds.

The characterization theorems of the Erdős spaces were based on a homeomorphism theory as well (Dijkstra and van Mill [55, 56]).

Our focus here is on homogeneous spaces that are not assumed to be close to a manifold; in fact, to homogeneous spaces on which very few assumptions are imposed and these assumptions are of quite general nature (like compactness). On the other hand, we also consider homogeneous spaces with some sort of pathology. There are many very interesting results on homogeneous metrizable continua which consequently will not be discussed by us (for example, the Bing-Borsuk Conjecture, see e.g. Halverson and Repovš [77]). It turns out that for the class of spaces we are interested in, many fundamental problems remain unsolved. Cardinal invariants and their theory are shown to be one of the main instruments in their study.

Clearly, the definition of homogeneity is a very transparent and natural definition. Informally, it means that a space is homogeneous if its topological structure at each point is the same, i.e. doesn't depend on the point. Of course, there is more than one way to formalize this idea; however, the definition we have adopted is, undoubtedly, the most natural one, and we will stick to it.

Of course, every topological group is a homogeneous space: translations are responsible for that. However, not every homogeneous space admits the structure of a topological group. Indeed, the Hilbert cube, i.e., the countable infinite product of copies of the unit interval $\mathbb{I} = [0, 1]$ is such an example. It is homogeneous by Kellers's Theorem from [98], but does not admit the structure of a topological group since it has the fixed-point property. Other examples are the spheres $\mathbb{S}^n$ for every $n \notin \{0, 1, 3\}$ (Samelson [151]). These examples are not so simple, but easier ones are readily available. Indeed, the 'double arrow' space of Alexandroff and Urysohn is first-countable, hereditarily separable and hereditarily Lindelöf, zero-dimensional, homogeneous, compact, but not metrizable. It follows that the 'double arrow' space is not homeomorphic to a topological group, since every first-countable topological group is metrizable.

There is a variety of known homogeneous compacta. Almost all of them belong to two subclasses. The first subclass is the class of homogeneous compacta which admit an algebraic structure of some sort. The second subclass consists of (products of) first-countable compacta that are homogeneous.

Examples in the second class include some of the ordered compacta constructed by Maurice [108, 109] and van Douwen [59]. Yet another example is the non-metrizable homogeneous Eberlein compact space constructed by van Mill in [111]. The cellularity (i.e. the Souslin number) of this compactum is 2^ω. It has to be uncountable, since every Eberlein compactum with countable Souslin number is metrizable (Arhangel'skii [17, Theorem III.5.8]).

It was shown by Dow and Pearl [62] that any infinite product of zero-dimensional first-countable spaces is homogeneous (for compact spaces, this was proved earlier by Motorov [134]). This is a highly non-trivial result which obviously generates many new examples of homogeneous spaces.

Homogeneity is not safe from set theory. It was shown by van Mill [118] that there is a compact space with countable π-weight and uncountable character which is homogeneous under MA+¬CH but not so under CH. A zero-dimensional variant of this space was constructed by Hart and Ridderbos [79]. So this is a consistent example of a homogeneous compactum that does not belong to the above two classes. See Milovich [129] for such an example in ZFC.

Jensen observed that it is easy to construct homogeneous compact Souslin lines from $\diamondsuit$. The square of such a space is another example of a compact homogeneous space with uncountable cellularity.

Semitopological groups are also homogeneous spaces, but compact semitopological groups are topological groups (see Arhangel'skii and Tkachenko [26, Theorem 2.3.13]). However, if we require only that left translations are continuous, then we obtain the definition of a left topological group, and compact left topological groups needn't be topological groups but are homogeneous spaces.

However, in private conversation with J. van Mill, Kunen remarked that it is not clear whether compact left topological group implies anything interesting which does not follow from just compact homogeneous. As he observed, not every compact homogeneous space is a left topological group. For example, the Hilbert cube is not a left topological group for the same reason that it is not a topological group.

There is also an example of a compact left topological group under $\diamondsuit$ which is first-countable and fails to have the countable chain condition (the square of the space in Kunen [102, Theorem 6.2]). So compact left topological groups need not be dyadic, and first-countable compact left topological groups need not be metrizable.

Kunen constructed under CH a compact L-space (i.e. perfectly normal and non-separable) which is even a left topological group, [102]. He asked whether there can be a homogeneous Souslin line which admits the structure of a left topological group. The square of Kunen's space satisfies the countable chain condition.

We have observed above that very often the homogeneity of a space is not easy to verify. We have also seen, however, that homogeneity can be an easy consequence of the existence of an algebraic structure nicely related to the topology of the space. We have already discussed some natural examples of this kind (semitopological and left topological groups). If this is the case, then the algebraic structure can be used to study topological properties of the homogeneous space. Keeping this in mind, it seems very natural to identify the most general situations in which homogeneity is algebraically generated.

It was shown by Bourbaki [38] that every homogeneous zero-dimensional compactum can be represented as a coset-space of some topological group (see [26]). Thus, the zero-dimensional homogeneous compactum constructed in [79] under MA+¬CH is a coset-space of a topological group. It follows that consistently *the π-character of a compact coset-space of a topological group may be countable while this coset-space is not first-countable.* Note that every topological group with countable π-character is first-countable, and hence, is metrizable.

There are many unsolved questions on homogeneous compacta, and some of them are famous problems formulated in quite elementary terms. To illustrate our ignorance in the matters of homogeneity, let us recall that it is not known whether every compact space can be represented as an image of some homogeneous compactum under a continuous mapping. This natural and fundamental problem has appeared in print for the first time in Arhangel'skii [16].

The first non-trivial results and problems on homogeneity date back to the fifties and sixties of the preceding century. We already noted that Keller has proved that the product of infinitely many copies of the closed unit interval $\mathbb{I}$ is homogeneous, [98]. Note that $\mathbb{I}$ itself is a standard example of a nonhomogeneous space.

In this connection, we say that a space X is *power-homogeneous* if X^μ is homogeneous for some cardinal number μ. Thus, $\mathbb{I}$ is power-homogeneous, but not homogeneous.

Among the first problems on homogeneity was the next question: does every infinite compact homogeneous space contain a non-trivial convergent sequence (this is W. Rudin's Problem [148]).

This problem has been posed in an attempt to prove that the Čech-Stone remainder of the discrete space of natural numbers is not homogeneous. This last task has been accomplished by different means by W. Rudin [147] and Frolík [75]. We will say more about this in the discussion to follow. However, Rudin's problem remains unsolved. Now it is more than 50 years old, and is one of the most challenging problems on homogeneity.

To E. van Douwen belongs another famous old question: does every compact homogeneous space have cellularity at most 2^ω (this we call *van Douwen's Problem*[1]).

Observe that a counterexample to one of these questions cannot be obtained as a continuous image of the product of some family of first-countable compacta. It also cannot be a topological group.

Indeed, any infinite compact group satisfies the countable chain condition and contains a non-trivial convergent sequence. Both statements follow easily from the Ivanovskij-Kuz′minov's Theorem in [104] and [90] that every compact group is a dyadic compactum.

Also, every first-countable compactum has cardinality at most 2^ω by Arhangel′skii [9] and hence has cellularity at most 2^ω. It is easy to see that an arbitrary product of compact topological groups and first-countable compacta does not yield a counterexample to van Douwen's Problem for basically the same reasons, cf., [92, p. 107].

[1] We are not aware of any paper of Eric van Douwen where this question was asked explicitly. But we know that he asked this in a letter to Jan van Mill from about 1980.

3 Some older results on homogeneity

One of the first surveys of older results on homogeneity was given in Arhangel'skii [16]. We refer the reader to it for the bulk of the material and for many problems posed there. Here we mention only a few of them which turned out to be most relevant to the later developments, including the most recent of them.

Observe also that in the translation to English of the article Arhangel'skii [16] the expression 'compact sequential' has been misinterpreted as 'sequentially compact" which has a very different meaning. This made impossible to properly understand some important statements in the English version of Arhangel'skii [16].

One of the first general results on cardinal invariants in homogeneous compacta was obtained in Arhangel'skii [12]: if X is a homogeneous compact sequential space, then $|X| \leq 2^\omega$. Under CH, this theorem implies that every homogeneous compact sequential space is first-countable [12]. One of the main pieces of technique on which the proofs of these results are based is the concept of free sequence (see [10, 14]). The next basic fact has been established in [12]; its role in the arguments is crucial.

Theorem 3.1. *Every non-empty sequential compactum X contains a non-empty closed G_δ-subset F such that $|F| \leq 2^\omega$.*

Sequential spaces constitute a large natural subclass of the class of spaces of countable tightness. The problem to extend the above results to homogeneous compacta of countable tightness has been posed in 1970 in Arhangel'skii's article in the Proceedings of the International Congress of mathematicians in Nice, [11, Problem 11][2]. This problem remained open until 2005 when it was solved by de la Vega. We give a detailed discussion of this remarkable achievement in one of the sections to follow. However, another question dating back to the same period of time remains open, [12]. Here it is:

Problem 3.2. Is it true in ZFC that every homogeneous compact sequential space is first-countable?

Another open problem (from [13]):

Problem 3.3. Is it true in ZFC that every homogeneous compact space of countable tightness is first-countable?

Observe that the next modification of Theorem 3.1 holds for compacta of countable tightness. It has been presented explicitly in [12].

[2] There is a typo in the original wording of this problem, *separable* should be be *countable tightness*. It was already known at that time that every compact group of countable tightness is metrizable, and hence of cadinality at most 2^ω. So asking this for homogeneous compacta was very natural.

Theorem 3.4. *Every non-empty compactum X with countable tightness contains a non-empty closed G_δ-subset F and a countable subset A such that $F \subset \overline{A}$.*

Thirty years later this result turned out to be one of the key elements in the proof of de la Vega's Theorem [170].

Let us now briefly consider some other older results on cardinal invariants of homogeneous spaces.

Recall that a subset U of a space X is said to be *canonically open* if it is the interior of its closure in X, that is, if U is the largest open set contained in the closure of U. We denote by $RO(X)$ the family of all canonically open subsets of X.

A prominent task in the theory of homogeneous spaces is to determine their cardinalies. One of the first estimates of this kind was established by M. Ismail in [89]:

Theorem 3.5. *(M. Ismail) Suppose that X is a homogeneous Tychonoff space. Then $|X| \leq |RO(X)|^{\pi\chi(X)}$.*

It has been observed in [16] that if X is a homogeneous Tychonoff space, and the Souslin number of X and the π-character of X are countable, then $|X| \leq 2^\omega$. This result was generalized by Carlson and Ridderbos [44] to power homogeneous spaces. In fact, they proved, using the Erdős-Rado Theorem, that if X is power homogeneous, then $|X| \leq 2^{\pi\chi(X)c(X)}$ (interestingly, no assumptions on separation axioms are needed in this result; for regular spaces the same inequality was first proved by Ridderbos [142]).

Here is another result of M. Ismail [89]:

Theorem 3.6. *If X is a homogeneous Tychonoff space of point-countable type, then $|X| \leq 2^{c(X)t(X)}$.*

Some strong results on the structure of homogeneous spaces of point-countable type were obtained under special assumptions. In particular, the next statement was established in 1970 in [10]:

Theorem 3.7. *If $2^\omega < 2^{\omega_1}$, then every homogeneous sequential space X of point-countable type is first-countable.*

In the proof of the above statement, as well as in the proofs of many other results mentioned in this survey, the following fact is used (see [14, Theorem 2.2.4]):

Theorem 3.8. *If X is a compact space with $t(X) \leq \kappa$, then the G_κ-density does not exceed κ at some point $e \in X$.*

Observe that under Martin's Axiom (MA) a weaker statement than Theorem 3.7 has been established in [13]:

Theorem 3.9. *If* MA *holds, then every homogeneous compact sequential space is Fréchet-Urysohn.*

It is not clear whether the last statement can be extended to compact homogeneous space of countable tightness. It is also not clear whether Theorem 3.9 can be proved in ZFC.

Observe, that M. Ismail in [89] established the following result which follows from Theorem 3.6 and which is now a very special case of de la Vega's Theorem:

Theorem 3.10. *The cardinality of an arbitrary hereditarily separable homogeneous compact space does not exceed* 2^ω.

However, the next question remains open:

Problem 3.11. Suppose that X is a homogeneous compact hereditarily separable space. Then does it follow in ZFC that X is first-countable?

Problem 3.12. Suppose that X is a homogeneous compact hereditarily separable space. Then does it follow in ZFC that X is perfectly normal?

We have already mentioned the very interesting and still unsolved problem on the existence of non-trivial convergent sequences in infinite homogeneous compacta posed by Walter Rudin in [148]. Here is a version of this problem which might have better chances to be solved soon:

Problem 3.13. Is it true that every infinite homogeneous compact space X with $w(X) \leq 2^\omega$ contains a non-trivial convergent sequence?

Of course, a complementary question to the last problem is also in line to be considered:

Problem 3.14. Is it true that every infinite homogeneous compact space contains an infinite homogeneous compact subspace Y such that $w(Y) \leq 2^\omega$?

A question related to this problem, which has been posed in [16], also remains unanswered:

Problem 3.15. Is it true that every infinite homogeneous compact space contains an infinite homogeneous compact subspace Y such that $|Y| \leq 2^\omega$?

We mention here another open question related to W. Rudin's Problem. Is it true in ZFC that every infinite homogeneous compact space of countable tightness contains a non-trivial convergent sequence?

Several classical results on homogeneity involve the space of closed subsets in the Vietoris topology. This functor, whci we will denote by Exp, preserves compactness, and it is natural to investigate when the result is a homogeneous compactum.

It is not difficult to prove that if X is the Cantor set, then $\mathsf{Exp}(X)$ is homeomorphic to X, and therefore, is a homogeneous compactum. A similar statement holds for D^{ω_1}, [159]. D.W. Curtis and R.M. Schori have established that if X is any Peano continuum without isolated points, then $\mathsf{Exp}(X)$ is homeomorphic to the Hilbert cube and hence, is homogeneous [50]. It follows that if X is any locally connected metrizable compactum without isolated points, then $\mathsf{Exp}(X)$ is the topological sum of finitely many Hilbert cubes, and is hence homogeneous as well. However, if $X = D^{\omega_2}$, then $\mathsf{Exp}(X)$ is no longer homogeneous [154]. In [155] a complete list of dyadic compacta, for which $\mathsf{Exp}(X)$ is homogeneous, is given.

Several necessary conditions for homogeneity of the space $\mathsf{Exp}(X)$ of closed subsets of a compact space X had been obtained in [16]. The next question, which seems to remains open, has been posed there:

Problem 3.16. Is it true that the space $\mathsf{Exp}(X)$ of closed subsets of the 'double arrow' space X (in the Vietoris topology) is homogeneous?

Problem 3.17. Suppose that X is a compact space such that $\mathsf{Exp}(X)$ is homogeneous. Then is it true that $c(X) \leq 2^\omega$?

See also the discussion in Nadler [135, Chapter 17].

4 The Cartesian product and some general questions on homogeneity

Clearly, every topological space X can be represented as an image of a homogeneous space under a continuous mapping: it suffices to give the set X the discrete topology.

V.V. Uspenskiy obtained a much more delicate result: in the same direction: he proved that for every Tychonoff space Y there exists a Tychonoff space Z such that the topological product $Y \times Z$ is homogeneous [167]. However, a similar statement for compacta doesn't hold. Indeed, D. B. Motorov [132] established that there exists a connected metrizable compact space Y such that $Y \times Z$ is not homogeneous, for any non-empty compact space Z. In fact, this Z is not a retract of any homogeneous compactum. See about this and related results [16] where the details are given.

However, in the class of zero-dimensional compacta the corresponding problem remains open. We wish to draw the reader's attention to some intriguing questions of this kind which seem to be of fundamental nature. In particular, the next question is open for any uncountable cardinal τ.

Problem 4.1. Given an infinite cardinal number τ, does there exist a zero-dimensional homogeneous compact space $B(\tau)$ such that $B(\tau) \times Y$ is homogeneous, for every zero-dimensional compact space Y of the weight $\leq \tau$?

Below τ stands for an infinite cardinal τ, and let $A(\tau)$ be the Alexandroff compactification of a discrete space of the cardinaliy τ.

Problem 4.2. Does there exists a non-empty compact space Y_τ such that the product space $A(\tau) \times Y_\tau$ is homogeneous?

The two questions above are obviously related.

Note that if the answer to the last question is positive for some cardinal number τ such that $2^\omega < \tau$, then we immediately obtain a homogeneous compact space X such that the Souslin number of X is greater than 2^ω. This would solve the old and famous problem posed by Eric van Douwen.

As the experiments show, we can use the product operation to construct natural examples of homogeneous compacta from non-homogeneous ones. For example, the topological product of any infinite number of copies of the closed unit interval is a homogeneous compactum (see [98]).

A result of D. B. Motorov [134] should be mentioned in this connection: X^ω is homogeneous, for any zero-dimensional first-countable compact space X.

Unfortunately, no other natural way to construct homogeneous compacta, besides Motorov's approach and taking compact topological groups, is known. This makes especially difficult to handle many long standing open questions on homogeneous compacta.

The full force of the product operation in constructing homogeneous compacta has not been well understood yet. Though some remarkable special results in this direction have been obtained, we do not really know what happens in some pretty standard situations. Note that Souslin number rather easily increases under products. Therefore, one may put some trust in the product operation when trying to solve van Douwen's problem.

Here are some concrete questions the answers to which are unknown. Recall that a space X is *extremally disconnected* if the closure of an arbitrary open subset of X is open.

Problem 4.3. Does there exist an infinite extremally disconnected compact space X and a non-empty compact space Y such that the product space $X \times Y$ is homogeneous?

Problem 4.4. Is there a non-empty compact space Y such that the product space $\beta\omega \times Y$ is homogeneous?

The second question is a special case of the first one. If a space Y satisfies the requirements in the last question, then Y is not metrizable. Indeed, if $\beta\omega \times Y$ is homogeneous, then Y is not first-countable at any of its points, see [19] (we will discuss this in the sections to come).

Problem 4.5. Is there a non-empty compact space Y such that the product space $(\beta\omega \setminus \omega) \times Y$ is homogeneous?

The last three problems are open. However, some closely related results are available. In particular, A. Dow and J. van Mill have shown [61] that if Y is a compact space such that the Souslin number of Y is countable, then the product space $(\omega_1+1)\times Y$ is not homogeneous. In fact, they have established the following theorem:

Theorem 4.6. *If Y is a compact space with the countable Souslin number, and X is a compact space with a non-isolated P-point, then the product space $X \times Y$ is not homogeneous.*

The last statement easily follows from another theorem established in [61]: no compact space can be covered by nowhere dense P-sets with the countable Souslin number.

So it is now easy to see that, under CH, $(\beta\omega \setminus \omega)\times Y$ is not homogeneous, for any non-empty compact space Y with the countable Souslin number. We do not know if this can be proved in ZFC.

A compact space X is an *F-space* if every cozero-set in X is C^*-embedded in X. Every closed subspace of an extremally disconnected compactum is an F-space. In particular, $\beta\omega$ and $\beta\omega \setminus \omega$ are F-spaces. K. Kunen proved that the product of any non-empty collection of infinite compact F-spaces is not homogeneous [101]. It easily follows from this result that, for any infinite compact F-space Y and any infinite cardinal τ, the product space $Y \times D^\tau$ is not homogeneous.

We recall that Z. Frolík was the first to prove that any infinite extremally disconnected compactum is not homogeneous [74], see also [14] and [49].

A slightly more general class of compacta than the class of compact F-spaces was introduced by E. van Douwen in [58]. A compact space X is said to be a *$\beta\omega$-space* if the closure of any countable discrete in itself subspace of X is homeomorphic to $\beta\omega$. This class of compacta is closed-hereditary and contains all extremally disconnected compacta.

Farah has shown in [66] that if X is a connected compact $\beta\omega$-space, then $X \times Y$ is inhomogeneous, for any non-empty compact space Y. Besides, Farah [66] noticed that Kunen's result on non-homogeneity of products of non-trivial compact F-spaces generalizes to products of compact $\beta\omega$-spaces.

Problem 4.7. Is it true that, for every connected locally connected metrizable compactum Y there exists a compact space Z such that the product space $Y \times Z$ is homogeneous?

Note that in Motorov's example discussed earlier the compactum is not locally connected (see [16]).

If $X \times Y$ is a homogeneous space, then, for any $b \in Y$, the topological copy $X \times \{b\}$ of the space X is a retract of the homogeneous space $X \times Y$ under an open continuous retraction. In this way the homogeneity problems for products considered above connect with the more general question: when can a space be represented as a retract of a homogeneous space?

An important step in this direction has been taken by V. K. Bel'nov. He introduced in [32] the concept of the free homogeneous space $H_B(X)$ of an arbitrary topological space X. Bel'nov [32] proved that if X has one of the following properties: X is Tychonoff, normal, paracompact, hereditarily normal, or hereditarily paracompact, then its free homogeneous space $H_B(X)$ has the same property.

Okromeshko developed the technique of free homogeneous spaces and used it to establish in [138] the following facts:

Every space X is a retract under an open continuous mapping of its free homogeneous space $H_B(X)$.

Every regular Lindelöf space X is a retract under a closed continuous retraction of some homogeneous regular Lindelöf space $H_L(X)$.

Every compact space X is a retract under a closed continuous retraction of some homogeneous regular σ-compact k_1-space $H_L(X)$.

Any space X is an image under an open continuous mapping of some homogeneous stratifiable zero-dimensional space.

A related result is the one by van Mill [116] that every homogeneous continuum is an open retract of an indecomposable homogeneous continuum of the same dimension. It is unknown whether there is a homogeneous indecomposable metrizable continuum of dimension greater than 1. This is due to Rogers [145], and is open since 1985.

5 Some amazing results of E. van Douwen on homogeneity, their generalizations and corollaries

We discuss here some results of E. van Douwen on homogeneity from his article [57]. The method he uses has its roots in Z. Frolík's proof of the nonhomogeneity of the compactum $\beta\omega \setminus \omega$. Recall that the first proof of its nonhomogeneity was under CH; it was given by W. Rudin in [147]. Frolík's proof was in ZFC, see [75].

The next statement is a special case of Theorem 4.1 from [57]. In its proof, we use the concept of a semi-open mapping that was introduced in [57].

A mapping $f\colon X \to Y$ will be called *semi-open at a point* $x \in X$ if, for every open neighbourhood U of x, the set $f(U)$ contains a non-empty open subset of Y. That is, $f(U)$ has nonempty interior.

Theorem 5.1. *Suppose that X is a homogeneous compact space which admits a continuous mapping f onto a compact space Y with a countable π-base $\mathscr{P}$. Suppose further that $w(X) \leq 2^\omega$. Then $|Y| \leq 2^\omega$.*

Proof. Let $\mathscr{K}(X)$ be the family of all cozero-sets in X.

Claim 1: $|\mathscr{K}(X)| \leq 2^\omega$.

This is so, since X is compact and $w(X) \leq 2^\omega$.

We treat ω as the set of natural numbers. The set of all mappings $\phi : \omega \to \mathscr{K}(X)$ of ω into $\mathscr{K}(X)$ will be denoted by $\mathscr{M}$. It follows from Claim 1 that the next inequality holds:

Claim 2: $|\mathscr{M}| \leq 2^\omega$.

Clearly, we may assume that every member of the countable π-base $\mathscr{P}$ for Y is a cozero-set in Y. This assumption easily implies that $f^{-1}(V) \in \mathscr{K}(X)$, for every $V \in \mathscr{P}$.

Claim 3: For each $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that the mapping f is semi-open at x_y.

Indeed, otherwise, using compactness of X, we would be able to find an open neighbourhood W of the set $f^{-1}(y)$ in X such that the interior of $f(W)$ is empty. However, this is impossible, since X is compact, f is continuous, and $f(X) = Y$.

Let us fix $x_y \in f^{-1}(y)$, such as in Claim 2, for each $y \in Y$. Put $S = \{x_y : y \in Y\}$. Clearly, the restriction of f to S is a one-to-one mapping of S onto Y. Therefore, $|Y| = |S|$.

For any $p \in X$ and any $\phi \in \mathscr{M}$, put $\eta(p, \phi) = \{A \subset \omega : p \in \overline{\bigcup\{\phi(n) : n \in A\}}$ and $\mathscr{W}(p) = \{\eta(p, \phi) : \phi \in \mathscr{M}\}$. Put also $\mathscr{W} = \bigcup\{\mathscr{W}(p) : p \in X\}$.

Observe that the next inequality follows from Claim 2:

Claim 4: $|\mathscr{W}(p)| \leq 2^\omega$, for every $p \in X$.

The homogeneity of X and the invariance of the family $\mathscr{K}(X)$ under homeomorphisms of X onto X imply

Claim 5: $\mathscr{W}(p) = \mathscr{W}(q)$, for any $p, q \in X$.

It follows from the definition of the family $\mathscr{W}$ and Claims 4 and 5 that $|\mathscr{W}| \leq 2^\omega$. Since we also have $|Y| = |S|$, to prove that $|Y| \leq 2^\omega$ it suffices to show that $|S| \leq |\mathscr{W}|$.

Let us do that. We will define an injection of S into $\mathscr{W}$. Fix an arbitrary $\phi \in \mathscr{M}$. Let us define a mapping g_ϕ of S into $\mathscr{W}$ as follows: put $g_\phi(p) = \eta(p, \phi)$, for every $p \in S$.

Claim 6: There exists $\phi \in \mathscr{M}$ such that the mapping $g_\phi : S \to \mathscr{W}$ is one-to-one.

Indeed, we can assume that $\mathscr{P} = \{V_n : n \in \omega\}$, and let $\phi : \omega \to \mathscr{K}(X)$ be defined as follows:

$$\phi(n) = f^{-1}(V_n),$$

for each $n \in \omega$.

Let us show that the function g_ϕ is one-to-one.

Take any distinct $p, q \in S$. Then $p = x_y$, for some $y \in Y$. Observe that $f(p) = y \neq f(q)$. Let $F = f^{-1}(y)$. Clearly, F is a compact subset of X and $q \notin F$. Since F is closed in X, we can find an open neighbourhood $O(F)$ of F such that $q \notin \overline{O(F)}$.

Since X is compact, we can fix an open neighbourhood $O(y)$ of y in Y such that $f^{-1}(O(y)) \subset O(F)$.

Put $A = \{n \in \omega : V_n \subset O(y)\}$.

Claim 7: $A \in \eta(p, \phi)$.

We have to show that $p \in \overline{\bigcup\{\phi(n) : n \in A\}}$. Take any open neighbourhood $O(p)$ of p. Since f is continuous at p, there exists an open neighbourhood $O_1(p)$ of p such that $O_1(p) \subset O(p)$ and $f(O_1(p)) \subset O(y)$. The mapping f is also semi-open at p, since $p \in S$. Therefore, there exists a non-empty open set E in Y such that $E \subset f(O_1(p)) \subset O(y)$.

Since $\mathscr{P} = \{V_n : n \in \omega\}$ is a π-base for Y, we can find $k \in \omega$ such that $V_k \subset E \subset O(y)$. Then $k \in A$ and $\phi(k) = f^{-1}(V_k) \subset f^{-1}(E) \subset f^{-1}(O(y)) \subset O(F)$.

Clearly, $V_k \subset E \subset f(O(p))$. We also have $f(f^{-1}(V_k)) = V_k$, since f maps X onto Y. It follows that $V_k \subset f(O(p) \cap f^{-1}(V_k))$. Therefore, $O(p) \cap \phi(k) = O(p) \cap f^{-1}(V_k) \neq \varnothing$.

Thus, we have shown that $O(p) \cap (\bigcup\{\phi(n) : n \in A\}) \neq \varnothing$, for every open neighbourhood $O(p)$ of p. Hence, $p \in \overline{\bigcup\{\phi(n) : n \in A\}}$, that is, $A \in \eta(p, \phi)$.

Claim 8: $A \notin \eta(q, \phi)$.

Indeed, $\bigcup\{\phi(n) : n \in A\} = \bigcup\{f^{-1}(V_n) : n \in A\} \subset O(F)$. Therefore, $\overline{\bigcup\{\phi(n) : n \in A\}} \subset \overline{O(F)}$, by the definition of A. Since $q \notin \overline{O(F)}$, it follows that $q \notin \overline{\bigcup\{\phi(n) : n \in A\}}$. Hence, $A \notin \eta(q, \phi)$.

It follows immediately from Claims 7 and 8 that $\eta(p, \phi) \neq \eta(q, \phi)$. Therefore, the mapping $g_\phi : S \to \mathscr{W}$ is one-to-one, and so, $|S| \leq |\mathscr{W}|$. □

The last theorem immediately implies the next statement:

Corollary 5.2. *The space $\beta\omega$ cannot be represented as an image of a homogeneous compactum X such that $w(X) \leq 2^\omega$ under a continuous mapping.*

However, as we have already observed, it is not yet known whether the assumption in the above statement that $w(X) \leq 2^\omega$ can be dropped. The proof of the previous theorem is as far as we know the simplest way to arrive at the conclusion of Corollary 5.2, and that is why we presented it.

The following result is a strengthening of Theorem 5.1 with a more complicated proof.

Theorem 5.3. *Suppose that X is a homogeneous compact space, and f is a continuous mapping of X onto a compact space Y with countable π-character. Suppose further that $w(X) \leq 2^\omega$. Then $|Y| \leq 2^\omega$.*

Proof. Again, $\mathscr{K}(X)$ is the family of all cozero-sets in X, and $|\mathscr{K}(X)| \leq 2^\omega$.

The set of all mappings $\phi : \omega \to \mathscr{K}(X)$ of ω into $\mathscr{K}(X)$ is denoted by $\mathscr{M}$. Clearly, $|\mathscr{M}| \leq 2^\omega$.

Since the π-character of Y is countable, we can fix a countable π-base γ_y for Y at y, for each $y \in Y$. Clearly, we may assume that every member of the family γ_y is a cozero-set in Y. This assumption implies that $f^{-1}(V) \in \mathscr{K}(X)$, for every $V \in \gamma_y$.

We will establish the next statement which plays a key role in the proof of the present theorem:

Fact 1: For any countable subset B of Y, the cardinality of the closure of B in Y does not exceed 2^ω.

This statement will follow from a series of other facts established below.

Put $C = \overline{B}$ and $\mathscr{P} = \bigcup\{\gamma_y : y \in B\}$.The family $\mathscr{P}$, clearly, is countable, and $f^{-1}(V) \in \mathscr{K}(X)$, for every $V \in \mathscr{P}$.

Now we need the next fact which coincides with Claim 3 in the proof of the preceding theorem:

Fact 2: For each $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that the mapping f is semi-open at x_y.

Let us fix $x_y \in f^{-1}(y)$, such as in Fact 2, for each $y \in Y$. Put $S = \{x_y : y \in C\}$. Clearly, the restriction of f to S is a one-to-one mapping of S onto C. Therefore, $|C| = |S|$.

For any $p \in X$ and any $\phi \in \mathscr{M}$, we again put $\eta(p, \phi) = \{A \subset \omega : p \in \overline{\bigcup\{\phi(n) : n \in A\}}$ and $\mathscr{W}(p) = \{\eta(p, \phi) : \phi \in \mathscr{M}\}$. Moreover, $\mathscr{W} = \bigcup\{\mathscr{W}(p) : p \in X\}$.

Clearly, we also have the next inequality:

Fact 3: $|\mathscr{W}(p)| \leq 2^\omega$, for every $p \in X$.

The homogeneity of X and the invariance of the family $\mathscr{K}(X)$ under homeomorphisms of X onto X imply

Fact 4: $\mathscr{W}(p) = \mathscr{W}(q)$, for any $p, q \in X$.

It follows from the definition of the family $\mathscr{W}$ and Facts 3 and 4 that $|\mathscr{W}| \leq 2^\omega$. Since we also have $|C| = |S|$, to prove that $|C| \leq 2^\omega$ it suffices to show that $|S| \leq |\mathscr{W}|$.

We will define an injection of S into $\mathscr{W}$. Fix an arbitrary $\phi \in \mathscr{M}$. We define a mapping g_ϕ of S into $\mathscr{W}$ as follows: $g_\phi(p) = \eta(p, \phi)$, for every $p \in S$.

Fact 5: There exists $\phi \in \mathscr{M}$ such that the mapping $g_\phi : S \to \mathscr{W}$ is one-to-one.

Indeed, we can assume that $\mathscr{P} = \{V_n : n \in \omega\}$, and let $\phi : \omega \to \mathscr{K}(X)$ be defined as follows:

$$\phi(n) = f^{-1}(V_n),$$

for each $n \in \omega$.

Let us show that the mapping g_ϕ is one-to-one.

Take any distinct $p, q \in S$. Then $p = x_y$, for some $y \in Y$. Observe that $f(p) = y \neq f(q)$. Let $F = f^{-1}(y)$. Clearly, F is a compact subset of X and $q \notin F$. Since F is closed in X, we can find an open neighbourhood $O(F)$ of F such that $q \notin \overline{O(F)}$.

Since X is compact, we can fix an open neighbourhood $O(y)$ of y in Y such that $f^{-1}(O(y)) \subset O(F)$.

Put $A = \{n \in \omega : V_n \subset O(y)\}$.

Fact 6: $A \in \eta(p, \phi)$.

We have to show that $p \in \overline{\bigcup\{\phi(n) : n \in A\}}$. Take any open neighbourhood $O(p)$ of p. Since f is continuous at p, there exists an open neighbourhood $O_1(p)$ of p such that $O_1(p) \subset O(p)$ and $f(O_1(p)) \subset O(y)$. The mapping f is also semi-open at p, since $p \in S$. Therefore, there exists a non-empty open set E in X such that $E \subset f(O_1(p)) \subset O(y)$. Note that $y = f(p) \in C$, since $C = f(S)$.

It follows from the definition of the family $\mathscr{P} = \{V_n : n \in \omega\}$ that $\mathscr{P}$ is a π-base for Y at the point y. Hence, we can find $k \in \omega$ such that $V_k \subset E \subset O(y)$. Then $k \in A$ and $\phi(k) = f^{-1}(V_k) \subset f^{-1}(E) \subset f^{-1}(O(y)) \subset O(F)$.

Clearly, $V_k \subset E \subset f(O(p))$. We also have $f(f^{-1}(V_k)) = V_k$, since f maps X onto Y. It follows that $V_k \subset f(O(p) \cap f^{-1}(V_k))$. Therefore, $O(p) \cap \phi(k) = O(p) \cap f^{-1}(V_k) \neq \varnothing$.

Thus, we have shown that $O(p) \cap (\bigcup\{\phi(n) : n \in A\}) \neq \varnothing$, for every open neighbourhood $O(p)$ of p. Hence, $p \in \overline{\bigcup\{\phi(n) : n \in A\}}$, that is, $A \in \eta(p, \phi)$.

Fact 7: $A \notin \eta(q, \phi)$.

Indeed, $\bigcup\{\phi(n) : n \in A\} = \bigcup\{f^{-1}(V_n) : n \in A\} \subset O(F)$. Therefore, $\overline{\bigcup\{\phi(n) : n \in A\}} \subset \overline{O(F)}$, by the definition of A. Since $q \notin \overline{O(F)}$, it follows that $q \notin \overline{\bigcup\{\phi(n) : n \in A\}}$. Hence, $A \notin \eta(q, \phi)$.

It follows from Facts 6 and 7 that $\eta(p, \phi) \neq \eta(q, \phi)$. Therefore, the mapping $g_\phi : S \to \mathscr{W}$ is one-to-one. Hence, $|S| \leq |\mathscr{W}|$. Thus, Fact 1 holds.

Fact 8: $w(Y) \leq w(X) \leq 2^\omega$.

Indeed, this is so, since X is compact and f is a continuous mapping of X onto Y.

Therefore, we can fix a set $M \subset Y$ such that M is dense in Y and $|M| \leq 2^\omega$.

Since M is dense in X and $\pi\chi(X) \leq \omega$, we have:

$$Y = \bigcup\{\overline{A} : A \subset M, |A| \leq \omega\}.$$

It follows from Fact 1 and this formula that $|Y| \leq 2^\omega$. □

The next result of Jan van Mill [122] immediately follows from the last theorem:

Corollary 5.4. *If X is a homogeneous compactum such that $w(X) \leq 2^\omega$ and $\pi_\chi(X) \leq \omega$, then $|X| \leq 2^\omega$.*

In fact, it was shown in [122] that if X is compact and power-homogeneous then $|X| \leq w(X)^{\pi_\chi(X)}$ from which it easily follows that $|X| \leq 2^{c(X)\cdot\pi_\chi(X)}$. In fact, both inequalities only require Hausdorff (and power homogeneous). In Carlson and Ridderbos [44] it is shown that the second inequality requires only Hausdorff, and in Ridderbos [142] it is shown that the first inequality requires only Hausdorff (in fact $w(X)$ can be replaced by $d(X)$).

Corollary 5.5. *Suppose that X is a homogeneous compact space, and f is a continuous mapping of X onto a compact space Y with countable tightness. Suppose further that $w(X) \leq 2^\omega$. Then $|Y| \leq 2^\omega$.*

Proof. By a well-known theorem of Shapirovskij [152], the π-character of an arbitrary compact space doesn't exceed the tightness of this space. Therefore, Corollary 5.5 follows from Theorem 5.3. □

Corollary 5.6. *Suppose that X is a homogeneous compact space with $w(X) \leq 2^\omega$, and that X does not admit a continuous mapping onto the Tychonoff cube I^{ω_1}. Then $|X| \leq 2^\omega$.*

Proof. By a theorem of Shapirovskij [153], there exists $a \in X$ such that $\pi\chi(a, X) \leq \omega$. Since the space X is homogeneous, it follows that $\pi\chi(x, X) = \pi\chi(a, X) \leq \omega$, for every $x \in X$. It remains to apply Theorem 5.3. □

Corollary 5.7. *Suppose that X is a homogeneous compact space with $w(X) \leq 2^\omega$, and that X is hereditarily normal. Then $|X| \leq 2^\omega$.*

Proof. Since the space X is compact and hereditarily normal, it doesn't admit a continuous mapping onto $\mathbb{I}^{\omega_1}$. Therefore, it follows from Corollary 5.6 that $|X| \leq 2^\omega$. □

Theorem 5.8. *If the weight of a homogeneous compactum X does not exceed 2^ω, then at least one of the following conditions is satisfied:*

1. *The cardinality of X is not greater than 2^ω;*
2. *The space X contains a topological copy of any extremally disconnected space Y the weight of which is ω_1.*

Proof. Indeed, if $|X| > 2^\omega$, then by Corollary 5.6 there exists a continuous mapping f of X onto the Tychonoff cube I^{ω_1}. Take any extremally disconnected space Y such that the weight of Y is ω_1. Clearly, we can assume that Y is a subspace of I^{ω_1}. Put $X_1 = f^{-1}(Y)$, and let f_1 be the restriction of f to X_1. Then f_1 is a perfect mapping of X_1 onto Y. Therefore, there exists a closed subspace Z of the space X_1 such that the restriction g of f_1 to Z is an irreducible perfect mapping of Z onto Y [25], [65]. Since Y is extremally disconnected, it follows that g is a homeomorphism. Hence, Z is homeomorphic to Y. Since Z is a subspace of X, we are done. □

An important common part of some of the above arguments can be specified as follows.

Proposition 5.9. *Suppose that X is a homogeneous compact space with $w(X) \leq 2^\omega$, f is a continuous mapping of X onto a space Y, and $B \subset Y$. Suppose further that $\mathscr{P}$ is a countable family of non-empty open sets in Y which is a π-base at every point of B. Then $|B| \leq 2^\omega$.*

A proof of this statement can easily be extracted from the proof of Theorem 5.1.

6 G_δ-modifications of compacta and homogeneity

Let $\mathscr{T}$ be a topology on a set X. Then the family of all G_δ-subsets of X is a base of a new topology on X, denoted by $\mathscr{T}_\omega$, and is called the G_δ-*modification* of $\mathscr{T}$. The space $(X, \mathscr{T}_\omega)$ is also denoted by X_ω and is called the G_δ-*modification* of the space $(X, \mathscr{T})$.

Clearly, the G_δ-modification X_ω of any topological space is a *P-space*, that is, every G_δ-subset of X_ω is open in X_ω.

We study how the assumption that the G_δ-modification X_ω is homogeneous influences properties of X.

Of course, if X is a homogeneous space, then the space $(X, \mathscr{T}_\omega)$ is also homogeneous. The converse is not true. Indeed, if X is any first-countable space, then X_ω is discrete and, hence, is homogeneous. Thus, X_ω is much more often homogeneous than X itself.

In general, the space $(X, \mathscr{T}_\omega)$ is very different from the space $(X, \mathscr{T})$. Many properties of $(X, \mathscr{T})$, such as compactness, Lindelöfness, paracompactness are usually lost under the G_δ-modification. Indeed, every countable subset of X is closed and discrete in the space $(X, \mathscr{T}_\omega)$. Therefore, if X is infinite, then the space $(X, \mathscr{T}_\omega)$ is not countably compact.

On the other hand, some properties of a space can greatly improve under this operation. We have already observed that if $(X, \mathscr{T})$ is first-countable, then the space $(X, \mathscr{T}_\omega)$ is discrete and hence, absolutely trivial.

It turns out that homogeneity of the G_δ-modification of a space has a deep influence on the structure of the original space, in particular, on the relationship between its cardinal invariants.

An important role in our study of G_δ-modifications of compacta belongs to the following theorem of E. G. Pytkeev [141, Theorem 4]:

Theorem 6.1. *The Lindelöf degree of the G_δ-modification of any compact space of countable tightness does not exceed 2^ω.*

This result was generalized in Theorem 3.5 of Carlson, Porter and Ridderbos [43]. The proof of this generalization represents an alternate closing-off argument to that given in Theorem 6.4 below (and results in a generalization of that theorem for any power homogeneous Hausdorff space).

The next theorem, rich with consequences, was obtained in [20]. Suffices to mention at this point that it obviously implies the old theorem on the cardinality of first-countable compacta [9] and provides one of the two key steps in a proof of the recent theorem of de la Vega [170].

Theorem 6.2. *Let X be a compact space of countable tightness such that the G_δ-modification X_ω of X is homogeneous. Then the weight of X, as well as the weight of X_ω, is not greater than 2^ω.*

Proof. There exists a non-empty open subspace U of X_ω such that $w(U) \leq 2^\omega$.

Indeed, since X is a non-empty compact space of countable tightness, there exists a non-empty G_δ-subset U of X such that the weight of the subspace U of X is not greater than 2^ω [12], [14]. Then U is an open subspace of X_ω and the weight of the subspace U of X_ω is also not greater than 2^ω.

Since X_ω is homogeneous, it follows that every point in X_ω has an open neighbourhood Ox in X_ω such that $w(Ox) \leq 2^\omega$. Now observe that by Pytkeev's Theorem 6.1, the Lindelöf degree of X_ω doesn't exceed 2^ω.

Since the local weight of X_ω does not exceed 2^ω, it follows that there exists an open covering γ of X_ω such that $w(U) \leq 2^\omega$, for each $U \in \gamma$, and $|\gamma| \leq 2^\omega$. Fixing a base of cardinality $\leq 2^\omega$ in each $U \in \gamma$, and taking the union of these bases, we obtain a base of cardinality $\leq 2^\omega$ in X_ω. Thus, $w(X_\omega) \leq 2^\omega$. Since, X is a continuous image of X_ω, we have $nw(X) \leq w(X_\omega) \leq 2^\omega$. However, since X is compact, $w(X) = nw(X) \leq 2^\omega$ [65]. □

The techniques presented above have far reaching applications and generalizations. Below we describe a few of them.

The following result of de la Vega [170] answers a long standing question raised in [11] (see also [13] and [16]).

Theorem 6.3. *Let X be a homogeneous compact space of countable tightness. Then the cardinality of X is not greater than 2^ω.*

Proof. It follows from Theorem 6.2 that $w(X) \leq 2^\omega$. Therefore, we can apply Corollary 5.5 and conclude that $|X| \leq 2^\omega$. □

The method of proof also yields a proof of Arhangel'skii's [9] Theorem: the cardinality of a first-countable compact space does not exceed 2^ω. Simply observe that if X is a first-countable compactum, then its G_δ-modification is discrete and hence homogeneous. Hence by Corollary 5.5, $w(X) \leq 2^\omega$ from which it easily follows that $|X| \leq 2^\omega$. This connection between Arhangel'skii's Theorem (and its variations) and De La Vega's Theorem (and its variations) are additionally explored in both [44] and [43].

Below we show how to evade the use of Pytkeev's Theorem in the proof of de la Vega's Theorem benefiting instead from the classical saturation argument.

Recall that, according to [14, Theorem 2.2.4] (see also [13]), *if X is a compact space of countable tightness, then the G_δ-density of X does not exceed ω at some point $e \in X$*. If, in addition, X is homogeneous, then the G_δ-density of X at any $x \in X$ is countable. Therefore, it is enough to prove the next statement from [20]:

Theorem 6.4. *Let X be a compact space with countable tightness. Suppose further that X has a covering γ satisfying the following conditions:*

1. *Each $F \in \gamma$ is a G_δ-subset of X, and*
2. *For every $F \in \gamma$ there exists a countable subset A of X such that $F \subset \overline{A}$.*

Then $w(X) \leq 2^\omega$.

Proof. (The proof is based on Buzyakova's version of the saturation argument (see [41], the proof of Theorem 3.10)).

By transfinite recursion, we define an increasing transfinite sequence $\{Y_\alpha : \alpha < \omega_1\}$ of closed subspaces of X, and an increasing transfinite sequence $\{\mathscr{P}_\alpha : \alpha < \omega_1\}$ of families of open sets in X satisfying the following conditions, for each $\alpha < \omega_1$:

1) $\mathscr{P}_\alpha$ is a pseudobase of X at Y_α, and $\mathscr{P}_\beta \subset \mathscr{P}_\alpha$ if $\beta < \alpha$;
2) $|\mathscr{P}_\alpha| \leq 2^\omega$;
3) $d(Y_\alpha) \leq 2^\omega$; and
4) if $Y_\alpha \subset \bigcup \eta$, where η is a finite subfamily of $\mathscr{P}_\alpha$ such that $X \setminus \bigcup \eta$ is non-empty, then $Y_{\alpha+1} \setminus \bigcup \eta$ is not empty.

We put $Y_0 = \varnothing$ and $\mathscr{P}_0 = \varnothing$. Take any $\beta < \omega_1$, and assume that Y_α and $\mathscr{P}_\alpha$ are already defined for all $\alpha < \beta$ in such a way that conditions 1), 2), 3), and 4) are satisfied for these values of α. Then we proceed as follows.

Case 1: β is a limit ordinal. Put $Y_\beta = \overline{\bigcup\{Y_\alpha : \alpha < \beta\}}$. Then $d(Y_\beta) \leq 2^\omega$, by 3). Therefore, the weight of Y_β is not greater than 2^ω (the closures of countable subsets of an appropriate dense subset of Y_β form a network S in the compactum Y_β such that $|S| \leq 2^\omega$; here is where we use that X has countable tightness). Now comes the main

Claim: The weight of Y_β in X does not exceed 2^ω.

Indeed, since Y_β is compact and $w(Y_\beta) \leq 2^\omega$, the number of compact G_δ-subsets of Y doesn't exceed 2^ω. Since γ covers Y_β, it follows that there is a subfamily μ of γ such that $|\mu| \leq 2^\omega$ and $Y_\beta \subset \bigcup \mu$. Since the weight in X of every $F \in \mu$ is not greater than 2^ω, it is enough to take the union of the appropriate external bases of elements of μ in X. The claim is proved.

Fix a base $\mathscr{S}_\beta$ of Y_β in X such that $|\mathscr{S}_\beta| \leq 2^\omega$ and put $\mathscr{P}_\beta = \mathscr{S}_\beta \cup \bigcup\{\mathscr{P}_\alpha : \alpha < \beta\}$. The construction in Case 1 is complete.

Case 2: $\beta = \alpha + 1$, for some $\alpha < \omega_1$. Let $\mathscr{E}_\alpha$ be the family of all finite subfamilies γ of the family $\mathscr{P}_\alpha$ such that $Y_\alpha \subset \bigcup \gamma$ and $X \setminus \bigcup \gamma$ is non-empty. Clearly, $|\mathscr{E}_\alpha| \leq 2^\omega$. For each $\gamma \in \mathscr{E}_\alpha$ fix a point $c(\gamma) \in X \setminus \bigcup \gamma$, and put $Y_\beta = \overline{\{c(\gamma) : \gamma \in \mathscr{E}_\alpha\} \cup Y_\alpha}$. Clearly, the density of Y_β is not greater than 2^ω, and, as in Case 1, we can define a base $\mathscr{S}_\beta$ of Y_β in X such that $|\mathscr{S}_\beta| \leq 2^\omega$. We put $P_\beta = \mathscr{P}_\alpha \cup \mathscr{S}_\beta$. The construction in Case 2 is complete.

Clearly, conditions 1)–4) are satisfied.

Put $Y = \bigcup\{Y_\alpha : \alpha < \omega_1\}$. Since the tightness of X is countable, and the sequence $\{Y_\alpha : \alpha < \omega_1\}$ is increasing, it follows that Y is closed in X. Obviously, $d(Y) \leq 2^\omega$. Let us show that $Y = X$.

Assume the contrary, and fix $z \in X \setminus Y$. Put $\mathscr{P} = \bigcup\{\mathscr{P}_\alpha : \alpha < \omega_1\}$ Clearly, $\mathscr{P}$ is a base of Y in X. Therefore, there exists a family $\eta \subset \mathscr{P}$ such that $Y \subset \bigcup \eta$ and $z \notin \bigcup \eta$. Since Y is compact, there is a finite subfamily γ of η such that $Y \subset \bigcup \gamma$. From the definition of $\mathscr{P}$ it follows that $\gamma \subset \mathscr{P}_\alpha$, for some $\alpha < \omega_1$. Since $z \in X \setminus \bigcup \gamma$, we have $\gamma \in \mathscr{E}_\alpha$ and $c(\gamma) \in Y_{\alpha+1} \subset Y$. On the other hand, $c(\gamma) \in X \setminus \bigcup \gamma \subset X \setminus Y$, a contradiction. Hence, $Y = X$, which implies that $d(X) = d(Y) \leq 2^\omega$. Hence (see Case 1), $w(X) \leq 2^\omega$. $\square$

Here is a direction in which one may try to generalize some of the above results.

Let us say that a space X is *homogeneous at points* of a subset A of X if for any $x, y \in A$ there exists a homeomorphism h of X onto itself such that $h(x) = y$.

Theorem 6.5. *Suppose that X is a compact space with countable tightness and that $w(X) \leq 2^\omega$. Suppose further that Y is a subset of X such that X is homogeneous at points of Y. Then $|Y| \leq 2^\omega$.*

Proof. First, we mention that the π-character of X at every point of X is countable, since X is a compact space of countable tightness [152].

The remaining part of the proof is a modification of the arguments given above. So we just sketch it.

Let $\mathscr{K}(X)$ be the family of all cozero-sets in X. Clearly, $|\mathscr{K}(X)| \leq 2^\omega$, and $\mathscr{K}(X)$ is invariant under homeomorphisms of X onto itself.

For every $x \in Y$, we fix a countable π-base γ_x of X at x. We may assume that $\gamma_x \subset \mathscr{K}(X)$. As before, the set of all mappings $\phi : \omega \to \mathscr{K}(X)$ of ω into $\mathscr{K}(X)$ is denoted by $\mathscr{M}$.

For any $p \in X$, the family $\eta(p, \phi)$ is defined in the same way as before. We also put $\mathscr{W}(p) = \{\eta(p, \phi) : \phi \in \mathscr{M}\}$. Clearly, $|\mathscr{W}(p)| \leq 2^\omega$.

Put $\mathscr{W} = \bigcup\{\mathscr{W}(p) : p \in Y\} = \{\eta(p, \phi) : \phi \in \mathscr{M}, p \in Y\}$.

Claim 1: $\mathscr{W}(p) = \mathscr{W}(q)$, for any $p, q \in Y$.

This is so, since X is homogeneous at points of Y, and since $\mathscr{K}(X)$ is invariant under homeomorphisms of X onto itself.

Since $|\mathscr{W}(p)| \leq 2^\omega$ for any $p \in X$, it follows from Claim 1 that that the next statement holds:

Claim 2: $|\mathscr{W}| \leq 2^\omega$.

Claim 3: For any countable subset C of Y, we have: $|\overline{C}| \leq 2^\omega$.

This is verified in the same way as the similar Fact 1 in the proof of Theorem 5.3.

Finally, fix a subset M of Y such that $Y \subset \overline{M}$ and $|M| \leq 2^\omega$. This is possible, since $w(Y) \leq 2^\omega$.

Since the tightness of X is countable, we have: $Y \subset \bigcup\{\overline{C} : C \subset M, |C| \leq \omega\}$. It follows from Claim 3 that $|Y| \leq 2^\omega$. □

The assumption in Theorem 6.5 that the weight of X does not exceed 2^ω cannot be dropped. Let X be Alexandroff's one-point compactification of a discrete space Y such that $|Y| > 2^\omega$. Then, clearly, X is homogeneous at points of Y, X is compact, and the tightness of X is countable. However, the cardinality of Y is greater than 2^ω.

Problem 6.6. Let τ be a cardinal number such that $\tau > 2^\omega$, and $A(\tau)$ be the Alexandroff one-point compactification of a discrete space τ. Can $A(\tau)$ be represented as a closed subspace of a compact space X with countable tightness such that X is homogeneous at points of $A(\tau)$?

Many results on homogeneous compacta presented above can be extended to power homogeneous compacta. In particular, the next fact had been established in [24]):

Theorem 6.7. *Suppose that X is a power-homogeneous compact space with $t(X) \leq \kappa$. Then the G_κ-density of X does not exceed κ at every point $x \in X$.*

This statement plays an essential role in [24] in extending de la Vega's Theorem to power-homogeneous compacta.

Another piece of technique which can be used very effectively is the following statement which generalizes Proposition 5.9:

Lemma 6.8. *Suppose that X is a power-homogeneous compact space with $w(X) \leq 2^\omega$, and $B \subset X$. Suppose further that $\mathscr{P}$ is a countable family of non-empty open sets in X which is a π-base at every point of B. Then $|B| \leq 2^\omega$.*

Proof. In this argument, we will use the notation, terminology and facts from the proof of Theorem 5.3. Clearly, we can assume that every member of $\mathscr{P}$ is a cozero-set in X.

Put $\tau = |B|$. We may assume that $B = \{q_\alpha : \alpha < \tau\}$ and that $q_\alpha \neq q_\beta$ whenever $\alpha \neq \beta$. Fix a cardinal μ such that the space X^μ is homogeneous. Clearly, we can also assume that $\tau \leq \mu$, since any power of a homogeneous space is homogeneous.

Fix an arbitrary mapping $\phi : \omega \to \mathscr{P}$ of the set ω onto $\mathscr{P}$. Using an argument from the proof of Theorem 5.3, we easily establish the following

Claim 1: $\eta(q_\alpha, \phi) \neq \eta(q_\beta, \phi)$ whenever $\alpha \neq \beta$.

We now define a point $q \in X^\mu$ as follows: if $\alpha < \tau$, then the α-th coordinate of q is the point $q_\alpha \in X$ which has been already defined. For $\tau \leq \alpha < \mu$ we define the α-th coordinate q_α of q to be an arbitrary point of X.

We also define a mapping ϕ_α of ω into the family $\mathscr{K}(X^\mu)$ of cozero-sets in X^μ as follows: $\phi_\alpha(n)$ is the largest open subset of X^μ such that the image of it under the projection of X^μ to the α-th coordinate space $X_\alpha = X$ is the set $\phi(n)$.

Claim 2: $\eta(q, \phi_\alpha) = \eta(q_\alpha, \phi)$, for every $\alpha < \tau$.

Claim 3: $|\mathscr{W}(q)| \geq \tau$.

This Claim follows from Claims 1 and 2, since $\eta(q, \phi_\alpha) \in \mathscr{W}(q)$, for every $\alpha < \tau$.

Now we fix a point $p \in X^\mu$ such that $p_\alpha = p_\beta$ for any $\alpha, \beta \in \mu$.

Claim 4: $\mathscr{W}(p) = \mathscr{W}(q)$ and hence, $|W(p)| \geq \tau$.

This is so, since X^μ is homogeneous and Claim 3 holds.

We also recall the following well-known fact [65]:

Claim 5: Every cozero-set in X^μ depends on countably many coordinates.

Now we use the special choice of the point p in X^μ: the fact that all coordinates of p are the same. Combining this fact with Claim 5, we conclude that $|\mathscr{W}(p)|$ coincides with $|\mathscr{W}(p|\omega)|$ calculated in the space X^ω.

Clearly, $w(X^\omega) \le 2^\omega$, since $w(X) \le 2^\omega$.Therefore, the cardinality of the family of cozero-sets in X^ω doesn't exceed 2^ω. It follows that $|\mathscr{W}(p|\omega)| \le 2^\omega$.

Hence, we have: $|\mathscr{W}(p)| = |\mathscr{W}(p|\omega)| \le 2^\omega$. This inequality and Claim 4 imply that $\tau \le |\mathscr{W}(p)| \le 2^\omega$. Thus, $|B| = \tau \le 2^\omega$. □

We will also need below the following statement of independent interest:

Theorem 6.9. *Suppose that X is a compact power-homogeneous space such that $w(X) \le 2^\omega$ and $\pi\chi(X) \le \omega$. Then $|X| \le 2^\omega$.*

Proof. Since the density of X doesn't exceed the weight of X, there exists a set $A \subset X$ such that $X = \overline{A}$ and $|A| \le 2^\omega$. Now it follows from $\pi\chi(X) \le \omega$ that X is the union of the closures of all countable subsets of A.

Claim 1: For every countable subset M of X, there exists a countable family $\mathscr{P}$ of non-empty open subsets of X such that $\mathscr{P}$ is a π-base for X at each $x \in \overline{M}$.

Indeed, suffices to fix a countable π-base γ_x for X at x, for each $x \in M$, and to put $\mathscr{P} = \bigcup\{\gamma_x : x \in M\}$.

Claim 2: The cardinality of the closure of an arbitrary countable subset of X doesn't exceed 2^ω.

This statement follows from Claim 1 and Lemma 6.8. Since X is the union of closures of countable subsets of A, and $|A| \le 2^\omega$, it follows from Claim 2 that $|X| \le 2^\omega$. □

Now we can establish the following result from [24] generalizing de la Vega's Theorem:

Theorem 6.10. *The cardinality of any power-homogeneous compact space X with countable tightness does not exceed 2^ω.*

Proof. First, we note that the weight of X doesn't exceed 2^ω. Indeed, it follows from Theorem 6.7 that the G_ω-density of X at every point is countable. Now we can apply Theorem 6.4 and conclude that $w(X) \le 2^\omega$.

Observe that $\pi\chi(X) \le t(X) \le \omega$. Since $w(X) \le 2^\omega$ and $\pi\chi(X) \le \omega$, it follows from Theorem 6.9 that $|X| \le 2^\omega$. □

Theorem 6.11. *Suppose $X = \prod\{X_\alpha : \alpha \in A\}$ where each X_α is first-countable at some point. Suppose further that Y is a compact space of cardinality ω_1, and that Y is not first-countable at some point. Then the space $X \times Y$ is not homogeneous.*

Proof. Assume the contrary. It follows from the assumptions about Y that Y is first-countable at a dense set M of points, [91, 2.22]. Since the product space $Y \times \prod\{X_\alpha : \alpha \in A\}$ is homogeneous, it follows from Theorem 7.8 in the next section, and from compactness of Y that the set of all G_δ-points of Y is closed in Y. However, Y is obviously first-countable at a dense set of points. Hence, as Y is compact, Y is first-countable, a contradiction. $\square$

Corollary 6.12. *Suppose that $X = \prod\{X_\alpha : \alpha \in A\}$ where each X_α is a non-first-countable compact space of cardinality ω_1. Then the space X is not homogeneous.*

Similarly to the last two results, many other concrete cases of non-homogeneity of a product space of two, or more, compacta can be identified with the help of the next general statement that is a corollary from Theorems 10 and 13 in [19] (see also Theorem 7.7 below).

Theorem 6.13. *Suppose that τ is an infinite cardinal number, and that $X = Y \times \prod\{X_\alpha : \alpha \in A\}$, where the character of X_α doesn't exceed τ at some $x_\alpha \in X_\alpha$, for each $\alpha \in A$. Suppose further that Y is compact, and that the set Z of all $y \in Y$ such that $\chi(y, Y) \leq \tau$ is not closed in Y. Then the space X is not power-homogeneous.*

We illustrate Theorem 6.13 by the following immediate corollary from it:

Corollary 6.14. *Suppose that $X = \prod\{X_\alpha : \alpha \in A\}$, where each X_α is a non-first-countable compact space which is first-countable at a dense set of points. Then X is not power-homogeneous.*

Observe that the assumptions in the last statement are satisfied if each X_α is a non-first-countable compact space with a dense set of isolated points. Corollary 6.14 also remains true if 'compact space' is replaced by 'space of point-countable type'.

Problem 6.15. Does there exist for a given arbitrary zero-dimensional compact space Y a non-empty zero-dimensional compact space Z such that $Y \times Z$ is homogeneous?

For more information on power homogeneous compacta, see e.g., [130], [143], [44].

7 Power-homogeneity and a weak algebraic structure

A different approach to power-homogeneous compacta is developed in the next two sections. It offers some new techniques and new possibilities in the study of such spaces.

In the preceding sections, we have discussed some results showing how homogeneity and power-homogeneity influence certain estimates for the weight and the cardinality of a space, especially, in the class of compacta.

In this section, we consider how power-homogeneity is reflected in the topological structure of the space, focusing our attention on G_δ-points.

The techniques presented in this section involve some algebraic operations.

Let us consider the following general question: given a topological space X, is it possible to introduce some 'helpful' algebraic structure on this space fitting nicely the topology of X? The explanation of 'helpfulness' of such structures lies in the fact that they usually behave very nicely under products.

Of course, these algebraic structures must be of a very general nature, if we want them to exist on spaces of rather general kind.

In what follows, τ is an infinite cardinal number. We say that *the π_τ-character of a space X at a point $e \in X$* is not greater than τ (and write $\pi_\tau\chi(e, X) \leq \tau$) if there exists a family γ of non-empty G_τ-sets in X such that $|\gamma| \leq \tau$ and every open neighbourhood of e contains at least one member of γ. Such a family γ is called a *π_τ-network* at e. If $\tau = \omega$, we rather use expressions *π_ω-character* and *π_ω-network*.

In particular, if X has a countable π-base at e, then $\pi_\omega\chi(e, X) \leq \omega$.

Suppose that $\mathscr{F} = \{X_a : a \in A\}$ is a family of spaces, and $X = \prod_{a\in A} X_a$ is the product of these spaces. *A τ-cube* in X is any subset B of X that can be represented as the product $B = \prod_{a\in A} B_a$, where B_a is a non-empty subset of X_a, for each $a \in A$, and the cardinality of $A_B = \{a \in A : B_a \neq X_a\}$ is not greater than τ. We put $X_K = \prod_{a\in K} X_a$, for every non-empty subset K of A, and denote by p_K the natural projection mapping of X onto X_K.

Let us say that the *G_τ-tightness* of a space X at a point $z \in X$ is not greater than τ (notation: $t_\tau(z, X) \leq \tau$) if, for every family γ of G_τ-subsets of X such that $z \in \overline{\bigcup\gamma}$, there is a subfamily η of γ such that $|\eta| \leq \tau$ and $z \in \overline{\bigcup\eta}$.

Theorem 7.1. *Suppose that $\{X_a : a \in A\}$ is a family of topological spaces, z_a is a point in X_a, for each $a \in A$, such that $\chi(z_a, X_a) \leq \tau$, and let $X = \prod_{a\in A} X_a$ be the topological product. Then the G_τ-tightness of X at the point $z = (z_a)_{a\in A}$ is not greater than τ.*

Proof. It is enough to show that, for any family γ of τ-cubes in X such that the point $z = (z_a)_{a\in A}$ is in the closure of the set $U = \bigcup\gamma$, there exists a subfamily η of γ such that $x \in \overline{\bigcup\eta}$ and $|\eta| \leq \tau$.

Let A_0 be any non-empty subset of A such that $|A_0| \leq \tau$. Assume that a subset A_n is already defined and satisfies the condition $|A_n| \leq \tau$. Put $K = A_n$ and $z_K = (z_a)_{a\in K}$. Obviously, $p_K(z) = z_K$. Since $\chi(z_K, X_K) \leq \tau$, there exists a subfamily γ_n of γ such that $|\gamma_n| \leq \tau$ and z_K is in the closure of $\bigcup\{p_K(V) : V \in \gamma_n\}$ in X_K. Put $A_{n+1} = A_n \cup \bigcup\{A_B : B \in \gamma_n\}$. The inductive step is complete.

Put $M = \bigcup\{A_n : n \in \omega\}$ and $\eta = \bigcup\{\gamma_n : n \in \omega\}$. Clearly, η is a subfamily of γ such that $|\eta| \leq \tau$. Let H be the closure of $\bigcup\eta$. Let us show that $z \in H$. This is established by showing that every standard open neighbourhood O_1 of z in X has a common point with H. Indeed, $O_1 = p_S^{-1}p_S(O_1)$, for some finite $S \subset A$. Put $F = S \cap M$ and $O = p_F^{-1}p_F(O_1)$. Then, clearly, $O_1 \subset O$ and

$O = p_F^{-1} p_F(O)$. The conditions $O \cap H \neq \varnothing$ and $O_1 \cap H \neq \varnothing$ are equivalent. To see this, assume that $O \cap H \neq \varnothing$, and fix $y \in O \cap H$. There exists $y' \in O_1$ such that $p_M(y') = p_M(y)$. Since $p_M^{-1} p_M(H) = H$ and $y \in H$, we have $y' \in H$. Therefore, $y' \in O_1 \cap H$ and $O_1 \cap H \neq \varnothing$. Since the sequence $\{A_n : n \in \omega\}$ is increasing, there exists $n \in \omega$ such that $F \subset A_n$. Then, by the choice of γ_n, $p_F(z)$ is in the closure of the set $\bigcup\{p_F(V) : V \in \gamma_n\}$ in the space X_F. Therefore, there exists a point $y \in \bigcup \eta$ such that $p_F(y) \in p_F(O)$. Since $O = p_F^{-1} p_F(O)$, it follows that y is in $O \cap \bigcup \eta$. Hence, $z \in H$. □

Recall that a mapping $f : X \to Y$ of a space X onto a space Y is said to be *pseudo-open* if for an arbitrary $y \in Y$ and any open neighbourhood U of $f^{-1}(y)$ in X, the set $f(U)$ contains some open neighbourhood of y in Y. Every open mapping is pseudo-open; each closed mapping is pseudo-open as well.

The proof of the next statement is omitted, since it is standard.

Proposition 7.2. *If $f : X \to Y$ is a pseudo-open continuous mapping of a space X onto a space Y, and the G_τ-tightness of X does not exceed τ, then the G_τ-tightness of Y does not exceed τ.*

Now we come to the concept introduced in [18] which plays the key role in this and the next section.

A *τ-twister at a point* e of a space X is a binary operation on X, written as a product operation xy for x, y in X, satisfying the following conditions:

a) $ex = xe = x$, for each $x \in X$;
b) for every $y \in X$ and every G_τ-subset V in X containing y, there exists a G_τ-subset P of X such that $e \in P$ and $xy \in V$, for each $x \in P$ (that is, $Py \subset V$) (this is the separate G_τ-continuity of the product operation at e on the right); and
c) if $e \in \overline{B}$, for some $B \subset X$, then $x \in \overline{xB}$ for every $x \in X$ (this is the separate continuity of the operation at e on the left).

If in the above definition condition b) is replaced by the following condition

b′) for every $y \in X$ and every open neighbourhood V of y, there is an open neighbourhood W of e such that $Wy \subset V$,

then the binary operation is called *a twister* on X at e.

Clearly, the concept of a twister is a very general one; this is a crucial feature for existence of twisters. Many examples of twisters are easily available.

Twisters have some nice stability properties which are easily established. Applications of twisters discussed below heavily depend on them.

Proposition 7.3. *If Z is a retract of X and $e \in Z$, and there exists a τ-twister (a twister) at e on X, then there exists a τ-twister (a twister) on Z at e.*

Proof. Fix a retraction r of X onto Z and a τ-twister on X at e, and define an operation ϕ on Z by the following rule: $\phi(z, h) = r(zh)$. Clearly, ϕ is a τ-twister (a twister) on Z at e. □

If a space X has a τ-twister (a twister) at a point $e \in X$, then we say that X is *τ-twistable (respectively, twistable) at e*. A space X is *τ-twistable* (*twistable*) if it is τ-twistable (respectively, twistable) at every point. Twistability has been introduced and studied in [19], [18].

Proposition 7.4. *Suppose that $\{X_a : a \in A\}$ is a family of spaces and $e_a \in X_a$, for each $a \in A$. Let $X = \prod_{a\in A} X_a$ and $e = (e_a)_{a\in A}$. Suppose further that X_a is τ-twistable (twistable) at e_a, for each $a \in A$. Then the product space X is τ-twistable (twistable) at e.*

Proof. Fix a τ-twister on X_a at e_a, and define the product operation on X coordinatewise. This is a τ-twister on X at e. □

Here comes another piece of technique based on twisters (see [19]).

Theorem 7.5. *Suppose that X is a τ-twistable space such that the G_τ-tightness of X does not exceed τ. Then the set A of all G_τ-points of X is closed in X.*

Proof. Take any $e \in \overline{A}$, and fix a τ-twister at e. Since the G_τ-tightness of X at e does not exceed τ, and each point in A is a G_τ-point in X, there exists a subset B of A such that $|B| \le \tau$ and $e \in \overline{B}$. For every $b \in B$ we can find a G_τ-subset P_b in X such that $e \in P_b$ and $P_b b = \{b\}$ (by the definition of a τ-twister). Put $P^* = \cap\{P_b : b \in B\}$. Then P^* is a G_τ-set in X, $e \in P^*$, and $xb = b$, for every $b \in B$ and every $x \in P^*$.
Claim: $P^* = \{e\}$. Assume the contrary, and fix $c \in P^*$ such that $c \neq e$. There is an open neighbourhood W of c such that e is not in the closure of W. We have $ce = c \in W$. By the continuity assumption, there exists an open neighbourhood V of e such that $cV \subset W$. We can also assume that $V \cap W = \varnothing$. Put $B_1 = B \cap V$. Then $B_1 \neq \varnothing$ and $cB_1 = B_1 \subset V$.

On the other hand, $cB_1 \subset cV \subset W$. It follows that $cB_1 \subset W \cap V$, a contradiction with $W \cap V = \varnothing$. Hence, $P^* = \{e\}$. Since P^* is a G_τ-set in X, it follows that e is a G_τ-point in X, that is, $e \in A$. Thus, the set A is closed. □

The quite elementary, simple, result below provides a very general sufficient condition for the existence of τ-twisters.

Proposition 7.6. *If e is a G_τ-point in a space X, then there exists a τ-twister on X at e.*

Proof. Put $ey = y$, for every $y \in X$, and $xy = x$ for every x and y in X such that $x \neq e$. It is easily verified that this operation is a τ-twister on X. □

The details of a mechanism described above can be unified in the proof of the next basic statement from [18]:

Theorem 7.7. *Suppose that X is a power-homogeneous space, and that μ is an infinite cardinal such that the character of X at least at one point is not greater than μ. Then, for any cardinal τ such that $\mu \leq \tau$, the set of all G_τ-points in X is closed.*

Proof. Fix an infinite cardinal number λ such that the space X^λ is homogeneous. It follows from Theorem 7.1 that the G_τ-tightness of X^λ at least at one point does not exceed τ. Hence, the G_τ-tightness of X^λ is not greater than τ at all points, since the space X^λ is homogeneous.

The natural projection of X^λ onto X is open and continuous. Therefore, by Proposition 7.2, the G_τ-tightness of X also does not exceed τ.

The set of G_τ-points in X is not empty. It follows from Proposition 7.6 that X is τ-twistable at some point. By Proposition 7.4, X^λ is τ-twistable at some point. Since X^λ is homogeneous, the space X^λ is τ-twistable. Since X is a retract of X^λ, it follows from Proposition 7.3 that X is τ-twistable. It remains to refer to Theorem 7.5. $\square$

Making obvious minor changes in the above argument (left to the reader) we can establish the following result essentially established in [18]:

Theorem 7.8. *Suppose that X is the product of a family $\gamma = \{X_\alpha : \alpha \in A\}$ of non-empty Hausdorff spaces X_α each of which is first-countable at least at one point, and that X is homogeneous. Then, for each $\alpha \in A$, the set of all G_δ-points of X_α is closed in X_α, for each $\alpha \in A$.*

The following facts were established in [18] with the help of the above statements.

Corollary 7.9. *If X is a power-homogeneous space with a dense set of isolated points, then every point in X is a G_δ-point.*

Corollary 7.10. *If X is a power-homogeneous compact space with a dense set of G_δ-points, then X is first-countable and $|X| \leq 2^\omega$.*

Under CH we have a more symmetric result:

Theorem 7.11. *Suppose that CH holds and that X is a zero-dimensional compact space such that $|X| \leq 2^\omega$. Then X is power-homogeneous if and only if X is first-countable.*

Proof. Indeed, because of the assumptions we made, the space X is first-countable at a dense in X set of points, [91, 2.22]. If X is power-homogeneous, then X is first-countable at every point, by the main theorem that we just proved.

Conversely, if X is first-countable, then X is power-homogeneous, since X is zero-dimensional [133] (see also [62]). $\square$

Clearly, the assumption in the last statement that X is compact can be weakened to the assumption that X is of point-countable type.

Yet another fact involving twisters and cardinal invariants is worth mentioning.

Theorem 7.12. *Let X be a space. Then $\psi(e, X) \leq \tau$ if and only if $\pi\tau\chi(e, X) \leq \tau$ and X is τ-twistable at e.*

Proof. If $\psi(e, X) \leq \tau$, then X is τ-twistable at e, by Proposition 7.6, and $\gamma = \{e\}$ is a $\pi\tau$-network at e. Therefore, $\pi\tau\chi(e, X) \leq \tau$.

Now assume that X is τ-twistable at e, and that $\pi\tau\chi(e, X) \leq \tau$. Fix a τ-twister at e and a $\pi\tau$-network γ at e. Take any $V \in \gamma$ and fix $y_V \in V$. There exists a G_τ-set P_V such that $e \in P_V$ and $P_V y_V \subset V$. Put $Q = \bigcap\{P_V : V \in \gamma\}$. Clearly, Q is a G_τ-set and $e \in Q$.

Claim: $Q = \{e\}$.

Assume the contrary. Then we can fix $x \in Q$ such that $x \neq e$. Since X is Hausdorff, there exist open sets U and W such that $x \in U$, $e \in W$, and $U \cap W = \varnothing$. Since $xe = x \in U$ and the multiplication on the left is continuous at e, we can also assume that $xW \subset U$.

Since γ is a $\pi\tau$-network at e, there exists $V \in \gamma$ such that $V \subset W$. Then for the point y_V we have: $y_V \in W$, $xy_V \in P_V y_V \subset V \subset W$ and $xy_V \in xV \subset xW \subset U$. Hence, $xy_V \in W \cap U$ and $W \cap U \neq \varnothing$, a contradiction. It follows that $Q = \{e\}$. ☐

Corollary 7.13. *Suppose that a space X is ω-twistable at some point $e \in X$. Suppose further that there exists a countable set A of G_δ-points of X such that $e \in \overline{A}$. Then e is also a G_δ-point in X.*

Corollary 7.14. *The space $\beta\omega$ is not ω-twistable at any point e of $\beta\omega \setminus \omega$.*

Now we can identify many other examples of non-ω-twistable spaces. In particular, the Alexandroff compactification of an uncountable discrete space is by Corollary 7.13 not ω-twistable at the non-isolated point.

Since the character and the pseudocharacter coincide in spaces of point-countable type [25], [65], the next statement follows from Theorem 7.12.

Proposition 7.15. *Suppose that X is a space of point-countable type, and $e \in X$. Then the following conditions are equivalent:*

a) X has a base of cardinality $\leq \tau$ at e; and
b) X is τ-twistable at e and has a π-base at e of cardinality $\leq \tau$.

The following elementary statement from [18] is also useful:

Proposition 7.16. *If X is a space of point-countable type, and the tightness of X is countable, then the $\pi\omega$-character of X is also countable.*

Proof. Take any $x \in X$, and fix a compact subspace F of X such that $x \in F$ and F is a G_δ-subset of X. Since $t(F) \leq \omega$ and F is compact, there exists a countable π-base η of the space F at x (by a theorem of Shapirovskij, see [152]). Every $P \in \eta$ is a G_δ-subset in X, since F is a G_δ-subset of X. Therefore, η is a $\pi\tau$-network of X at x. Hence, $\pi\omega\chi(x, X) \leq |\eta| \leq \omega$. □

Theorem 7.17 ([18]). *If X is an ω-twistable space of point-countable type, then the tightness of X is countable if and only if X is first-countable.*

Proof. Indeed, if the tightness of X is countable, then the $\pi\omega$-character of X is also countable, by Proposition 7.16, and it remains to apply Theorem 7.12. □

8 On compact G_δ-subspaces of semitopological groups

In this section we consider homogeneous spaces of a special kind - semitopological groups - and give some information on the structure of compact subspaces of such spaces.

A *semitopological group* is a group with a topology such that all left and all right translations in the group are homeomorphisms.

Clearly, every semitopological group is ω-twistable, and even twistable, at the neutral element by the product operation in the group. Since every semitopological group is a homogeneous space, it is ω-twistable (twistable) at every point.

M. M. Choban [48] and B. A. Pasynkov [140] established that every compact G_δ-subset of a topological group is a dyadic. The stronger result that such a G_δ-subset is even Dugundji is due to Pasynkov [140] (which also follows from the considerations in [48]). (For more information, see Uspenskiy [169]). This result provides a real motivation for the next theorem. For the sake of brevity, a compact space F will be called *Tychonoff small* if F cannot be mapped continuously onto the Tychonoff cube I^{ω_1}.

Theorem 8.1 ([18]). *Suppose that G is a semitopological group, and that F is a non-empty compact subspace of G with a countable base of neighbourhoods in G. Suppose further that F is Tychonoff small. Then G is first-countable, the diagonal in $G \times G$ is a G_δ-set, and F is metrizable.*

Proof. Indeed, the $\pi\omega$-character of the space F at some point of F is countable, since F is Tychonoff small [153].

Since F is a G_δ-set in G, and G is homogeneous, it follows that the $\pi\omega$-character of G is countable at every point of G.

The space G is ω-twistable, since G is a semitopological group. Now it follows from Theorem 7.12 that each point of G is a G_δ-point.

Clearly, G is a space of point-countable type. Therefore, G is first-countable.

However, every first-countable semitopological group G has a G_δ-diagonal (see [46] and [8, Theorem 3.1]). It remains to refer to the well-known fact that every compact space with a G_δ-diagonal is metrizable [65]. $\square$

Corollary 8.2 ([18]). *Suppose that G is a semitopological group of point-countable type such that the tightness of G is countable. Then the space G is first-countable, and the diagonal in $G \times G$ is a G_δ-set.*

Corollary 8.3. *Suppose that G is a semitopological group with countable tightness which admits a perfect mapping onto a first-countable space. Then the space G is first-countable, and the diagonal in $G \times G$ is a G_δ-set.*

Note that a modification of van Mill's [118] compactum discussed in [79] shows that a zero-dimensional compact coset-space needn't be ω-twistable. Indeed, this compactum has countable π-character but is not first-countable.

9 On twisters in $\beta\omega \setminus \omega$

The natural question, whether the compactum $\beta\omega \setminus \omega$ is ω-twistable at some point, turns out to be rather delicate. The following result is from [19]:

Proposition 9.1. *Suppose that z is a point in $X = \beta\omega\setminus\omega$ such that $z \in \overline{A}\setminus A$, for some countable discrete subspace A of X. Then X is not ω-twistable at z.*

Proof. Assume that X is ω-twistable at z. The subspace $Z = \overline{A}$ is a retract of $\beta\omega$ [113]. Therefore, Z is a retract of X.

Since X is ω-twistable at z, it follows from Proposition 7.3 that Z is ω-twistable at z. Since Z is homeomorphic to $\beta\omega$, and z is not a G_δ-point in Z, it follows that Z is not ω-twistable at z (see Corollary 7.13), a contradiction. $\square$

There are many points in $\beta\omega \setminus \omega$ such as in Proposition 9.1. Hence, $\beta\omega \setminus \omega$ is not ω-twistable at some points. However, it was shown in [18] that, consistently, $\beta\omega \setminus \omega$ is ω-twistable at some point.

A point x of a space X is *a chain-point* [18] if there exists a family γ of open subsets of X satisfying the following conditions:

a) $\bigcap \gamma = \bigcap\{\overline{V} : V \in \gamma\} = \{x\}$; and
b) γ is a chain, that is, for any $V, U \in \gamma$, either $V \subset U$ or $U \subset V$.

Any such family γ will be called *a strong chain* at x.

A slightly stronger version of twistability has been defined in [18], as follows.

A *point-continuous twister* at a point e of a space X is a binary operation on X satisfying the following conditions:

a) $ex = xe = x$, for each $x \in X$;
b) The multiplication is jointly continuous at (x, y) whenever $x = e$ or $y = e$.

A space X will be called *pc-twistable at* $e \in X$ if there exists a point-continuous twister on X at e. A space X is *pc-twistable* if it is pc-twistable at every point of X.

Proposition 9.2 ([18]). *Any space X is pc-twistable at any chain-point in X.*

Proof. Let e be a chain-point in X, and γ be a strong chain at e. Take any $x, y \in X$. Put $xy = y$ if there exists $V \in \gamma$ such that $x \in V$ and $y \notin \overline{V}$. Otherwise, put $xy = x$. In particular, it follows that $ey = y$, for each $y \in X$, and $xe = x$, for each $x \in X$. It cannot occur that, for some $V, U \in \gamma$ and $x, y \in X$, $x \in V$, $y \notin V$, $y \in U$, and $x \notin U$, since γ is a chain. Therefore, the definition of multiplication is correct. Let us check that the binary operation so defined is a point-continuous twister on X at e.

Case 1: $a \neq e$. Then there exists $V \in \gamma$ such that $a \in Oa = X \setminus \overline{V}$. Then $xy = y \in Oa$, for any $x \in V$ and any $y \in Oa$. Thus, the multiplication is jointly continuous at (e, a). It is also clear that $yx = y \in Oa$, for each $y \in Oa$ and each $x \in V$. Therefore, the multiplication is jointly continuous at (a, e) as well.

Case 2: $a = e$. The multiplication at (e, e) is continuous, since whenever W is an open neighbourhood of e and x, y are any elements of W, the product xy is either x or y and, therefore, xy belongs to W.

Hence, X is point-continuously twistable at e. □

Clearly, point-continuous twistability implies twistability.

We also have:

Proposition 9.3. *Every regular space X is point-continuously twistable at any G_δ-point e in X.*

Proof. Indeed, in a regular space X every G_δ-point is a chain-point. It remains to apply Proposition 9.2. □

Recall that a point $x \in X$ is *a P-point* in X if, for every countable family γ of open neighbourhoods of x, the intersection of γ contains an open neighbourhood of x.

Theorem 9.4 (CH). *$\beta\omega \setminus \omega$ is point-continuously twistable at any P-point in $\beta\omega \setminus \omega$. Therefore, $\beta\omega \setminus \omega$ is point-continuously twistable (and hence, ω-twistable) at some point.*

Proof. There is a P-point in $\beta\omega \setminus \omega$ under CH, [147]. Clearly, CH implies that every P-point in $\beta\omega \setminus \omega$ is a chain-point, since the weight of $\beta\omega \setminus \omega$ does not exceed ω_1 in this case. It follows from Proposition 9.2 that, under CH, $\beta\omega \setminus \omega$ is point-continuously twistable at any P-point. □

Problem 9.5. Is it true in ZFC that $\beta\omega\setminus\omega$ is ω-twistable (point-continuously twistable) at some point?

Here is a curious generalization of Theorem 9.4.

Proposition 9.6 (CH). *Suppose that X is a space admitting a one-to-one continuous mapping f onto $\beta\omega\setminus\omega$. Then, X is point-continuously twistable at some point.*

Proof. The proof of Theorem 9.4 shows that CH implies that there exists a chain-point y in $\beta\omega\setminus\omega$. The point $x \in X$, which is the preimage of y under f, is obviously a chain-point in X, since the preimage of a strong chain at y in $\beta\omega\setminus\omega$ under f is a strong chain in X at x. Therefore, X is point-continuously twistable at x, by Proposition 9.2. □

We also need the following simple fact [18]:

Proposition 9.7. *Suppose that Y is an open subspace of a space X, $e \in Y$, and Y is τ-twistable at e. Then X is also τ-twistable at e.*

Proof. Fix a τ-twister on Y at e. Take any $x, y \in X$. If both x, y are in Y, xy and yx are already defined and we stick to these definitions. Suppose that $x \notin Y$ and $y \in Y$. Then we put $xy = x$ and $yx = x$. If $x \notin Y$ and $y \notin Y$, then we put $xy = x$ and $yx = y$. Since Y is open in X and $e \in Y$, it is clear that the binary operation on X so defined is a τ-twister on X. Hence, X is τ-twistable at e. □

The following partial generalization of E. van Douwen's theorem that $\beta\omega\setminus\omega$ is not power-homogeneous [57] was established in [18]. Notice, that we need CH, while van Douwen proved his theorem in ZFC.

Theorem 9.8 (CH). *Suppose that $\beta\omega\setminus\omega$ is an open subspace of a space X. Then X is not power-homogeneous.*

Proof. By Proposition 9.7 and Theorem 9.4, the space X is ω-twistable at some point. If X is power-homogeneous, then X is ω-twistable at every point. Since X is Hausdorff, the compact subspace $\beta\omega\setminus\omega$ is closed in X. Since $\beta\omega\setminus\omega$ is open in X, it follows that $\beta\omega\setminus\omega$ is a retract of X. Hence, $\beta\omega\setminus\omega$ is ω-twistable at every point, which contradicts Proposition 9.1. □

Problem 9.9. Can one drop CH in the last theorem?

10 Murray Bell's Theorem

Results in this section are closely related to some deep theorems of M. Bell [30] and M.E. Rudin [146].

M. Bell has shown that if a compactum X is a continuous image of some linearly ordered compact space, and X is power-homogeneous, then X must be first-countable [30]. On the other hand, M.E. Rudin established [146] that a compactum X can be represented as a continuous image of some linearly ordered compact space if and only if X is monotonically normal. It follows that every power-homogeneous monotonically normal compactum is first-countable. Since the arguments in [30] and, especially, in [146] are not easy, we present below an elementary proof of the last result given in [18].

Theorem 10.1. *Suppose that X is a locally compact monotonically normal space, and that Y is a space which is ω-twistable at least at one point. Suppose also that $X \times Y$ is power-homogeneous. Then X is first-countable at a dense in X set of points.*

Every monotonically normal space is hereditarily normal, and to prove the above theorem, we need some results on hereditarily normal spaces.

It has been shown in [18] that hereditary normality and ω-twistability rarely go together in compacta.

The Tychonoff number of a space X will be said to be countable if every compact subspace of X is Tychonoff small (notation: $Tych(X) = \omega$).

Clearly, every compactum with countable tightness is Tychonoff small. Also every hereditarily normal compactum is Tychonoff small, since I^{ω_1} is not hereditarily normal.

Theorem 10.2 ([18]). *If X is an ω-twistable space of point-countable type, and the Tychonoff number of X is countable, then X is first-countable at a dense set of points.*

Proof. Take any $x \in X$ and any open neighbourhood Ox of x. Since X is a space of point-countable type, we can fix a compact subspace F of X such that $x \in F \subset Ox$ and F has a countable base of neighbourhoods in X. The space F is compact and cannot be mapped continuously onto the Tychonoff cube I^{ω_1}. By a theorem of Shapirovskij [152], it follows that the set H of all points $y \in F$ at which the space F has a countable π-base is dense in F. Fix any $y \in H$.

Clearly, $\pi\omega\chi(y, X) \leq \omega$. Since X is ω-twistable at y, it follows from Theorem 7.12 that y is a G_δ-point in X. Under the restrictions on X, this implies that X is first-countable at y. □

Corollary 10.3. *Every hereditarily normal ω-twistable space X of point-countable type is first-countable at a dense set of points.*

The ordinal space $\omega_1 + 1$ is a hereditaily normal ω-twistable compactum [19]. This space is first-countable at a dense set of points but not at all points.

Problem 10.4. Is every power-homogeneous hereditarily normal compact space X first-countable?

The answer to the last question seems to be not known even when the space X is homogeneous. It is known, though, that the cardinality of power homogeneous hereditarily normal compacta X is bounded by $2^{c(X)}$, [144].

Proof. (of Theorem 10.1) Fix a non-empty open set U in X such that $\overline{U}$ is compact. Since $\overline{U}$ is also monotonically normal, Theorem 3.12 (iii) in [93] implies that there exists a chain-point e in $\overline{U}$ such that $e \in U$. Then, clearly, e is a chain-point in X. It follows that X is ω-twistable at e. Since Y is also ω-twistable at some point y, the space $X \times Y$ is ω-twistable at (e, y). Since $X \times Y$ is power-homogeneous, it follows that $X \times Y$ is ω-twistable at all points. Therefore, by Proposition 7.3, the space X is ω-twistable at every point. It remains to apply Theorem 10.2. □

Notice, that the above proof of Theorem 10.1 also shows that the following statement is true:

Proposition 10.5. *Every non-empty monotonically normal locally compact space is ω-twistable at some point.*

A proof of the next theorem from [19] should be clear by now. It generalizes in a straightforward way some results on power-homogeneity we presented earlier.

Theorem 10.6. *Suppose that X is the product of a family $\gamma = \{X_\alpha : \alpha \in A\}$ of non-empty spaces X_α of point-countable type each of which is first-countable at least at one point, and that X is homogeneous. Then, for each $\alpha \in A$, the set of all points at which X_α is first-countable, is closed in X_α.*

The next two statements from [18] slightly generalize Bell's results.

Theorem 10.7. *If X is the product of a family $\gamma = \{X_\alpha : \alpha \in A\}$ of non-empty locally compact monotonically normal spaces X_α, and X is homogeneous, then every $X_\alpha \in \gamma$ is first-countable.*

Proof. Fix $\alpha \in A$, and put $B = A \setminus \{\alpha\}$, $Y = X_\alpha$, and $Z = \prod\{X_\beta : \beta \in B\}$. Then $X = Y \times Z$. It follows from Proposition 10.5 that the space Z is ω-twistable at some point. Therefore, by Theorem 10.1, the space $X_\alpha = Y$ is first-countable at a dense set of points. Since this is true for each $\alpha \in A$, it follows from Theorem 10.6 that each X_α is first-countable. □

Corollary 10.8 ([18]). *If a locally compact monotonically normal space X is power-homogeneous, then X is first-countable.*

Problem 10.9. Suppose that X is a monotonically normal power-homogeneous space X of point-countable type. Then is X first-countable?

In connection with Theorem 10.1, we mention again the next natural question:

Problem 10.10. Does there exist a compact space Y such that the product $(\omega_1+1)\times Y$ is homogeneous?

Recall that all linearly ordered spaces are monotonically normal [81], [76]. It was shown in [18] that every power-homogeneous linearly ordered space of point-countable type is first-countable. This result gives some hope that Problem 10.9 might get a positive answer.

In conclusion of this section, we refer the reader to [137]. It is stated there that if a homogeneous compactum X is a continuous image of a linearly ordered compactum, then either X is metrizable, or $\dim(X)=0$, or X is a union of a finite pairwise disjoint family of generalized simple closed curves.

We recall that an *arc* is a non-degenerate compact connected linearly ordered space. A space obtained from an arc by identifying the first and the last element of an arc is called a *generalized simple closed curve*.

11 Corson compacta and power-homogeneity

Recall that *a Corson compactum* is a compact subspace of the Σ-product of separable metrizable spaces (see [17]). It easily follows from this definition (and is well-known) that each Corson compactum X is *monolithic*, that is, the weight of the closure of an arbitrary subset A of X does not exceed the cardinality of A. The tightness of any Corson compactum is countable, since the tightness of the Σ-product of any family of separable metrizable spaces is countable (see 3.10.D in [65]).

Theorem 11.1. *Suppose that X is a Corson compactum such that $\mathsf{Exp}(X)$ is power-homogeneous. Then X is metrizable.*

Proof. If X is a Corson compactum, then there exists a dense subspace Y of X such that X is first-countable at every point of Y. Indeed, every monolithic compactum of countable tightness is first-countable at a dense set of points (see [17]). The set Z of all finite subsets of Y is a dense subspace of $\mathsf{Exp}(X)$, and $\mathsf{Exp}(X)$ is, obviously, first-countable at each $F\in Z$. Therefore, by Theorem 7.7, the space $\mathsf{Exp}(X)$ is first-countable, since it is compact and power-homogeneous.

By a result of Choban in [47], this implies that X is separable. Hence, X is metrizable, since X is monolithic. $\square$

Clearly, the above theorem holds for all monolithic compacta of countable tightness.

The above argument also shows that the next theorem from [18] holds:

Theorem 11.2. *Every power-homogeneous Corson compactum is first-countable and hence, the Souslin number of X doesn't exceed 2^ω.*

Clearly, this result can be generalized as follows: *if the product of some family of nonempty Corson compacta is homogeneous, then all compacta in the family are first-countable.*

12 Some further consistency results on homogeneous compacta

There are quite a few remarkable results on homogeneous compacta which could be established only consistently. A good example of a result of this kind provides an answer to the following question: does there exist a non-first-countable homogeneous compactum X such that $|X| \leq 2^\omega$? This question is especially interesting, since we have seen above many non-trivial sufficient conditions for a homogeneous compactum X to satisfy the inequality $|X| \leq 2^\omega$. Clearly, under CH the answer is 'no'. However, under $\mathsf{MA}+\neg\mathsf{CH}$ the answer is 'yes' (just take D^{ω_1}).

If X is a homogeneous compactum, then, under the Generalized Continuum Hypothesis GCH, the π-weight of X coincides with the character of X, that is, $\chi(X) = \pi(X)$. Hence CH implies that a homogeneous compactum of countable π-weight is first-countable.

There is a compact space X of countable π-weight and uncountable character which is homogeneous under $\mathsf{MA}+\neg\mathsf{CH}$, but not under CH. This example is due to van Mill [118]. K.P. Hart and G.J. Ridderbos have adapted van Mill's example so that, in addition, the compactum X became zero-dimensional.

See a description of this modification and a discussion of its properties in [79]. Observe that the space X is not ω-twistable at any point of X, since otherwise X would have been first-countable. Thus, countable π-character in compacta, unlike first-countability, doesn't imply ω-twistability.

Here is a curious inequality:

Theorem 12.1. *Let X be a homogeneous compactum. Then $2^{\chi(X)} \leq 2^{\pi(X)}$.*

There are two ingredients in the proof. The first one is the result of van Douwen [57] (see also [92, 2.38]) that $|X| \leq 2^{\pi(X)}$ for every homogeneous space X.

The second ingredient is the classical Čech-Pospišil Theorem, see [92, 3.16], that if X is compact and if for some κ, $\chi(x, X) \geq \kappa$ for every $x \in X$, then $|X| \geq 2^\kappa$. To complete the proof, we observe that the homogeneity of X implies that all points in X have the same character, hence $|X| \geq 2^{\chi(X)}$.

Theorem 12.1 has some interesting consequences.

Corollary 12.2. *Let X be a homogeneous compactum. Then $\chi(X) < 2^{\pi(X)}$.*

Simply apply Cantor's Theorem that $2^\kappa > \kappa$ for every cardinal κ.

Corollary 12.3 (GCH)**.** *If X is a homogeneous compactum then $\chi(X) \leq \pi(X)$.*

This inequality is much more appealing than the one in Theorem 12.1.

Corollary 12.4 ($2^\omega < 2^{\omega_1}$)**.** *Every homogeneous compactum of countable π-weight is first-countable.*

Theorem 12.5 (MA)**.** *Let X be a homogeneous compactum of countable π-weight. If X has weight less than 2^ω, then X is first-countable.*

The next result may be known (see e.g., [94]).

Theorem 12.6. *Let $\kappa < \mathfrak{p}$. If X is a compact space of weight at most κ and of countable π-weight then X is somewhere first-countable.*

The following question is quite natural. Let X be a compact homogeneous space of countable π-weight. Assume that X has weight less than 2^ω. Does X have countable weight under MA? The answer to this question is in the negative. Let G be a dense subgroup of $\mathbb{R}$ of cardinality ω_1 such that $1 \in G$. In the unit interval $\mathbb{I}$, split every point $g \in G \cap (0, 1)$ in two distinct points g^- and g^+. Order the set so obtained in the natural way, where x^- precedes x^+ if x is split. The ordered compact space that we obtain by this procedure has weight ω_1, has countable π-weight, and is homogeneous by the method of van Douwen [59]. (This example is in [118] and in Hart and Kunen [78]).

There are many nonhomogeneity results in the literature which in essence boil down to cardinality considerations. Frolík's Theorem in [75] that $\mathbb{N}^*$ is not homogeneous is such an example. The proofs of these results were sometimes replaced by better proofs, presenting explicit topological properties shared by some but not all points of the spaces under consideration. In the case of Frolík's Theorem this was done by Kunen in [100]: he showed that some but not all points in $\mathbb{N}^*$ are weak P-points. Van Douwen called such arguments 'honest' nonhomogeneity proofs.

For the space discussed by J. van Mill in [118] and for the space constructed in [79] it seems impossible to present an 'honest' proof of its nonhomogeneity in some model of set theory. Simply observe that it is homogeneous under MA+$\neg$CH. This is a very curious phenomenon which deserves a further study.

13 Homogeneity and actions of topological groups

Among other things, we are interested here in topological spaces X that admit a transitive continuous action of an interesting topological group G. Since the action is assumed to be transitive, the topological spaces X we are interested in are homogeneous.

For a homogeneous space X, its group of homeomorphisms $\mathscr{H}(X)$ endowed with the discrete topology acts transitively and continuously on X. But the discrete topology is not interesting. The compact-open topology on $\mathscr{H}(X)$ is better but only works well if X is compact (or if X is locally compact, by thinking of X as a subspace of its Alexandroff one-point compactification).

An *action* of a topological group G on a space X is a continuous function

$$(g, x) \mapsto gx: G \times X \to X$$

such that $ex = x$ for every $x \in X$ and $g(hx) = (gh)x$ for $g, h \in G$ and $x \in X$. It is easily seen that for each $g \in G$ the function $x \mapsto gx$ is a homeomorphism of X whose inverse is the function $x \mapsto g^{-1}x$. For every $x \in X$ let $\gamma_x: G \to X$ be defined by $\gamma_x(h) = hx$. Then γ_x is continuous and a surjection if G acts transitively. The action is *micro-transitive* if for every $x \in X$ and every neighbourhood U of e in G the set Ux is a neighbourhood of x in X.

The proof of the following simple result is left as an exercise to the reader.

Lemma 13.1. *Let G be a group acting transitively on a space X. Then the following statements are equivalent.*

1. *The action of G on X is micro-transitive.*
2. *For every $x \in X$ the function $\gamma_x: G \to X$ is open.*
3. *For some $x_0 \in X$ the function $\gamma_{x_0}: G \to X$ is open.*

Let G be a topological group with a closed subgroup H. If $x, y \in G$ and $xH \cap yH \neq \varnothing$ then $xH = yH$. Hence the collection of all *left cosets* $G/H = \{xH : x \in G\}$ is a partition of G in closed sets. Let $\pi: G \to G/H$ be defined by $\pi(x) = xH$. We endow G/H by the quotient topology. In other words, if $A \subset G$ then $\{xH : x \in A\}$ is open in G/H if and only if $\bigcup\{xH : x \in A\} = AH$ is open in G.

A space X is a *coset space* provided that there is a closed subgroup H of a topological group G such that X and G/H are homeomorphic. In this subsection we will consider the following basic question: which spaces are coset spaces of topological groups?

Let G be a topological group with a closed subgroup H. Then H is a subset of G and H is a point of G/H. This sometimes leads to a confusion.

Lemma 13.2. *Let G be a topological group with a closed subgroup H. Then*

1. *if $A \subset G$, then $\pi^{-1}\big(\pi(A)\big) = AH$,*
2. *if $U \subset G$ is open, then $\pi(U) = \{xH : x \in U\}$ is open in G/H.*

Proof. For 1, let $p \in AH$, say $p \in aH$ for certain $a \in A$. Then $\pi(p) = \pi(a) \in \pi(A)$. Conversely, if $\pi(p) = \pi(a)$ for some $a \in A$, then $p \in aH \subset AH$.

For 2, first observe that UH is open, hence so is $\pi(U)$, since by 1,

$$\pi^{-1}\big(\pi(U)\big) = UH$$

and G/H is endowed with the quotient topology. □

Let G be a topological group with a closed subgroup H. We let G act transitively on G in the standard way by $(g, x) \mapsto gx$. We also let G act on G/H by

$$G \times G/H \to G/H : (g, xH) \mapsto gxH.$$

We call this the *natural* action of G on G/H. We will check that this action is continuous and transitive. Consider the diagram

$$(*) \qquad \begin{array}{ccc} G \times G & \xrightarrow{(g,x)\mapsto gx} & G \\ {\scriptstyle 1_G \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times G/H & \xrightarrow[(g,xH)\mapsto gxH]{} & G/H \end{array}$$

and observe that it clearly commutes.

Corollary 13.3. *Let G be a topological group with a closed subgroup H. The natural action of G on G/H is continuous, transitive and micro-transitive. As a consequence, for every $g \in G$ the function $xH \mapsto gxH$ is a homeomorphism of G/H, i.e., G/H is a homogeneous space.*

Proof. This is clear by the commutativity of $(*)$ and the fact that $1_G \times \pi$ is open by Lemma 13.2. □

By Corollary 13.3, if X is a coset space then X must be homogeneous. It is a natural question to ask whether the converse is true.

(A) Characterizing coset spaces. Let G be a topological group acting transitively on X. For every $x \in X$, put

$$G_x = \{g \in G : gx = x\}.$$

It is clear that G_x is a closed subgroup of G. It is called the *stabilizer* of x. Observe that if $g \in G$ and $h \in G_x$, then

$$(gh)x = g(hx) = gx.$$

This means that the function $\bar{\gamma}_x : G/G_x \to X$ defined by

$$(*) \qquad \bar{\gamma}_x(gG_x) = \gamma_x(g) = gx$$

is well-defined. In addition, the diagram

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr, "\gamma_x"] & \\ G/G_x \arrow[rr, "\bar{\gamma}_x"'] & & X \end{tikzcd}$$

commutes. Since π is open (Lemma 13.2(2)) and γ_x is surjective, $\bar{\gamma}_x$ is a continuous surjection. We claim that $\bar{\gamma}_x$ is one-to-one. To this end, assume that $gG_x \neq g'G_x$ for certain $g, g' \in G$. Then $g^{-1}g' \notin G_x$, i.e., $g'x \neq gx$. So $\bar{\gamma}_x(gG_x) \neq \bar{\gamma}_x(g'G_x)$. This means that X is a coset space if $\bar{\gamma}_x$ is open.

Proposition 13.4. *Let G be a topological group acting transitively on X. The following statements are equivalent:*

1. *For some $x \in X$, $\bar{\gamma}_x\colon G/G_x \to X$ is open.*
2. *For all $x \in X$, $\bar{\gamma}_x\colon G/G_x \to X$ is open.*
3. *For some $x \in X$, $\gamma_x\colon G \to X$ is open.*
4. *For all $x \in X$, $\gamma_x\colon G \to X$ is open.*
5. *G acts micro-transitively.*

Proof. Take arbitrary $x, y \in X$, and pick $h \in G$ such that $hx = y$. The diagram

$$\begin{tikzcd}
G \arrow[rr, "g\mapsto gh^{-1}"] \arrow[d, "\pi"] \arrow[ddr, bend left, "\gamma_x"] & & G \arrow[d, "\pi"'] \arrow[ddl, bend right, "\gamma_y"'] \\
G/G_x \arrow[dr, "\bar{\gamma}_x"'] & & G/G_y \arrow[dl, "\bar{\gamma}_y"] \\
& X &
\end{tikzcd}$$

commutes. Now use that both functions π are open (Lemma 13.2(2)) and apply Lemma 13.1. $\square$

This yields a characterization of coset spaces.

Theorem 13.5. *Let X be a space. The following statements are equivalent:*

1. *X is a coset space.*
2. *There is a topological group acting transitively on X such that for some (equivalently: for all) $x \in X$ the function $\gamma_x\colon G \to X$ is open.*
3. *There is a topological group acting transitively and micro-transitively on X.*

Proof. Simply apply Corollary 13.3 and Proposition 13.4. $\square$

(B) The Effros Theorem. We now formulate the following important result, known as 'The Open Mapping Principle'.

Theorem 13.6 (Open Mapping Principle, Version A). *Suppose that a Polish group G acts transitively on a metrizable space X. Then the following statements are equivalent:*

1. *G acts micro-transitively on X.*
2. *X is Polish.*
3. *X is of the second category.*

The implication $2 \Rightarrow 3$ is simply the Baire Category Theorem for Polish spaces, and $1 \Rightarrow 2$ is a consequence of Hausdorff's theorem [80] that an open continuous image of a completely metrizable space is completely metrizable.

This extremely useful result was first proved by Effros [63] using a Borel selection argument. Simpler proofs were found independently by Ancel [5], Hohti [84], and Toruńczyk (unpublished). The proof of Ancel and Toruńczyk is based on an ingenious technique of Homma [85], while Hohti uses an open mapping theorem due to Dektjarev [52].

The Open Mapping Principle implies Effros's Theorem 2.1 of [63] as well as the classical Open Mapping Theorem of Functional Analysis (for separable Banach spaces). For let B and E be separable Banach spaces, and let $\alpha\colon B \to E$ be a continuous linear surjection. We think of B as a topological group, and define an action of B on E by $(x, y) \mapsto \alpha(x)+y$. This action is transitive, since if y and y' in E and x in B are such that $\alpha(x) = y'-y$, then $(x, y) \mapsto y'$. So by Theorem 13.6, the map $B \to E$ defined by $x \mapsto \alpha(x) + 0$ is open.

The Open Mapping Principle also implies that for every homogeneous metrizable compactum (X, ϱ) and every $\varepsilon > 0$ there exists $\delta > 0$ such that, if x and y in X satisfy $\varrho(x, y) < \delta$, then there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$ and f moves no point more than ε. (This goes part way towards explaining the word micro-transitive.) This interesting and surprising fact, first discovered by Ungar [165], was used with great success by continuum theorists in their study of homogeneous metrizable continua. See Ancel [5] and Charatonik and Maćkowiak [45] for details and further references.

A space is *analytic* if it has countable weight and is a continuous image of a Polish space. It is well-known that every Borel subset of the Hilbert cube is analytic, and that a Borel subspace of an analytic space is analytic. For information on analytic spaces see, for example, Kechris [96].

Theorem 13.6 was generalized in [120], as follows.

Theorem 13.7 (Open Mapping Principle, Version B). *Suppose that an analytic group G acts transitively on a metrizable space X. If X is of the second category, then G acts micro-transitively on X.*

This result is similar but much stronger than the result of Charatonik and Maćkowiak [45] asserting that a Borel subgroup of the group of all homeomorphisms of a compact metrizable space acts micro-transitively provided that it acts transitively.

We are now in a position to identify our first important class of coset spaces.

Theorem 13.8. *Let X be a locally compact separable metrizable homogeneous space. Then X is a coset space.*

Proof. Consider X to be a subspace of its Alexandroff one-point compactification $\alpha X = X \cup \{\infty\}$. The subgroup

$$\{g \in \mathscr{H}(\alpha X) : g(\infty) = \infty\}$$

is a closed subgroup of the Polish group $\mathscr{H}(\alpha(X))$. Consider the standard action of $\mathscr{H}_\alpha(X)$ on X. By Theorem 13.6, this action is micro-transitive. Hence we are done by Theorem 13.5. $\square$

This result is due to Ungar [165] and answered questions raised by the work of Ford [72], Mostert [131] and Ungar [164].

The question naturally arises whether all homogeneous spaces are coset spaces. The answer to this question is in the negative (see Theorem 13.12). An important class of homogeneous spaces that are coset spaces will be identified in the remaining part of this section.

(C) Strongly locally homogeneous spaces. A space X is called *strongly locally homogeneous* if it has a base $\mathscr{B}$ such that for all $B \in \mathscr{B}$ and $x, y \in B$ there is a homeomorphism $f: X \to X$ that is supported on B (that is, f is the identity outside B) and moves x to y.

The notion of an SLH-space is due to Ford [72].

Ford [72] essentially proved that every Tychonoff homogeneous and SLH-space X is a coset space (see also Mostert [131, Theorem 3.2]). The proof goes as follows. One thinks of X as a subspace of its Čech-Stone compactification βX. The subgroup

$$G = \{g \in \mathscr{H}(\beta X) : g(X) = X\}$$

of the homeomorphism group $\mathscr{H}(\beta X)$ of βX endowed with the compact-open topology acts transitively on X, and by strong local homogeneity, $\gamma_x: G \to X$ is open for every $x \in X$. Ford [72] also gave an example of a homogeneous Tychonoff space which is not a coset space. His example is not metrizable.

Corollary 13.9. *Every homogeneous and* SLH*-space is a coset space.*

Since any zero-dimensional homogeneous space is, obviously, SLH, we obtain:

Corollary 13.10. *Let X be zero-dimensional and homogeneous. Then X is a coset space.*

In fact, any homogeneous zero-dimensional space is a coset space of some zero-dimensional topological group. This follows from the above result, since every topological group is a quotient of some zero-dimensional topological group (Arhangel'skii [15]).

(D) A homogeneous space that is not a coset space. We will now discuss examples of homogeneous space that are not coset spaces. This shows that in Corollary 13.9 some extra condition is essential.

The first example of such a space is due to Ford [72]. But his space is neither compact, nor metrizable. We will now describe two examples of homogeneous spaces that are not coset spaces. The first one is compact (and hence is not metrizable by Theorem 13.8), and the second one has countable weight (and hence is not locally compact, again by Theorem 13.8).

Actions on compact spaces can be 'characterized' rather easily. To see this, let X be a compact space and let G be a topological group acting on X. For each $g \in G$ the function $x \mapsto gx$ is a homeomorphism of X. We denote this homeomorphism by $\varphi(g)$. Clearly, $\varphi\colon G \to \mathscr{H}(X)$ is a continuous homomorphism. Conversely, if $\varphi\colon G \to \mathscr{H}(X)$ is a continuous homomorphism then the composition

$$G \times X \xrightarrow{\varphi \times 1_X} \mathscr{H}(X) \times X \to X$$

is an action of G on X.

So on a compact space X there is basically only one action: the natural action $\mathscr{H}(X) \times X \to X$. All 'other' actions come from continuous homomorphisms from topological groups into the group $\mathscr{H}(X)$.

The first example of a homogeneous separable compact space X that is not a coset space is due to Fedorchuk [68]. It has the property that $\dim X = 1$ and $\operatorname{ind} X = \operatorname{Ind} X = 2$. Interestingly, Pasynkov [139] showed that if Y is a compact coset space of some locally compact group, then the dimension functions dim, ind and Ind take the same values on Y. Another example of a homogeneous continuum which is not a coset space was constructed by Bellamy and Porter [31]. They showed that there is a homogeneous continuum X such that for some neighbourhood V of e in $\mathscr{H}(X)$ and some $p \in X$ we have that Vp is nowhere dense. To see that G is not a coset space, let G be a topological group acting transitively on X, and let $\varphi\colon G \to \mathscr{H}(X)$ be the continuous homomorphism that is associated with this action. Then $W = \varphi^{-1}(V)$ is an open neighbourhood of the neutral element e of G, and $Wp = Vp$ is nowhere dense, hence the action is not micro-transitive.

Theorem 13.11. *There is a homogenous continuum which is not a coset space.*

We will now describe the second example. Let $Q = \prod_{n=1}^{\infty}[-1,1]_n$ denote the Hilbert cube. For each i let

$$W_i = \prod_{j \neq i}[-1 + 2^{-i}, 1 - 2^{-i}]_j \times \{1\}_i \subset Q.$$

Then W_i is a 'shrunken' endface in the i-th coordinate direction.

It was shown by Anderson, Curtis and van Mill [7] that $Y = Q \setminus \bigcup_{i=1}^{\infty} W_i$ is homogeneous. It can be shown that Y is a coset space. Put $W = \bigcup_{i=1}^{\infty} W_i$.

Theorem 13.12 ([119]). *W is homogeneous, but not a coset space.*

Observe that Q is a compactification of W with the following property: for all $x, y \in W$ there exists $h \in \mathscr{H}(Q)$ such that $h(x) = y$ and $h(W) = W$. This implies that there is a topological group G acting transitively on W. Simply let

$$G = \{g \in \mathscr{H}(Q) : g(W) = W\}.$$

Observe also that G has countable weight, and that Y is σ-compact. So there are separable metrizable spaces on which some nice group acts transitively but that are not coset spaces.

The space in Theorem 13.14 is an example of a homogeneous Polish space that is not a coset space.

Of course, coset spaces can be also formed on the basis of semitopological groups. We could even start with left topological groups. The spaces so obtained are, clearly, homogeneous, and Theorem 13.12 provides a motivation for the next problem.

Problem 13.13. Is every homogeneous Tychonoff space homeomorphic to a coset space of some semitopological (left topological) group with respect to some closed subgroup?

It is also natural to ask whether there are homogeneous spaces without transitive actions of nice groups.

(E) A homogeneous Polish space on which no nice topological group acts transitively. We address here the question whether every homogeneous Polish space is a coset space, preferably of some Polish group. This is related to Question 3 in Ancel [5]. He asked whether for every homogeneous Polish space X there is an admissible topology on its homeomorphism group $\mathscr{H}(X)$ which makes X a coset space of $\mathscr{H}(X)$.

Theorem 13.14 ([126]). *There is a homogeneous Polish space Z with the following property. If G is a topological group acting on Z, then there are an element $z \in Z$ and a neighbourhood U of the neutral element e of G such that Uz is meager in Z.*

So an arbitrary homogeneous Polish space X need not be a coset space since no action on Z by a topological group is micro-transitive. This answers Question 3 in Ancel [5] in the negative.

A topological group G is called $\aleph_0$-*bounded* (or ω-*narrow*) provided that for every neighbourhood U of the identity e there is a countable subset F of G such that $G = FU$. It was proved by I.I. Guran that a topological group G is $\aleph_0$-bounded if and only if it is topologically isomorphic to a subgroup of a product of separable metrizable groups. For a proof, see Uspenskiy [168].

Corollary 13.15 ([126]). *If G is an $\aleph_0$-bounded topological group acting on Z, then there is an element $z \in Z$ such that its orbit Gz is meager in Z.*

It was asked in [123, Question 4.2] whether for every homogeneous Polish space X there is a separable metrizable topological group acting transitively on X. Hence Z is a counterexample to this question. It was also asked by Aarts and Oversteegen [1] whether every homogeneous Polish space is the product of one of its quasi-components and a totally disconnected space. This question was answered in the negative in [124] by using highly non-trivial results of Bing and Jones [37] and Lewis [106]. It can be shown that

Z is a much better (and simpler) counterexample. So Z is a counterexample to several natural questions on homogeneity in the literature.

The space Z is a subspace of the product S of a Cantor set and the unit interval. The complement $S \setminus Z$ of Z in S is an F_σ-subset that is chosen in such a way that the components of Z are in a sense as wildly distributed as possible. This extremal behavior of the components of Z on the one hand guarantees homogeneity, but on the other hand kills transitive group actions by $\aleph_0$-bounded topological groups. So the pathology of Z is based upon connectivity.

14 Countable dense homogeneity

A separable space X is *countable dense homogeneous* (abbreviated: CDH) if given any two countable dense subsets D and E of X, there is a homeomorphism $f : X \to X$ such that $f(D) = E$. This notion is of interest only if X is separable, so we include separability in its definition.

The first result in this area is due to Cantor [42], who showed by his now famous 'back-and-forth' method that the reals $\mathbb{R}$ are CDH. Fréchet and Brouwer, independently, proved that the same is true for the n-dimensional Euclidean space $\mathbb{R}^n$. In 1962, Fort [73] proved that the Hilbert cube is also CDH.

All these spaces have in common that they are strongly locally homogeneous (abbreviated SLH).

There are very few topological operations under which the classes of CDH-spaces and SLH-spaces are stable. Consider for example the product $X = \triangle \times \mathbb{R}$, where $\triangle$ denotes the Cantor set in $\mathbb{I}$. Then $\triangle$ is obviously SLH, and hence CDH by Theorem 14.1. But X is neither SLH, nor CDH, as can easily be seen from the fact that any homeomorphism of X permutes the components of X. Observe that X is even a topological group. Since any compact metrizable space is a continuous image of $\triangle$, this clearly implies that continuous images of CDH-spaces need not be CDH. Similarly for SLH. So both concepts behave much worse than the classical notion of homogeneity.

Still, there are many CDH-spaces, as the following result shows.

Theorem 14.1 (Bessaga and Pełczyński [34]). *Let X be Polish and strongly locally homogeneous. Then X is countable dense homogeneous.*

Proof. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be faithfully indexed dense subsets of X. The hypothesis of strong local homogeneity implies that for each neighbourhood U of a point $x \in X$, and for any dense $G \subset X$, there exists a homeomorphism of X which is supported on U and takes x into G (use that $G \cap U \neq \varnothing$). We construct a sequence $(h_n)_n$ of homeomorphisms of X such that its infinite left product $h = \lim_{n\to\infty} h_n \circ \cdots \circ h_1$ is a homeomorphism and such that the following conditions (which ensure $h(A) = B$) are satisfied:

(1) $h_n \circ \cdots \circ h_1(a_i) = h_{2i} \circ \cdots \circ h_1(a_i) \in B$ for each i and $n \geq 2i$,
(2) $(h_n \circ \cdots \circ h_1)^{-1}(b_i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(b_i) \in A$ for each i and each $n \geq 2i+1$,

Assume $h_1, \ldots, h_{2i-1}$ have been defined for certain i.

If $h_{2i-1} \circ \cdots \circ h_1(a_i) \in B$, take h_{2i} to be the identity on X. Otherwise, choose a small neighbourhood U_{2i} of $h_{2i-1} \circ \cdots \circ h_1(a_i)$ which is disjoint from the finite set

$$\{b_1, \ldots, b_{i-1}\} \cup h_{2i-1} \circ \cdots \circ h_1(\{a_1, \ldots, a_{i-1}\}).$$

Take f_{2i} to be a homeomorphism of X supported on U_{2i} and such that

$$f_{2i} \circ h_{2i-1} \circ \cdots \circ h_1(a_1) \in B.$$

If $\big(h_{2i} \circ \cdots \circ h_1(b_i)\big)^{-1} \in A$, take h_{2i+1} to be the identity on X. Otherwise, choose a small neighbourhood U_{2i+1} of b_i which is disjoint from the finite set

$$\{b_1, \ldots, b_{i-1}\} \cup h_{2i} \circ \cdots \circ h_1(\{a_1, \ldots, a_{i-1}\}).$$

Take f_{2i+1} to be a homeomorphism of X supported on U_{2i+1} and such that

$$f_{2i+1}^{-1}(b_i) \in (h_{2i} \circ \cdots \circ h_1)(A).$$

If the neighbourhoods U_{2i} and U_{2i+1} are chosen small enough, the conditions of the Inductive Convergence Criterion of Anderson and Bing [6, p. 777] are satisfied ensuring that the infinite left product of the sequence of constructed homeomorphisms converges to a homeomorphism. □

So all of the CDH spaces that we get from this result are *Polish.* This is not by accident: consider the following result, obtained by Hrušák and Zamora-Avilés [86] in 2005.

Theorem 14.2. *If X is a* CDH *Borel space then X is Polish. Under* $\mathrm{MA} + \neg\mathrm{CH} + \omega_1 = \omega_1^L$, *there exists an analytic* CDH *space that is not Polish.*

The second half of Question 387 from Fitzpatrick and Zhou [70] asks for which zero-dimensional subsets X of $\mathbb{R}$ the infinite power X^ω is CDH. The following partial answer from [86] is a nice application of a couple of theorems that we have mentioned so far.

Corollary 14.3. *Let $X \subseteq 2^\omega$ be Borel. Then X^ω is* CDH *if and only if X is G_δ.*

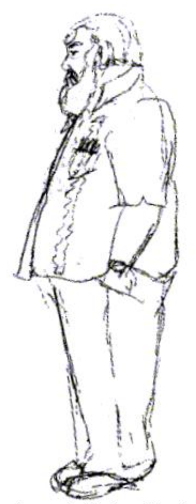
Proof. The left-to-right implication follows immediately from the above theorem. Now assume that X is G_δ, hence Polish. So X^ω is Polish as well. Since X^ω is homogeneous (by Dow and Pearl [62]) and zero-dimensional, it is SLH. So X^ω is CDH by Theorem 14.1. □

Hrušák and Zamora-Avilés then ask if there exists a non-G_δ subset X of 2^ω such that X^ω is CDH. Medini and Milovich [110] constructed such an example under MA for countable posets. Their example is a non-principal ultrafilter on ω, viewed as a subspace of 2^ω under the obvious identification. Subsequently, Hernández-Gutiérrez and Hrušák [83] showed that for every non-meager P-filter $\mathscr{F}$ on ω, both $\mathscr{F}$ and $\mathscr{F}^\omega$ are CDH. Since non-principal non-meager filters on ω cannot be analytic or co-analytic, the following question from [110] seems natural.

Question 14.4. Is there an analytic non-G_δ subset X of 2^ω such that X^ω is CDH? Coanalytic?

The topological sum of the 1-sphere $\mathbb{S}^1$ and $\mathbb{S}^2$ is an example of a CDH-space that is not homogeneous. R. Bennett [33] proved in 1972 that a *connected* first-countable CDH-space is homogeneous. (The converse is not true, and the assumption on first-countability is superfluous, see below.) Hence for *connected metrizable* spaces, countable dense homogeneity can be thought of as a strong form of homogeneity.

After 1972, the interest in CDH-spaces was kept alive mainly by Fitzpatrick.



He obtained many interesting results on CDH-spaces. For example, in [71] he proved that if X is a connected, locally compact metrizable space and is countable dense homogeneous, then X is locally connected. This result suggests the following interesting question that has been open for a long time.

Problem 14.5 ([70]). Is every connected Polish CDH-space locally connected?

Moreover, Fitzpatrick and Lauer proved in [69] that every component of a CDH-space is again CDH. In that same paper it was also shown that a connected CDH-space is homogeneous, thereby generalizing a well-known result due to Bennett [33].

Not all known separable and metrizable CDH-spaces are obtained from the Bessaga and Pełczyński Theorem 14.1. Farah, Hrušák and Martínez Ranero [67] proved in 2005 that there is a subspace of $\mathbb{R}$ of size $\aleph_1$ that is CDH. Kawamura, Oversteegen and Tymchatyn [95] proved that the complete Erdős space is CDH. (The *complete Erdős space* is the set of all vectors $x = (x_n)_n$ in Hilbert space ℓ^2 such that x_n is irrational for every n.) There is a connected and locally connected (Polish) CDH-space which is not SLH (under CH, Saltsman [149], in ZFC, van Mill [121]). There is a connected and locally connected (Polish) CDH-space S with a dense open rigid connected subset (under CH, Saltsman [150], in ZFC, van Mill [125]). (A space is *rigid* if the identity is its only homeomorphism.) In fact, $S \times S$ is homeomorphic to the separable Hilbert space ℓ^2. See also Baldwin and Beaudoin [27] for an example of a CDH Bernstein subspace of $\mathbb{R}$ under MA for countable partial orders, and the ultrafilters of Medini and Milovich [110], the P-filters of Hernández-Gutiérrez and Hrušák [83] and the analytic CDH-space of Hrušák and Zamora-Avilés [86] that we discussed above. (For CDH-spaces that are not metrizable, see e.g. §15).

As we said before, R. Bennett [33] proved that a first-countable connected CDH-space is homogeneous. Much more is known today.

A space X is *n-homogeneous*, where $n \geq 1$, if for all n-point subsets F and G of X, there is a homeomorphism $f\colon X \to X$ such that $f(F) = G$. Moreover, X is *strongly n-homogeneous*, where $n \geq 1$, if given any two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of distinct points of X, there exists a homeomorphism g of X such that $g(x_i) = y_i$ for every $i \leq n$.

The basic tool for obtaining the new general homogeneity results in countable dense homogeneity seems to be [127, Proposition 3.1] of which we include the simple proof for the sake of completeness. The intuitive idea is that a CDH-space must have 'many' homeomorphisms in order to deal with all countable dense sets, and that these homeomorphisms imply more structure than one would expect at first glance.

Proposition 14.6. *Let X be CDH. If $F \subset X$ is finite and $D, E \subset X \setminus F$ are countable and dense in X, then there is a homeomorphism $f\colon X \to X$ such that $f(D) \subset E$ and f restricts to the identity on F.*

Proof. Let h_0 be an arbitrary homeomorphism of X, for example the identity function. Suppose $\{h_\beta : \beta < \alpha\} \subset \mathscr{H}(X)$ have been constructed for some $\alpha < \omega_1$. Now by CDH, pick $h_\alpha \in \mathscr{H}(X)$ such that

$$h_\alpha(F \cup E) = \bigcup_{\beta<\alpha} h_\beta(D). \tag{†}$$

For $1 \le \alpha < \omega_1$, let T_α be a nonempty finite subset of $[1,\alpha)$ such that $h_\alpha(F) \subset \bigcup_{\beta\in T_\alpha} h_\beta(D)$. By the Pressing Down Lemma, for the function $T\colon [1,\omega_1) \to [\omega_1]^{<\omega}$ defined by $T(\alpha) = T_\alpha$, the fiber $B = T^{-1}(A)$ is uncountable for some $A \in [\omega_1]^{<\omega}$. Then $h_\alpha(F) \subset \bigcup_{\beta\in A} h_\beta(D)$ for every $\alpha \in B$. Since $\bigcup_{\beta\in A} h_\beta(D)$ is countable, and B is uncountable, we may consequently assume without loss of generality that $h_\alpha{\restriction}F = h_\beta{\restriction}F$ for all $\alpha, \beta \in B$. Hence if $\alpha, \beta \in B$ are such that $\beta < \alpha$, then $h_\alpha{\restriction}F = h_\beta{\restriction}F$ and by (†), $(h_\alpha^{-1} \circ h_\beta)(D) \subset E$. □

This leads to the following results.

Theorem 14.7 ([128]). *Let X be a non-trivial connected* CDH*-space. Then X is n-homogeneous for every n.*

It was asked in Problem 136 of Watson [171] in the Open Problems in Topology Book whether every connected CDH-space is strongly 2-homogeneous. Observe that the real line $\mathbb{R}$ is an example of a space that is CDH but not strongly 3-homogeneous. It was shown in [128] that every connected CDH-space is strongly 2-homogeneous provided it is locally connected. Moreover, an example is presented there of a connected Lindelöf CDH-space that is not strongly 2-homogeneous. We will sketch this example.

The example is of course Tychonoff, but not metrizable. It may be possible to construct a separable metrizable space with similar properties using the methods of Saltsman [149, 150]. However, his methods need the Continuum Hypothesis, while our result requires no additional set theoretic assumptions.

As usual, λ denotes Lebesgue measure on $\mathbb{R}$. For every $x \in \mathbb{R}$ we will define a certain collection of subsets $\mathscr{F}_x$ of $(\leftarrow, x)$, as follows: $F \in \mathscr{F}_x$ iff F is closed in $(-\infty, x)$, and

$$\sum_{n=0}^{\infty} 2^n \lambda([x-2^{-n}, x-2^{-n-1}] \cap F) < \infty.$$

Observe that $\mathscr{F}_x$ is closed under finite unions and contains all closed subsets of $(-\infty, x)$ of measure 0. Topologize $\mathbb{R}$ as follows: a basic neighbourhood of $x \in \mathbb{R}$ has the form $U \setminus F$, where U is an open subset of $\mathbb{R}$ containing x, and $F \in \mathscr{F}_x$. Let $\mathscr{B}(x)$ denote all sets of this form. Then $\mathscr{B}(x)$ is a neighbourhood system for x, and the space with the topology τ generated by these neighbourhood systems will be denoted by X. Clearly, τ is stronger than the euclidean topology on $\mathbb{R}$.

Lemma 14.8. *X is regular and Lindelöf.*

Proof. Let $x \in X$, $U \subset \mathbb{R}$ open such that $x \in U$, and $F \in \mathscr{F}_x$. There is an open neighbourhood A of F in $(-\infty, x)$ such that the closure G of A in $(-\infty, x)$ belongs to $\mathscr{F}_x$. Let V be an open neighbourhood of x in $\mathbb{R}$ such that $\overline{V} \subset U$, and consider $W = V \setminus G$. We claim that the closure of W in X is contained in $U \setminus F$. To check this, let p be an arbitrary element of that closure. Clearly, $p \in \overline{V} \subset U$. Assume that $p \in F$. Then A is a neighbourhood of p in $\mathbb{R}$ and hence in X which misses W which is a contradiction. Hence $p \in U \setminus F$. Hence X is regular.

To see that X is Lindelöf, it suffices to observe that the topology on X is weaker than the Sorgenfrey topology on $\mathbb{R}$, which is Lindelöf ([65, 3.8.14]). $\square$

We conclude from this that X is normal, and hence Tychonoff ([65, 3.8.2]).

Lemma 14.9. *X is* CDH.

Proof. Let D and E be any two countable dense subsets of X. Then D and E are countable dense subsets of $\mathbb{R}$, and hence by Zamora Avilés [172] (see also [54]), there is a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ having the following properties:

1. $f(D) = E$,
2. for all distinct $x, y \in \mathbb{R}$, $1/2 \leq \frac{|f(x)-f(y)|}{|x-y|} \leq 2$.

This implies that for every measurable subset S of $\mathbb{R}$ we have that

$$1/2\lambda(S) \leq \lambda(f(S)) \leq 2\lambda(S).$$

Hence $f\colon X \to X$ is a homeomorphism as well since it maps for every $x \in X$ every element of $\mathscr{F}_x$ onto an element of $\mathscr{F}_{f(x)}$, etc. $\square$

Lemma 14.10. *X is connected but not strongly 2-homogeneous.*

Proof. The proof that X is connected follows the same pattern as the standard proof that $\mathbb{R}$ is connected, but is slightly more complicated.

Since the identity $X \to \mathbb{R}$ is a bijection, this implies that X and $\mathbb{R}$ have the same connected sets by Kok [99, Theorem 3 on page 5].

It will be convenient for every $x \in \mathbb{R}$ to denote $(\leftarrow, x)$ and $(x, \rightarrow)$ by L_x and R_x, respectively.

Take $p, q \in \mathbb{R}$ such that $p < q$. We claim that there does not exist a homeomorphism $f\colon X \to X$ such that $f(p) = q$ and $f(q) = p$. Striving for a contradiction, assume that such a homeomorphism f exists. Since X and $\mathbb{R}$ have the same connected sets, a moments reflection shows that $f(R_q \cup \{q\}) = L_p \cup \{p\}$. There is a sequence $(q_n)_n$ in R_q such that $q_n \to q$. Hence $(f(q_n))_n$ is a sequence in L_p such that $f(q_n) \to p$. But this is clearly impossible since no sequence in L_p converges to p, being of measure 0. $\square$

Problem 14.11. Is there a separable and metrizable connected space X which is CDH but not strongly 2-homogeneous?

The question is natural whether the homogeneity notions considered here are actually equivalent to CDH-ness for certain classes of spaces. For locally compact spaces of countable weight, this is indeed the case, as the following elegant result shows.

Theorem 14.12 (Ungar [165, 166]). *Let X be a locally compact separable metrizable space such that no finite set separates X. Then the following statements are equivalent:*

1. *X is* CDH.
2. *X is n-homogeneous for every n.*
3. *X is strongly n-homogeneous for every n.*

Let us comment on Ungar's proof. First of all, the equivalence $2 \Leftrightarrow 3$ follows from Corollary 3.10 in his earlier paper [165]. The assumption there on local connectivity is superfluous since all one needs for the proof is the existence of a Polish group which makes the space under consideration n-homogeneous for all n. That no finite set separates X is essential for (2) $\Leftrightarrow$ (3) as $\mathbb{S}^1$ demonstrates. Ungar's proofs of the implications $1 \Rightarrow 3$ and $3 \Rightarrow 1$ were both based (among other things) on the celebrated Open Mapping Principle 13.6. In the proof of $3 \Rightarrow 1$ the Open Mapping Principle controls the inductive process, and in the proof of $1 \Rightarrow 3$ it allows the use of the Baire Category Theorem. Moreover, $3 \Rightarrow 1$ is true for all locally compact spaces, additional connectivity assumptions are not needed for the proof, and $\mathbb{S}^1$ again demonstrates that $1 \Rightarrow 3$ is false without them.

The question whether in Ungar's Theorem 14.12, the assumption on local compactness can be relaxed to that of completeness, is natural.

Theorem 14.13 ([127]). *There are a Polish space $\mathscr{X}$ and a (separable metrizable) topological group (G, τ) such that*

1. *(G, τ) acts on $\mathscr{X}$ by a continuous action, and makes $\mathscr{X}$ strongly n-homogeneous for every n,*
2. *$\mathscr{X}$ is not* CDH.

By the transitivity of the action, this example is not the same as the one described in Corollary 13.15, the tricky subspace of the product of the Cantor set and the unit interval. But it is a variation of it. The group (G, τ) cannot be chosen to be complete, and $\mathscr{X}$ is totally disconnected and 1-dimensional. So its components are points, so its pathology is therefore not based upon connectivity. The space $\mathscr{X}$ is a tricky subspace of the product of the Cantor set and the so-called *complete Erdős space* $\mathfrak{E}_{\mathrm{c}}$.

In 1940 Erdős proved that the 'rational Hilbert space' space $\mathfrak{E}$, which consists of all vectors in the real Hilbert space ℓ^2 that have only rational coordinates, has dimension one, is totally disconnected, and is homeomorphic to its own square. This answered a question of Hurewicz [88] who proved that for every compact space X and every 1-dimensional space Y we have that $\dim(X \times Y) = \dim X + 1$.

It is not difficult to prove that $\mathfrak{E}$ has dimension at most 1. Erdős proved the surprising fact that every nonempty clopen subset of $\mathfrak{E}$ is unbounded, and hence that for no $x \in \mathfrak{E}$ and no $t > 0$ the open ball $\{y \in \mathfrak{E} : \|x - y\| < t\}$ contains a nonempty clopen subset of $\mathfrak{E}$. This implies among other things that $\mathfrak{E}$ is nowhere zero-dimensional. This is the crucial property that makes the Erdős spaces so interesting. Erdős also proved that the closed subspace $\mathfrak{E}_c$ of ℓ^2 consisting of all vectors such that every coordinate is in the convergent sequence $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ has the same property. The space $\mathfrak{E}_c$ is called *complete Erdős space* and was shown by Dijkstra [53] to be homeomorphic to the 'irrational' Hilbert space, which consists of all vectors in the real Hilbert space ℓ^2 that have only irrational coordinates. All nonempty clopen subsets of $\mathfrak{E}_c$ are unbounded just as the nonempty clopen subsets of $\mathfrak{E}$ are.

The space $\mathfrak{E}_c$ surfaces at many places. For example, as the set of endpoints of certain dendroids (among them, the Lelek fan), the set of endpoints of the Julia set of the exponential map, the set of endpoints of the separable universal $\mathbb{R}$-tree, line-free groups in Banach spaces and Polishable ideals on $\mathbb{N}$. Metric and topological characterizations of $\mathfrak{E}_c$ were proved by Kawamura, Tymchatyn, Oversteegen [95], Dijkstra and van Mill [55]. Moreover, Dijkstra and van Mill [56] topologically characterized $\mathfrak{E}$ and proved that if M is a topological n-manifold for $n \geq 2$, then the homeomorphism group of all homeomorphisms of M that fixes a given countable dense set is homeomorphic to $\mathfrak{E}$.

15 Countable dense homogeneous spaces and set theory

Countable dense homogeneity was studied mainly within the class of separable metrizable spaces. Actually, except for some general homogeneity results, there are very few results known for countable dense homogeneous spaces that are Tychonoff but not metrizable. Steprāns and Zhou [160] proved that every separable manifold of weight less than $\mathfrak{b}$ is CDH, and also that the Cantor cube 2^{ω_1} is CDH under $\mathsf{MA}+\neg\mathsf{CH}$. (For information on small uncountable cardinals, see [60]). In fact, Hrušák and Zamora Avilés [86] show that the least κ such that 2^κ is not CDH is exactly the pseudointersection number $\mathfrak{p}$. But to obtain interesting examples from these results, one needs additional set theoretic assumptions.

That CDH-spaces do not seem to be free from set theory, was also demonstrated by Baldwin and Beaudoin [27]. Call a topological space Y *compressed* if every non-empty open subset of Y has the same cardinality as Y. If X is a space and $x \in X$, then x is called a *point of compression of* X if it has a compressed neighbourhood. Finally, X is *locally compressed* if every point of X is a point of compression.

Theorem 15.1 (Baldwin and Beaudoin [27]). *Assume Martin's Axiom for σ-centered posets, and let X be a space of countable weight. If X has size less than 2^ω, then X is* CDH *if and only if X is locally compressed and every open subset of X is uncountable.*

They asked whether there is a CDH-subset of $\mathbb{R}$ of size $\aleph_1$ in ZFC. As we mentioned earlier, this question was answered in the affirmative in Farah, Hrušák and Martínez Ranero [67]. The following question in [27] still seems to be open.

Problem 15.2. Can one prove from ZFC the existence of a CDH Bernstein subspace of $\mathbb{R}$?

In [22], the authors addressed a very basic problem: what is the cardinality of a CDH-space?

Let us call a space X a c_1*-space* if the following condition (c_1) is satisfied: (c_1) *X is separable, and every two countable dense subspaces of X are homeomorphic.*

Sierpiński [157] proved that the space of rational numbers $\mathbb{Q}$ is topologically the unique nonempty regular countable and first-countable space without isolated points. Hence every compact space with countable weight is a c_1-space. This means that there are many compact spaces that are c_1 but not CDH, for example the product of the Cantor set and the circle, which is even a topological group. There does not exist a compact zero-dimensional space of countable weight which is c_1 but not CDH. This follows easily from the topological characterization of the Cantor set due to Brouwer [39].

In [22], the authors proved:

Theorem 15.3.

1. *If X is a c_1-space, then $|X| \le 2^\omega$.*
2. *Suppose that X is a c_1-space which contains a dense first-countable subspace E. Moreover, assume that X is regular. Then X is first-countable.*

Corollary 15.4 ($2^\omega < 2^{\omega_1}$). *Every c_1-compactum is first-countable.*

By Steprāns and Zhou [160], the Cantor cube 2^{ω_1} is a CDH-space under $\mathsf{MA}_{\aleph_1}$. Hence this corollary cannot be proved in ZFC. But it can be considerably generalized, with essentially the same proof, as follows. Recall that a space X is of *pointwise countable type* if every $x \in X$ is contained in a compact subspace F of X with a countable base of open neighbourhoods in X.

Corollary 15.5 ($2^\omega < 2^{\omega_1}$). *Every c_1-space of pointwise countable type is first-countable.*

This statement covers locally compact c_1-spaces, Čech-complete c_1-spaces, and even c_1-spaces that are p-spaces.

The following fundamental problem remains open.

Problem 15.6. Does there exist in ZFC a compact CDH-space that is not metrizable?

The first natural counterexample that comes to mind is the Alexandroff-Urysohn double arrow space. But it was shown in [22] that this space is not CDH. (This was recently generalized in Hernández-Gutiérrez [82] who proved that if $\mathbb{A}$ denotes the Alexandroff-Urysohn double arrow space, then $\mathbb{A} \times 2^\omega$ and $\mathbb{A}^\omega$ are not CDH, while $\mathbb{A}$ has exactly 2^ω types of countable dense sets). That this is really a question in ZFC was also shown in [22] by the construction of a compact CDH-space X of uncountable weight under CH. In fact, X is both hereditarily separable and hereditarily Lindelöf.

Problem 15.7. Let X be a compact space such that $X \times X$ is CDH. Does it follow under CH that X has countable weight?

16 Spaces with few countable dense sets

Let X be a space, and let $\alpha \geq 1$ be a cardinal number not exceeding 2^ω. We say that a space X has α *types of countable dense sets* provided that α is the least cardinal for which there is a collection $\mathscr{A}$ of countable dense sets such that $|\mathscr{A}| \leq \alpha$ while moreover for any given countable dense set B of X there exist $A \in \mathscr{A}$ and a homeomorphism $f\colon X \to X$ such that $f(A) = B$. The topological sum of n copies of $[0,1)$ is ${}^1\!/_{n+1}$-CDH. Moreover, the topological sum of countably many copies of $[0,1)$ has countably many types of countable dense sets and the pseudoarc is an example of a homogeneous continuum that has 2^ω types of countable dense sets, which is the maximum number possible. Hrušák and van Mill [87], generalized Theorem 14.2 as follows:

Theorem 16.1. *If X is Borel and has fewer than 2^ω types of countable dense sets, then X is Polish.*

They also proved the following structure theorem, the proof of which is based on the Effros Theorem from [63] as well as Ungar's [165, 166] analysis of various homogeneity notions.

Theorem 16.2. *Let X be a locally compact, separable metrizable and dense-in-itself space. Assume that X has at most countably many types of countable dense sets. Then there is a closed and scattered subset S of X of finite Cantor-Bendixson rank which is invariant under all homeomorphisms of X while moreover $X \setminus S$ is* CDH. *Moreover, $|S| \leq n-1$ if X has at most n types of countable dense sets.*

So the topological sum of copies of $[0,1)$ are typical examples of such spaces.

The question remains of whether there can be a Polish (or locally compact) space X which has α types of countable dense sets for some cardinal α such that $\omega_1 \leq \alpha < 2^\omega$. This does not seem to be a simple problem, since, this was shown by Hrušák and van Mill [87] to be related to the well-known

Topological Vaught Conjecture which says that if G is any Polish group then any Polish G-space either has countably many orbits or has perfectly many orbits. For details, see Becker and Kechris [29].

17 Unique homogeneity

A space X is called *uniquely homogeneous* provided that for all $x, y \in X$ there is a unique homeomorphism of X that takes x onto y. This concept is due to Burgess [40] who asked in 1955 whether there exists a uniquely homogeneous metrizable continuum.

Theorem 17.1 (Barit and Renaud [28]). *Let X be a locally compact space of countable weight. If X is uniquely homogeneous, then $|X| \leq 2$.*

Proof. Let G be the Polish group acting transitively on X that was constructed in the proof of Theorem 13.8. For a given $x \in X$, the evaluation mapping $\gamma_x \colon G \to X$ is a continuous bijection, since X is uniquely homogeneous, and is open by the Open Mapping Principle 13.6. Hence G and X are homeomorphic. Since all inner isomorphisms of G are trivial, G is Abelian. Moreover, the function $x \mapsto -x$ is trivial as well. Hence we conclude that all elements of G have order 2. Since G is torsion and locally compact, it contains a compact open subgroup H. This subgroup is topologically isomorphic to $\{0\}$ or to D^κ, for certain $\kappa \leq \omega$. Unique homogeneity easily implies that in the second case $\kappa \leq 1$. □

This is a very interesting result, and immediately leads us to the following basic problem of which the solution currently seems beyond reach.

Problem 17.2. Is there a compact uniquely homogeneous space?

Before thinking about this, one should of course know whether the class of uniquely homogeneous spaces contains something of interest.

(A) A non-trivial uniquely homogeneous space that is a topological group. Let G be a Boolean topological group. We call G *rigid* if every continuous $f \colon G \to G$ is either constant or a translation. Observe that if G is rigid, then G is connected, the group of isomorphisms of G is trivial (which explains the terminology), and G is uniquely homogeneous. Indeed, for $x, y \in X$ the translation $p \mapsto p - x + y$ is the unique homeomorphism of G which moves x to y.

Theorem 17.3. *Let G be a Polish group which is Boolean and has the property that any pair of points of G is contained in a disc D in G. Then G contains a dense rigid subgroup H. Suppose that G has the additional property that for all open $U \subset G$ and $x \in U$, there is an open $V \subset U$ such that $x \in V \subset U$ and any pair of points in V is contained in a disc D in U. Then H can be chosen to be locally connected.*

This result is due to van Mill [112]. The same technique can be used to prove the existence of a uniquely homogeneous space that does not admit the structure of a topological group [114] and an infinite-dimensional linear subspace L of ℓ^2 such that L and $L \times \mathbb{R}$ are not homeomorphic [115].

In the remainder of this subsection, we will present the proof of this result. We will use the so-called 'technique of killing homeomorphisms' of Sierpiński [158]. It was widely used by various authors for various purposes. For example, Sierpiński [158] and Kuratowski [103] used it for the construction of a rigid subspace of $\mathbb{R}$ (thereby implicitly constructing the first example of a rigid Boolean Algebra since if $X \subset \mathbb{R}$ is rigid, then so is its Čech-Stone compactification βX), van Douwen [59] used it for the construction of a compact space with a measure that 'knows' which sets are homeomorphic, Shelah [156] and van Engelen [64] used it to prove that $\mathbb{R}$ can be partitioned into two homeomorphic rigid sets, Todorčević [161] used it for the construction of various interesting examples on cardinal functions, Marciszewski [107] used it for the construction of a compact space K such that the Banach space $C(K)$ is not weakly homeomorphic to its own square, Keesling and Wilson [97] used it to construct an 'almost uniquely homogeneous' subgroup of $\mathbb{R}^n$, and van Mill [112, 115, 117] used it for the construction of various examples. This list of the use of Sierpiński's technique is highly incomplete.

Throughout, let (G, ϱ) be a Polish Boolean group. If $x \in G$ and $\varepsilon > 0$, then $B(x, \varepsilon) = \{y \in G : \varrho(x, y) < \varepsilon\}$. We say that a subset A of G is *(algebraically) independent* if for all $a_1, \ldots, a_n \in A$ such that $a_i \neq a_j$ if $i \neq j$ we have $\sum_{i=1}^{n} a_i \neq e$ (this corresponds of course to the usual notion of linear independence in a vector space).

Lemma 17.4. *If $\{a_1, \ldots, a_n\}$ is independent in the Boolean topological group G then there exists $\varepsilon > 0$ such that if $y_i \in B(x_i, \varepsilon)$ for every $i \leq n$ then $\{y_1, \ldots, y_n\}$ is independent.*

Proof. This is a simple consequence of the continuity of the algebraic operations on G. $\square$

Let $\mathscr{K}(G)$ denote the collection of all homeomorphisms $h: K_1 \to K_2$ between disjoint Cantor sets in G such that $K_1 \cup K_2$ is independent.

Theorem 17.5. *There is a subgroup H of G with the following property: for each $h \in \mathscr{K}(G)$ there exists $x \in \operatorname{dom}(h)$ such that $x \in H$ but $h(x) \notin H$.*

Proof. It is clear that the collection of Cantor subsets of G is of cardinality at most 2^ω. Moreover, if K and L are Cantor sets then the collection of all homeomorphisms $K \to L$ has size 2^ω. From this we see that $|\mathscr{K}(G)| \leq 2^\omega$. List $\mathscr{K}(G)$ as $\{g_\alpha : \alpha < 2^\omega\}$ (repetitions permitted).

We show by transfinite induction that for all $\alpha < 2^\omega$, there exist subgroups Y_α and Z_α in G such that

1. $Y_\alpha \cap Z_\alpha = \{e\}$,

2. for $\beta < \alpha$, $Y_\beta \subset Y_\alpha$ and $Z_\beta \subset Z_\alpha$,
3. $|Y_\alpha|, |Z_\alpha| \le \omega{\cdot}|\alpha|$,
4. there exists $x \in \operatorname{dom}(h_\alpha)$ such that $x \in Y_\alpha$ and $h_\alpha(x) \in Z_\alpha$.

Then $H = \bigcup_{\alpha<2^\omega} Y_\alpha$ is the required subgroup of G.

Suppose that we completed the construction for all $\beta < \alpha$. Let $Y^\alpha = \bigcup_{\beta<\alpha} Y_\beta$ and $Z^\alpha = \bigcup_{\beta<\alpha} Z_\beta$, respectively. Then, clearly, $|Y^\alpha|, |Z^\alpha| \le \omega{\cdot}|\alpha|$, and $Y^\alpha \cap Z^\alpha = \{e\}$. Put

$$S = \big\{x \in \operatorname{dom}(h_\alpha) : \langle\!\langle \{x\} \cup Y^\alpha \rangle\!\rangle \cap \langle\!\langle \{h_\alpha(x)\} \cup Z^\alpha \rangle\!\rangle = \{e\}\big\}.$$

We claim that $|S| = 2^\omega$. Indeed, let $x \in \operatorname{dom}(h_\alpha) \setminus S$. Observe that

$$\langle\!\langle \{x\} \cup Y^\alpha \rangle\!\rangle = (x + Y^\alpha) \cup Y^\alpha,\ \ \langle\!\langle \{h_\alpha(x)\} \cup Z^\alpha \rangle\!\rangle = (h_\alpha(x) + Z^\alpha) \cup Z^\alpha.$$

The are several subcases. Assume first that for some $y \in Y^\alpha$ and $z \in Z^\alpha$ we have $x + y = h_\alpha(x) + z$. As a consequence,

$$x + h_\alpha(x) = y + z \in Y^\alpha + Z^\alpha.$$

Since $|Y^\alpha + Z^\alpha| < 2^\omega$, the independence of $\operatorname{dom}(h_\alpha) \cup \operatorname{range}(h_\alpha)$ implies that there are only fewer than 2^ω such x's. Assume next that for some $y \in Y^\alpha$ and $z \in Z^\alpha$ we have $x + y = z$. But then $x \in Y^\alpha + Z^\alpha$. Since $|Y^\alpha + Z^\alpha| < 2^\omega$, there are again only fewer than 2^ω such x's. Since h_α is one-to-one, a similar reasoning can be used for the situation that for some $y \in Y^\alpha$ and $z \in Z^\alpha$ we have $y = h_\alpha(x) + z$. Since $Y^\alpha \cap Z^\alpha = \{e\}$, this exhausts all possible cases.

So $|S| = 2^\omega$ since $|\operatorname{dom}(h_\alpha)| = 2^\omega$. Pick an arbitrary $x \in S$, and put $Y_\alpha = \langle\!\langle \{x\} \cup Y^\alpha \rangle\!\rangle$, and $Z_\alpha = \langle\!\langle \{h_\alpha(x_\alpha)\} \cup Z^\alpha \rangle\!\rangle$, respectively. Then Y_α and Z_α are clearly as required. □

Let $g\colon A \to G$ be a function defined on a subset of G. A subset P of A is said to be *g-independent* if the following conditions are satisfied:

1. $g{\restriction}P$ is injective,
2. $P \cap g(P) = \varnothing$,
3. $P \cup g(P)$ is independent.

Lemma 17.6. *Let $g\colon A \to G$ be continuous such that $A \subset G$ is Polish. If A contains an uncountable g-independent set, then A contains a g-independent Cantor set.*

Proof. Let d be an admissible complete metric on A. For each $x \in A$ and $\varepsilon > 0$, let $\hat{B}(x,\varepsilon) = \{a \in A : d(a,x) \le \varepsilon\}$. Since each space is the union of a countable set and a dense-in-itself set (this is the so-called Cantor-Bendixson Theorem, see [65, 1.7.11] for details), the hypothesis implies that A contains a dense-in-itself g-independent set P. Using finite disjoint unions of balls about points of P, we may construct a Cantor set K in the Polish space A by the standard procedure; a little extra care will ensure that K is g-independent. It suffices to describe the first two steps in the inductive construction.

Pick any $p_1 \in P$. Since $g(p_1) \neq p_1$, there exists $0 < \varepsilon_1 < 1$ such that $\hat{B}(p_1, \varepsilon) \cap g(\hat{B}(p_1, \varepsilon_1)) = \varnothing$. Let $B_1 = \hat{B}(p_1, \varepsilon_1)$. Since the set $\{p_1, g(p_1)\}$ is independent, we may assume by Lemma 17.4 that ε_1 is sufficiently small so that, for any $F \subset B_1 \cup g(B_1)$ such that F contains at most a single point from each of B_1 and $g(B_1)$, F is independent. Let $K_1 = B_1$.

Since P is dense-in-itself, there exist distinct points $p_{1,0}$ and $p_{1,1}$ in $P \cap B_1$. Choose $0 < \varepsilon_2 < 1/2$ such that, for $B_{1,0} = \hat{B}(p_{1,0}, \varepsilon_2)$ and $B_{1,1} = \hat{B}(p_{1,1}, \varepsilon_2)$, we have $B_{1,0} \cup B_{1,1} \subset B_1$, $B_{1,0} \cap B_{1,1} = \varnothing$ and $g(B_{1,0}) \cap g(B_{1,1}) = \varnothing$. Since the set $\{p_{1,0}, p_{1,1}, g(p_{1,0}), g(p_{1,1})\}$ is independent and has size 4, we may also assume that ε_2 is small enough so that for any $F \subset B_{1,0} \cup B_{1,1} \cup g(B_{1,0}) \cup g(B_{1,1})$ such that F contains at most a single point from each of $B_{1,0}, B_{1,1}, g(B_{1,0})$ and $g(B_{1,1})$, F is independent. Set $K_2 = B_{1,0} \cup B_{1,1}$.

Continuing with this procedure in the standard manner, we obtain a nested sequence $(K_n)_n$ of closed sets in A. Let $K = \bigcap_{n=1}^{\infty} K_n$. The requirements of the type $B_{1,0} \cup B_{1,1} \subset B_1$ and $B_{1,0} \cap B_{1,1} = \varnothing$, together with the requirement that $\varepsilon_n \to 0$ and the fact that ϱ is a complete metric, show K is a Cantor set. The requirements of the type $g(B_{1,0}) \cap g(B_{1,1}) = \varnothing$ show that $g \restriction K$ is injective. Since $g(K) \subset g(B_1)$ and $K \subset B_1$, $K \cap g(K) = \varnothing$. And finally, the independence requirement at the nth-stage of the construction ensures that each finite subset of $K \cup g(K)$ is independent, hence $K \cup g(K)$ is independent. Thus K is g-independent. $\square$

The same proof yields:

Corollary 17.7. *Let $A \subset G$ be analytic and uncountable. Then then A contains an independent Cantor set.*

Let G be a topological group. We say that a function f such that $\mathrm{dom}(f)$ and $\mathrm{range}(f)$ are contained in G has *countable type* provided that there is a countable set $Z \subset G$ such that

$$f(x) \in \langle\!\langle \{x\} \cup Z \rangle\!\rangle, \qquad x \in \mathrm{dom}(f).$$

This is equivalent to the following statement: there is a countable subgroup A of G such that for every $x \in \mathrm{dom}(f)$, $f(x) \in A \cup (\{x\} + A)$.

Proposition 17.8. *Let G be a Boolean topological group. A function f such that* $\mathrm{dom}(f)$ *and* $\mathrm{range}(f)$ *are subsets of G has countable type if and only if every f-independent set is countable.*

Proof. Suppose first that f has countable type; let Z be a countable subset of G such that $f(y) \in \langle\!\langle \{y\} \cup Z \rangle\!\rangle$ for each y. Put $T = \langle\!\langle Z \rangle\!\rangle$. Striving for a contradiction, assume that there is an uncountable f-independent $B \subset \mathrm{dom}(f)$. Since $f \restriction B$ is one-to-one and T is countable, we may assume without loss of generality assume that $f(B) \cap T = \varnothing$. Observe that if $x \in G$ is arbitrary, then

$$f(x) \in \langle\!\langle \{x\} \cup Z \rangle\!\rangle = T \cup (\{x\} + T).$$

Hence since $f(B) \cap T = \varnothing$, for $b \in B$ we may pick $t_b \in T$ such that $f(b) = b + t_b$. Since B is uncountable, and T is countable, there are distinct $b, b' \in B$ such that $t_b = t_{b'}$. But then

$$b + b' + f(b) + f(b') = b + b' + b + t_b + b' + t_{b'} = t_b + t_b = e$$

contradicts the fact that $\{b, b', f(b), f(b')\}$ is independent.

Conversely, assume that every f-independent set is countable. It is easily seen that in the collection of f-independent sets, partially ordered by inclusion, every chain has an upper bound. Thus there exists a maximal f-independent set Q, which by hypothesis is countable. If $Q = \varnothing$ then $f(x) \in \{e, x\}$ for every $x \in \mathrm{dom}(f)$ and f obviously has countable type. Otherwise, put

$$A = \langle\!\langle Q \cup f(Q) \rangle\!\rangle, \quad Z = A + \langle\!\langle f(A \cap \mathrm{dom}(f)) \rangle\!\rangle.$$

Then Z is countable, and we claim that $f(x) \in \langle\!\langle \{x\} \cup Z \rangle\!\rangle$ for every $x \in \mathrm{dom}(f)$. This is clear for $x \in Q$, so take $x \in \mathrm{dom}(f) \setminus Q$. The set $Q \cup \{x\}$ is not f-independent, and one of the following occurs:

1. $f(x) = f(q)$ for some $q \in Q$. Then $f(x) \in Z$.
2. $x = f(q)$ for some $q \in Q$. Then $x \in f[Q] \cap \mathrm{dom}(f) \subset A \cap \mathrm{dom}(f)$, hence $f(x) \in Z$.
3. $\{x, f(x)\} \cup Q \cup f[Q]$ is not independent. In this case either

$$x \in \langle\!\langle Q \cup f(Q) \rangle\!\rangle \cap \mathrm{dom}(f) = A \cap \mathrm{dom}(f),$$

hence $f(x) \in Z$, or

$$f(x) \in \langle\!\langle \{x\} \cup Q \cup f(Q) \rangle\!\rangle \subset \langle\!\langle \{x\} \cup Z \rangle\!\rangle.$$

Thus f has countable type. □

Theorem 17.9. *Let H be the subgroup of G we get from Theorem 17.5. In addition, let $f: H \to H$ be continuous, and let S be a G_δ-subset of G containing H such that f can be extended to a continuous function $\bar{f}: S \to G$. Then*

1. *H intersects every Cantor set in G (hence $G \setminus S$ is countable, H is dense in G, H is locally uncountable and a Baire space),*
2. *$\bar{f}$ has countable type.*

Proof. Let K be a Cantor set in G. We may assume by Corollary 17.7 that K is independent. There are disjoint Cantor sets K_1 and K_2 in K. Let $g: K_1 \to K_2$ be any homeomorphism. Then $g \in \mathscr{K}(G)$ and so by construction, $H \cap \mathrm{dom}(g) = H \cap K_1 \neq \varnothing$.

To prove that $G \setminus S$ is countable, write $G \setminus S$ as $\bigcup_{n=1}^{\infty} F_n$, where each F_n is closed. If some F_n is uncountable then it contains a Cantor set by

Theorem 17.7 and hence it intersects H. So every F_n is countable, i.e., $S \setminus G$ is countable.

To prove that H is dense and locally uncountable, observe that every nonempty open subset of G contains a Cantor set and consequently intersects H. In fact, every Cantor set can be split in an uncountable disjoint collection Cantor sets which proves that H intersects every Cantor set in an uncountable set. Hence G is locally uncountable. Since every dense G_δ-subset of G is uncountable, the same proof shows that H intersects every such set, i.e., H is a Baire space.

It remains to prove that $\bar{f}: S \to G$ has countable type. Suppose that S contains an uncountable $\bar{f}$-independent set. Then by Lemma 17.6, S contains a $\bar{f}$-independent Cantor set K. Then $\bar{f} \restriction K$ belongs to $\mathscr{K}(G)$, and so by hypothesis there exists $x \in K \cap H$ such that $\bar{f}(x) \notin H$. But this contradicts the fact that $\bar{f}$ extends f since $\bar{f}(x) = f(x) \in H$. Thus, every f-independent subset of H is countable, i.e., f has countable type by Proposition 17.8. $\square$

Proof (of Theorem 17.3). Let H be the subgroup of G we get from Theorem 17.9. Suppose that $f: H \to H$ is continuous. Since G is Polish, there is a G_δ-subset S of G containing H such that f can be extended to a continuous function $\bar{f}: S \to G$, [65, 4.3.21]. There is a countable subgroup A of G such that $\bar{f}(x) \in \langle\!\langle \{x\} \cup A \rangle\!\rangle$ for every $x \in S$.

For each $a \in A$, put

$$S_a = \{x \in S : \bar{f}(x) = x + a\}, \quad T_a = \{x \in S : \bar{f}(x) = a\}.$$

Then each S_a and T_a is closed in S, and the collections $\{S_a : a \in A\}$ and $\{T_a : a \in A\}$ are evidently pairwise disjoint. Suppose that for a and a' in A we have $S_a \cap T_{a'} \neq \varnothing$. Pick $x \in S_a \cap T_{a'}$. Then

$$a' = \bar{f}(x) = x + a,$$

i.e., $x \in A$. Put $D = A \cup (G \setminus S)$, and $Y = G \setminus D$. Then D is countable, and the collection

$$\mathscr{E} = \{S_a \cap Y : a \in A\} \cup \{T_a \cap Y : a \in A\}$$

is a closed partition of Y. We claim that at most one element of $\mathscr{E}$ is nonempty. Striving for a contradiction, assume that there exist distinct $E, E' \in \mathscr{E}$ and elements x and y in Y such that $x \in E$ and $y \in E'$. There is a disc D in G which contains both x and y. Since $G \setminus Y$ is countable, there is an arc J in Y which connects x and y and misses $G \setminus Y$, i.e., $J \subset Y$. So the collection

$$\{E \cap J : E \in \mathscr{E}\}$$

is a countable closed partition of J, and has at least two nonempty members. But this violates the Sierpiński Theorem, [65, 6.1.27].

Case 17.10. For some $a \in A$, $T_a \cap Y \neq \varnothing$.

Then $\bar{f}(x) = a$ for every $x \in Y$. Hence $F = \{x \in H : f(x) = a\}$ is closed in H, and $H \setminus F \subset A$. So $H \setminus F$ is empty, H being locally uncountable. We conclude that f is constant.

Case 17.11. For some $a \in A$, $S_a \cap Y \neq \varnothing$.

With the same technique as in Case 1, it follows easily that f is a translation.

So H is rigid, and hence is connected. The proof of the local connectivity of H if G satisfies the condition mentioned at the end of Theorem 17.3 is simple and is left to the reader. □

The question remains of course whether there is a topological group that satisfies the conditions in Theorem 17.3. There is by Bessaga and Pełczyński [35, VI, 7.2] a Boolean group structure on ℓ^2 which is compatible with its topology. Hence this group satisfies the conditions of Theorem 17.3, and so its subgroup H is rigid, Baire, connected and locally connected. From this we conclude that there is a connected and locally connected, Baire, uniquely homogeneous space of countable weight. In view of Theorem 17.1, the following problem is quite natural.

Problem 17.12. Is there a non-trivial Polish uniquely homogeneous space?

Several other examples of uniquely homogeneous spaces were constructed. In [114] there is an example of a uniquely homogeneous space that does not admit the structure of a topological group. Moreover, the authors have constructed in [23] a family $\mathscr{A}$ consisting of 2^{2^ω} uniquely homogeneous spaces of countable weight such that its product $\prod_{A \in \mathscr{A}} A$ is uniquely homogeneous. Hence there is a uniquely homogeneous space which contains a copy of the Cantor cube of weight 2^{2^ω}. This means that a space such as $\beta\omega$ can be embedded in a uniquely homogeneous space. This leads us to the following problems.

Problem 17.13. Are there uniquely homogeneous spaces of arbitrarily large weight? Can every compact space be embedded in a uniquely homogenous space?

There is a third example of a uniquely homogeneous space that was constructed by the authors in [21]. The reason why it was constructed will be explained in the next subsection.

(B) Recent results on unique homogeneity. Recent work on uniquely homogeneous spaces was done by the authors.

Theorem 17.14 ([23]). *Every infinite uniquely homogeneous space is connected. No infinite ordered space is uniquely homogeneous.*

A consequence of this is that no infinite uniquely homogeneous space is CDH. Simply observe that it is connected, and hence by Proposition 14.6 has many homeomorphisms that fix a given point but are not the identity homeomorphism.

Let X be uniquely homogeneous, and fix an element $e \in X$.
For every $x \in X$ let f_x be the unique homeomorphism taking e onto x.
Define a binary operation '$\cdot$' and an operation '$^{-1}$' on X by

$$x{\cdot}y = f_x(y), \qquad x^{-1} = f_x^{-1}(e).$$

It is easy to see that this makes X into a left topological group. That is, '$\cdot$' is a group operation on X, and all left translations of X are homeomorphisms. This is called the *standard group operation* on X. It is natural to ask whether this operation gives X the structure of a topological group. This is so if X is locally compact, separable and metrizable, Theorem [28], and for separable metrizable spaces this need not be true, [114]. What about the structure of a semitopological group? Or a *quasitopological group*? That is, a semitopological group such that the inverse operation is continuous. These questions will be considered now.

A space X is *2-flexible* if, for all $a, b \in X$ and open neighbourhood $O(b)$ of b, there is an open neighbourhood $O(a)$ of a such that, for any $z \in O(a)$, there is a homeomorphism h of X satisfying the following conditions: $h(a) = z$ and $h(b) \in O(b)$.

A space X will be called *Abelian* if all homeomorphisms of X commute pairwise.

A space X will be called *skew-2-flexible* if, for any a, b in X and any open neighbourhood $O(b)$ of b, there is an open neighbourhood $O(a)$ of a such that, for every $z \in O(a)$, there is a homeomorphism g of X satisfying the following conditions: $g(a) = z$ and $b \in g(O(b))$.

A space X will be called *Boolean* if every homeomorphism of X is an *involution*. That is, a homeomorphism f such that $f \circ f$ is the identity.

These notions are related as follows within the class of uniquely homogeneous spaces.

Theorem 17.15 ([23]). *Let X be a uniquely homogeneous space. Then the following statements are equivalent.*

1. *X is 2-flexible,*
2. *the standard group structure on X is semitopological,*
3. *X is homeomorphic to a semitopological group,*
4. *X is Abelian,*
5. *the standard group structure on X is semitopological and Abelian,*
6. *X is homeomorphic to an Abelian semitopological group.*

Theorem 17.16 ([23]). *Let X be a uniquely homogeneous space. Then the following statements are equivalent.*

1. X is skew-2-flexible,
2. X is 2-flexible and skew-2-flexible,
3. the standard group structure on X is quasitopological,
4. X is homeomorphic to a quasitopological group,
5. X is Boolean,
6. the standard group structure on X is quasitopological and Boolean,
7. X is homeomorphic to a Boolean quasitopological group.

Hence for uniquely homogeneous spaces, skew-2-flexibility implies 2-flexibility. The example in Corollary 13.15 is a homogeneous Polish space which is skew-2-flexible but not 2-flexible. There is also a uniquely homogeneous space which is Abelian but not Boolean, [21]. Hence there is a uniquely homogeneous space which is 2-flexible but not skew-2-flexible.

As we have seen in this section, there are quite a few very interesting open questions on unique homogeneity. Here are a few more such problems.

Problem 17.17. Is every first-countable uniquely homogeneous compactum X trivial? What if, in addition, we require X to be perfectly normal?

Problem 17.18. Is there a (Tychonoff) uniquely homogeneous space which is not homeomorphic to any semitopological group?

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