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Monotone partitions and almost partitions

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ABSTRACT

In this paper we are interested in monotone versions of partitionability of topological spaces and weak versions thereof. We identify several classes of spaces with these properties by constructing trees of open sets with various properties.

1. Introduction

Monotone versions of covering properties have been extensively studied the last decade. For details, see e.g., [2,4–8,12,14]. A space is *zero-dimensional* if it has a base consisting of open-and-closed (abbreviated clopen) sets. It is well-known, and easy to prove, that a zero-dimensional Lindelöf space X has the following covering property: every open cover $\mathscr{U}$ of X can be refined by a clopen partition $r(\mathscr{U})$ of X. In [11] Levy and Matveev introduce *monotone partionability* in zero-dimensional spaces in the following way: a zero dimensional space X is *monotonically partition-Lindelöf* (abbreviated mpL) if one can assign to every open cover $\mathscr{U}$ a countable partition $r(\mathscr{U})$ of X into clopen sets so that $r(\mathscr{U})$ refines $\mathscr{U}$, and $r(\mathscr{U})$ refines $r(\mathscr{V})$

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whenever $\mathscr{U}$ refines $\mathscr{V}$. In [11], it is shown that every zero-dimensional second countable space is mpL, and it is noted that the one-point Lindelöfication of the discrete space of cardinality ω_1, L_{ω_1}, is an example of a non-metrizable mpL-space. Also, it is proved that every countable mpL-space is second countable [11, Corollary 16].

In this paper we prove that in ZFC every Lindelöf P-spaces of weight ω_1 is mpL; this result was proved in [11] under CH for mL-spaces (see below) instead of mpL. Then we discuss several natural generalizations of monotone partition-Lindelöfness which brings spaces of countable π-weight, ordered ccc spaces and monotonically normal compact ccc spaces into the picture.

A space X is called *monotonically (weakly) Lindelöf* (abbreviated: m(w)L) if for every open cover $\mathscr{U}$ of X there is a countable open cover $r(\mathscr{U})$ of X (countable open collection $r(\mathscr{U})$ such that $\bigcup r(\mathscr{U})$ is dense) such that $r(\mathscr{U})$ refines $\mathscr{U}$, and if $\mathscr{U}$ refines the open cover $\mathscr{V}$ of X, then $r(\mathscr{U})$ refines $r(\mathscr{V})$.

It is clear that every mpL-space is mL. In [10], Levy and Matveev showed that no dense subspace of 2^{ω_1} is mL. Hence if X is any countable dense subspace of 2^{ω_1}, then X is an example of a countable zero-dimensional space that is not mpL. It is a more challenging question whether there exists an mL zero-dimensional space that is not mpL [11, Question 17]. We prove that a countable example of Levy and Matveev [10] which requires the Continuum Hypothesis, does the job.

We end with several open problems.

In this paper a family $\mathscr{U}$ of subsets of a space X is called *open* if every $U \in \mathscr{U}$ is open in X, a family of sets is called *cellular* if its elements are pairwise disjoint and all given spaces are assumed to be regular.

2. Monotone partition-Lindelöfness

At least two important classes of spaces are monotonically partition-Lindelöf (for more or less obvious reasons): the separable metrizable zero-dimensional spaces (see also [11]) and the Lindelöf P-spaces of weight ω_1. In both cases the proofs are identical. In the metrizable case, one can find a tree of clopen partitions of height ω which forms a basis for the topology, as follows. Consider the open cover $\mathscr{U}_n$ of a zero-dimensional separable metrizable space of all open sets of diameter less than 2^{-n}. By the assumptions and the comment at the start of Section 1, this cover can be refined by a clopen partition $\mathscr{P}_n$. It is trivial to construct clopen partitions $\mathscr{Q}_n$ of X such that $\mathscr{Q}_n$ refines the common refinement of $\mathscr{P}_n$ and $\mathscr{Q}_{n-1}$. Then the partitions $\mathscr{Q}_n$ form the tree of clopen partitions that we are after. In the Lindelöf P-case, the proof is similar, except that the tree has height ω_1.

Theorem 2.1. *Every Lindelöf P-space of weight ω_1 is* mpL.

Proof. Let X be a Lindelöf P-space of weight ω_1. For every $\alpha < \omega_1$ let $\mathscr{B}_\alpha$ be a countable clopen partition of X such that if $\beta < \alpha$, then $\mathscr{B}_\alpha$ refines $\mathscr{B}_\beta$, and $\bigcup_{\alpha<\omega_1} \mathscr{B}_\alpha$ is a base for X. Now for an open cover $\mathscr{U}$ of X, by induction on $\alpha < \omega_1$ we define (possibly empty) subcollections $\mathscr{U}_\alpha$ of $\mathscr{B}_\alpha$, as follows. Put $\mathscr{U}_0 = \{E \in \mathscr{B}_0 : (\exists U \in \mathscr{U})(E \subseteq U)\}$. Observe that $\mathscr{U}_0$ may be the empty collection. Having defined for some $\alpha < \omega_1$ the collections $\mathscr{U}_\beta$ for all $\beta < \alpha$, put $\mathscr{V}_\alpha = \bigcup_{\beta<\alpha} \mathscr{U}_\beta$, and let

$$\mathscr{U}_\alpha = \Big\{E \in \mathscr{B}_\alpha : \Big(E \cap \bigcup \mathscr{V}_\alpha = \emptyset\Big) \,\&\, (\exists U \in \mathscr{U})(E \subseteq U)\Big\}.$$

This completes the construction.

Now put $r(\mathscr{U}) = \bigcup_{\alpha<\omega_1} \mathscr{U}_\alpha$. It is clear that $r(\mathscr{U})$ is a cellular (and hence countable) clopen collection.

Claim 1. *$r(\mathscr{U})$ refines $\mathscr{U}$.*

This is clear by the construction.

Claim 2. *$\bigcup r(\mathscr{U})$ covers X.*

Take an arbitrary $x \in X$. Since $\mathscr{U}$ covers, we may take $U \in \mathscr{U}$ such that $x \in U$. There exists some $\alpha < \omega_1$ and an element $B \in \mathscr{B}_\alpha$ such that $x \in B \subseteq U$. If $B \cap \bigcup \mathscr{V}_\alpha = \emptyset$, then $B \in \mathscr{V}_\alpha$ and hence we are done. If $B \cap \bigcup \mathscr{V}_\alpha \neq \emptyset$, then there exists $V \in \mathscr{V}_\beta$ for some $\beta < \alpha$ such that $B \subseteq V$, hence we are done as well.

Claim 3. *If $\mathscr{U}$ refines $\tilde{\mathscr{V}}$, then $r(\mathscr{U})$ refines $r(\tilde{\mathscr{V}})$.*

Indeed, for some $\alpha < \omega_1$ take an arbitrary element $E \in \mathscr{U}_\alpha$. There exist $U \in \mathscr{U}$ and $V \in \tilde{\mathscr{V}}$ such that $E \subseteq U \subseteq V$. Let $\mathscr{W}_\alpha = \bigcup_{\beta<\alpha} \tilde{\mathscr{V}}_\beta$. If $E \cap \bigcup \mathscr{W}_\alpha = \emptyset$, then $E \in \tilde{\mathscr{V}}_\alpha$, and hence we are done. If not, then for some $\beta < \alpha$ we have that $E \cap \bigcup \tilde{\mathscr{V}}_\beta \neq \emptyset$. But $E \in \mathscr{B}_\alpha$, and since $\beta < \alpha$ there consequently exists $F \in \tilde{\mathscr{V}}_\beta$ such that $E \subseteq F$, which finishes the proof. □

Our aim now is to present an example of a countable zero-dimensional mL-space that is not mpL.

Let $\kappa > 0$ be a cardinal. Say that a sequence $T = \{T_\alpha : \alpha < \kappa\}$ of infinite subsets of ω is a *pretower* if $T_\beta \subseteq^* T_\alpha$ and $T_\beta \neq^* T_\alpha$ whenever $\alpha < \beta < \kappa$. Let $p \notin \omega$. Denote by X_T the set $\omega \cup \{p\}$ with the topology $\mathscr{T}_T$ generated by the base $\{\{n\} : n < \omega\} \cup \{\{p\} \cup (T_\alpha \setminus A) : \alpha < \kappa$ and $A \in [\omega]^{<\omega}\}$. Here $[\omega]^{<\omega}$ stands for the collection of finite subsets of ω. The following result is obvious.

Proposition 2.2. *If $T = \{T_\alpha : \alpha < \kappa\}$ is a pretower and κ has uncountable cofinality, then X_T is not second countable.*

In Levy and Matveev [10, §3] it was shown that under the Continuum Hypothesis, there is a pretower $T = \{T_\alpha : \alpha < \omega_1\}$ such that X_T is mL. Hence X_T is mL but not mpL by [11, Corollary 16] and Proposition 2.2.

3. Generalizations

The following generalization of monotone partition-Lindelöfness is quite natural.

Definition 3.1. A space X is *monotonically weakly partition-Lindelöf* (abbreviated: mwpL) if for every open cover $\mathscr{U}$ of X there is a family $r(\mathscr{U})$ of open sets of X such that:

(1) $r(\mathscr{U})$ is countable;
(2) $r(\mathscr{U})$ is cellular;
(3) $r(\mathscr{U})$ refines $\mathscr{U}$;
(4) $\bigcup r(\mathscr{U})$ is dense in X;
(5) if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U})$ refines $r(\mathscr{V})$.

There is also a natural 'hereditary' version of this notion that is of interest.

Definition 3.2. A space X is *hereditarily monotonically weakly partition-Lindelöf* (abbreviated: hmwpL) if for every family $\mathscr{U}$ of open sets there is a family of open sets $r(\mathscr{U})$ such that:

(1) $r(\mathscr{U})$ is countable;
(2) $r(\mathscr{U})$ is cellular;
(3) $r(\mathscr{U})$ refines $\mathscr{U}$;
(4) $\bigcup r(\mathscr{U})$ is dense in $\bigcup \mathscr{U}$;
(5) if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U})$ refines $r(\mathscr{V})$.

The terminology is justified by observing that every open subspace of an hmwpL-space is mwpL. For closed subspaces this is not true: consider any Isbell–Mrówka space Ψ (see [3, 3.6.I.(a)]).

Observe that mpL $\Rightarrow$ mwpL $\Rightarrow$ mwL. However, an mpL-space need not be hmwpL. Indeed, the one-point Lindelöfication L_{ω_1} of a discrete space D of cardinality ω_1 is an mpL-space (see Section 1) but the open subset D is not weakly Lindelöf, hence L_{ω_1} is not hmwpL.

By [11, Corollary 16] and Corollary 4.2 below, we have that the space $\omega \cup \{p\}$, where p is a nonprincipal ultrafilter on ω, is an hmwpL-space which is not mpL.

Lemma 3.3. *If Y is a dense subspace of X, and if X is* hmwpL*, then so is Y.*

Proof. Let r be an operator witnessing the fact that X is hmwpL. For every relatively open subset U of Y, put

$$E(U) = X \setminus \overline{Y \setminus U}.$$

Here closure means closure in X. So $E(U)$ is the largest open subset of X that extends U. Now for a family of open subsets $\mathscr{U}$ of Y, put

$$\varrho(\mathscr{U}) = \bigl\{V \cap Y : V \in r\bigl(\bigl\{E(U) : U \in \mathscr{U}\bigr\}\bigr)\bigr\}.$$

Then ϱ demonstrates that Y is hmwpL. $\square$

It is clear that if X is hmwpL, then X is ccc. The one-point Lindelöfication of ω_1 is not ccc, yet is mwpL. Hence there is a space that is mwpL but not hmwpL. This leads us to the following problem.

Question 3.4. Is there a ccc space that is mwpL but not hmwpL?

Question 3.5. Is there a ccc mwL-space that is not mwpL?

4. Spaces with a special π-base

Theorem 4.1. *Let $\lambda \le \omega_1$. Let X be a ccc space containing for every $\alpha < \lambda$ an open cellular family $\mathscr{B}_\alpha$ such that*

(1) *if $\alpha < \beta < \lambda$, then $\mathscr{B}_\beta$ refines $\mathscr{B}_\alpha$,*
(2) *$\mathscr{B} = \bigcup_{\alpha<\lambda} \mathscr{B}_\alpha$ is a π-basis for X.*

Then X is hmwpL*.*

Proof. Let $\mathscr{U}$ be any collection of nonempty open sets. By induction on $\alpha < \lambda$ we define (possibly empty) subcollections $\mathscr{U}_\alpha$ of $\mathscr{B}_\alpha$, as follows. Put $\mathscr{U}_0 = \{E \in \mathscr{B}_0 : (\exists U \in \mathscr{U})(E \subseteq U)\}$. Observe that $\mathscr{U}_0$ may be the empty collection. Having defined for some $\alpha < \lambda$ the collections $\mathscr{U}_\beta$, put $\mathscr{V}_\alpha = \bigcup_{\beta<\alpha} \mathscr{U}_\beta$, and let

$$\mathscr{U}_\alpha = \Bigl\{E \in \mathscr{B}_\alpha : \Bigl(E \cap \bigcup \mathscr{V}_\alpha = \emptyset\Bigr) \,\&\, (\exists U \in \mathscr{U})(E \subseteq U)\Bigr\}.$$

This completes the construction. The proof can now be completed as in the proof of Theorem 2.1. $\square$

Corollary 4.2. *Every space X with countable π-weight is* hmwpL*.*

Proof. Let $\mathscr{B} = \{B_n : n < \omega\}$ be a countable π-base for X. For each $n < \omega$ we will construct a subcollection $\mathscr{E}_n$ of $\mathscr{B}$ satisfying the following conditions:

(1) If $A, B \in \mathscr{E}_n$ are distinct, then $\overline{A} \cap \overline{B} = \emptyset$,
(2) $\bigcup \mathscr{E}_n$ is dense in X,
(3) there is an element $E \in \mathscr{E}_n$ such that $E \subseteq B_n$,
(4) if $m < n$, then $\mathscr{E}_m$ refines $\mathscr{E}_n$.

Let $\mathscr{E}_0$ be a maximal collection of elements of $\mathscr{B}$ such that its elements have pairwise disjoint closures and which contains B_0. Clearly, $\bigcup \mathscr{E}_0$ is dense.

Having defined $\mathscr{E}_n$ for some $n < \omega$, consider B_{n+1}. There exists $E \in \mathscr{E}_n$ such that $E \cap B_{n+1} \neq \emptyset$. There moreover exists $m < \omega$ such that $B_m \subseteq E \cap B_{n+1}$. Now let $\mathscr{F}$ be a maximal collection of elements of $\mathscr{B}$ such that its elements have pairwise disjoint closures which are all contained in E and which contains B_m. Put $\mathscr{E}_{n+1} = (\mathscr{E}_n \setminus \{E\}) \cup \mathscr{F}$. The inductive hypotheses are clearly satisfied.

Hence we are done by the $\lambda = \omega$ case of Theorem 4.1. $\square$

These results suggest (at least) two questions. The first one is whether Corollary 4.2 can be generalized to spaces with larger π-weight. This is not possible unfortunately. In [2] it was shown on page 591 that no countable dense set in the Cantor cube 2^{ω_1} is mwL. As a consequence, no countable dense set in 2^{ω_1} is hmwpL. Having said that, the second question is whether there is an example of an hmwpL-space of uncountable π-weight. There is one, if one assumes enough set theory. We will deal with this in the next section.

5. Monotonically normal spaces

Recall that a *linearly ordered space* (LOTS) is a triple $(X, <, \mathscr{I})$, where $<$ is a linear ordering of X and $\mathscr{I}$ is the open-interval topology of that ordering. A *generalized ordered space* (GO-space) is a triple $(X, <, \mathscr{T})$, where $<$ is a linear ordering of X and $\mathscr{T}$ is a Hausdorff topology on X that has a base of order-convex sets. Here a subset $C \subseteq X$ is called *order-convex* if $x \le y \le z$ and $\{x, z\} \subseteq C$ implies $y \in C$. E. Čech proved that X is a GO-space if and only if X is a subspace of some LOTS.

In the proof of Theorem 5.1 below we need the fact that the density of any ccc GO-space is at most ω_1. For ordered spaces, a proof of this can be found in Juhász [9, p. 14]. It was explained to us by David J. Lutzer that from this one can quite easily get the result for GO-spaces; he ascribes it to folklore. If $(X, <, \tau)$ is a GO-space that satisfies ccc, let λ be the usual open-interval topology of the order $<$. Then $\lambda \subseteq \tau$ so that $(X, <, \lambda)$ is a LOTS that satisfies ccc. Consequently, there is a dense subset D of (X, λ) having $|D| \le \omega_1$. Let J be the set of all isolated points of the original GO-space (X, τ). Then, by ccc, J is countable. Then $D \cup J$ is dense in (X, τ) and has cardinality $\le \omega_1$.

The following result is related to Bennett, Lutzer and Matveev [1, §3].

Theorem 5.1. *Every ccc GO-space is* hmwpL.

Proof. Let X be a ccc GO-space. First observe that X is first countable and has as we discussed above a dense subset D of size at most ω_1. If X is separable, then X has countable π-weight and so we are done by Corollary 4.2. So we may assume that X is not separable. Let A be the closure of the isolated points I of X (possibly, $A = \emptyset$). List $D \setminus A$ as $\{d_\alpha : \alpha < \omega_1\}$. For every $\alpha < \omega_1$ put $K_\alpha = A \cup \overline{\{d_\beta : \beta < \alpha\}}$. Then $\bigcup_{\alpha<\omega_1} K_\alpha$ is closed in X since X is first countable, and it is obviously dense, hence $X = \bigcup_{\alpha<\omega_1} K_\alpha$.

For every $\alpha < \omega_1$, let $\mathscr{T}_\alpha$ be the family of convex components of $X \setminus K_\alpha$. Observe that $\mathscr{T}_\alpha$ is countable by ccc, and $\mathscr{T}_\alpha$ refines $\mathscr{T}_\beta$ if $\beta < \alpha$. We claim that $\mathscr{T} = \{\{x\} : x \in I\} \cup \bigcup_{\alpha<\omega_1} \mathscr{T}_\alpha$ is a π-basis of X. Indeed, let (a, b) be an arbitrary interval. We may assume that (a, b) contains no isolated points. Hence there exists $\alpha < \omega_1$ such that $|K_\alpha \cap (a, b)| \geq 2$. This implies that some member of $\mathscr{T}_\alpha$ is contained in (a, b). Hence by replacing $\mathscr{T}_0$ by $\{\{x\} : x \in I\} \cup \mathscr{T}_0$, we see that the π-basis satisfies the conditions in Theorem 4.1. As a consequence, X is hmwpL. □

Corollary 5.2. *A Souslin continuum is* hmwpL *and has π-weight ω_1.*

Question 5.3. Is there in ZFC an example of an hmwpL-space of uncountable π-weight?

Let X be a compact monotonically normal space. By a result of Rudin [13] there is a compact LOTS L that maps onto X, say $f: L \to X$ is a continuous surjection. There is a closed subset L' of L such that $f \restriction L': L' \to X$ is an irreducible surjection [3, Exercise 3.1.C]. Hence we may assume without loss of generality that f is irreducible. Observe that if $U \subseteq L$ is nonempty and open, and $U^\# = X \setminus f(L \setminus U)$, then $f^{-1}(U^\#)$ is a dense open subset of U. Hence if X is ccc, then so is L. This leads us to the following result:

Theorem 5.4. *Let X be a compact monotonically normal ccc space. Then X is* hmwpL.

Proof. Let $f: L \to X$ be a continuous mapping such as the one we described above. By Theorem 5.1, L is hmwpL; let r be the operator witnessing this. For every collection of open subsets $\mathscr{U}$, put $\mathscr{U}_L = \{f^{-1}(U) : U \in \mathscr{U}\}$. Moreover, put

$$\varrho(\mathscr{U}) = \{V^\# : V \in r(\mathscr{U}_L)\}.$$

We claim that the operator ϱ is as required. Indeed, the collection $\varrho(\mathscr{U})$ is open and cellular since $r(\mathscr{U}_L)$ is. Also, since $\bigcup \varrho(\mathscr{U})$ is dense in $\bigcup \mathscr{U}_L$, and for every $V \in \varrho(\mathscr{U})$, $f^{-1}(V^\#)$ is dense in V, we are done once we showed that ϱ is monotone. To check this, suppose that $\mathscr{U}$ and $\mathscr{V}$ are open collections of X such that $\mathscr{U}$ refines $\mathscr{V}$. Observe that $r(\mathscr{U}_L)$ refines $r(\mathscr{V}_L)$. Pick an arbitrary member $U^\#$ in $\varrho(\mathscr{U})$, where $U \in r(\mathscr{U}_L)$. There exists $V \in r(\mathscr{V}_L)$ such that $U \subseteq V$. Clearly, $U^\# \subseteq V^\#$, and hence we are done. □

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