FUNDAMENTA
MATHEMATICAE
226 (2014)

Countable dense homogeneity and λ-sets

by

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Abstract. We show that all sufficiently nice λ-sets are countable dense homogeneous (CDH). From this fact we conclude that for every uncountable cardinal $\kappa \leq \mathfrak{b}$ there is a countable dense homogeneous metric space of size κ. Moreover, the existence of a meager in itself countable dense homogeneous metric space of size κ is equivalent to the existence of a λ-set of size κ. On the other hand, it is consistent with the continuum arbitrarily large that every CDH metric space has size either ω_1 or $\mathfrak{c}$. An example of a Baire CDH metric space which is not completely metrizable is presented. Finally, answering a question of Arhangel'skii and van Mill we show that that there is a compact non-metrizable CDH space in ZFC.

1. Introduction. A separable topological space X is *countable dense homogeneous* (CDH) if, given any two countable dense subsets D and E of X, there is a homeomorphism $f : X \to X$ such that $f[D] = E$. This is a classical notion [5] that can be traced back to the work of Cantor, Brouwer, Fréchet, and others (see [3] and [11]).

Examples of CDH spaces are the Euclidean spaces, the Hilbert cube and the Cantor set. In fact, every strongly locally homogeneous Polish space is CDH, as was shown by Bessaga and Pełczyński [6]. Recall that a space X is *strongly locally homogeneous* if it has a basis $\mathscr{B}$ of open sets such that for every $U \in \mathscr{B}$ and all $x, y \in U$ there is an autohomeomorphism h of X such that $h(x) = y$ and h restricts to the identity on $X \setminus U$. All these results are based on the completeness of the spaces involved.

Countable dense homogeneity had for a long time been studied mostly as a geometrical notion ([5], [10]) and only relatively recently did set-theoretic methods enter the picture. At first the use of set theory was restricted to constructions of non-completely metrizable CDH spaces assuming special set-theoretic axioms such as CH ([12], [32], [33]) or versions of Martin's

2010 *Mathematics Subject Classification*: Primary 54H05; Secondary 03E15, 54E50.
Key words and phrases: countable dense homogeneous, λ-set.

DOI: 10.4064/fm226-2-5

Axiom ([4]). In fact, it was only recently that a ZFC example of a non-complete CDH space was given by Farah, Hrušák and Martínez Ranero [9]. They proved that there exists a CDH subset of $\mathbb{R}$ of size $\aleph_1$. The proof given in [9] uses forcing combined with an absoluteness argument involving infinitary logic (the so-called Keisler Compactness Theorem), while giving no hint as to how to produce an "honest" ZFC proof. One of the main purposes of our paper is to provide just that.

Note that every countable CDH space is discrete, hence $\aleph_1$ is the first cardinal where anything of CDH interest can happen. Since $\mathbb{R}$ is CDH and has size $\mathfrak{c}$, it is an interesting problem as to what can happen for cardinals greater than $\aleph_1$ but below $\mathfrak{c}$. In this paper we will address this problem too.

It is easy to see that every CDH space is a disjoint sum of a Baire CDH space and a meager in itself CDH space. The example given in [9] is meager in itself. It is a result of Fitzpatrick and Zhou [12] that every meager in itself CDH space is, in fact, a λ-set. Recall that a separable metric space X is a *λ-set* if every countable subset of X is a relative G_δ-set. Here we show that for every λ-set there is another λ-set of the same cardinality which is CDH. This not only provides an honest construction of a separable metric CDH space which is not completely metrizable, but also shows that there can be many distinct cardinalities of CDH metric spaces, as there is a meager in itself CDH metric space of size κ if and only if there is a λ-set of the same cardinality. In particular, there is a CDH metric space of size κ for every $\kappa \leq \mathfrak{b}$. On the other hand, we show that it is also consistent with the continuum arbitrarily large that every meager in itself CDH metric space has size ω_1 while every Baire CDH metric space has size $\mathfrak{c}$.

We present a ZFC construction of a Baire CDH metric space which is not completely metrizable, thus providing a very different absolute example of a CDH metric space which is not completely metrizable. The space is the complement (in a completion) of a carefully chosen λ-set (in fact, a λ'-set).

Finally, we deal with the existence of non-metrizable CDH compact spaces. We again use λ'-sets to define linearly ordered non-metrizable CDH compacta (variants of the *double arrow* space). This provides the first ZFC examples of non-metrizable CDH compacta and answers a question of Arhangel'skii and van Mill [3].

2. λ-sets. A *λ-set* is a subset X of 2^ω such that every countable subset of X is relative G_δ. This notion is due to Kuratowski [20]. The existence of uncountable λ-sets was proved by Luzin [21] and was later improved by Rothberger [31]: *There exist λ-sets of size κ for every $\kappa \leq \mathfrak{b}$.* A subset X of 2^ω is a *λ'-set* if for every countable subset of $Y \subseteq 2^\omega$, Y is relative G_δ in $X \cup Y$. Rothberger [31] has also shown that there is a λ-set of size $\mathfrak{b}$ which is not a λ'-set, while every set of size less than $\mathfrak{b}$ is a λ-set, hence also a λ'-set. Sierpiński [34] noted that a union of countably many λ'-sets is a λ'-set.

Every λ-set is meager in itself. Topologically, λ-sets are characterized as follows: they are precisely the zero-dimensional spaces having the property that all countable sets are G_δ. We will call such spaces λ-sets as well.

LEMMA 2.1. *Let X and Y be λ-sets. Then so is $X \times Y$.*

Proof. Let $A \subseteq X \times Y$ be countable. Let $\pi : X \times Y \to X$ denote the projection. Then $X \setminus \pi[A]$ is F_σ by assumption, hence so is $(X \setminus \pi[A]) \times Y$. Since for every $p \in \pi[A]$ the set $(\{p\} \times Y) \setminus A$ is F_σ in $\{p\} \times Y$ and hence in $X \times Y$, we are done. ∎

LEMMA 2.2. *If a space X has a countable closed cover by λ-sets, then X is a λ-set.*

Proof. First observe that X is zero-dimensional by the Countable Closed Sum Theorem [8, 1.3.1]. Since subspaces of λ-sets are λ-sets, we may assume by zero-dimensionality that X is covered by a disjoint countable family $\mathscr{L}$ of relatively closed λ-sets. Now, if $A \subseteq X$ is countable, then for every $L \in \mathscr{L}$, $L \setminus A$ is F_σ in L and hence in X. Hence $X \setminus A$ is F_σ. ∎

Meager in themselves CDH subspaces of $\mathbb{R}$ are λ-sets, as can be seen from the following observation. If X is first category, then it contains a countable dense set D which is G_δ in X. This simple but useful fact was proved by Fitzpatrick and Zhou [12] (and was put to good use in Hrušák and Zamora Avilés [15]). Every countable subset A of X can be extended to a countable dense subset of X which consequently must be G_δ since D is G_δ, and X is CDH. Hence A is G_δ.

For $x, y \in 2^\omega$ we say that

$$x \sim y \quad \text{if} \quad \exists m, n \in \omega\ \forall k \in \omega\ x(m+k) = y(n+k).$$

The relation $\sim$, known as *tail equivalence*, is a Borel equivalence relation on 2^ω with countable and dense equivalence classes. Given a set $X \subseteq 2^\omega$ we define its *saturation* $X^* = \{y \in 2^\omega : \exists x \in X\ y \sim x\}$. We will call a set $X \subseteq 2^\omega$ *saturated* if it is saturated with respect to $\sim$, i.e. if $X = X^*$.

Given $s \in 2^{<\omega}$, we denote by $[s] = \{x \in 2^\omega : s \subseteq x\}$ the basic clopen set (the *cone*) determined by s. Given $s, t \in 2^{<\omega}$ we let $h_{s,t} : [s] \to [t]$ be defined by $h_{s,t}(s^\frown x) = t^\frown x$ for every $x \in 2^\omega$. Then $h_{s,t}$ is a natural (even monotone with respect to the lexicographic order on 2^ω) homeomorphism between the clopen sets $[s]$ and $[t]$. The crucial property we use is that $h_{s,t}$ *respects* the equivalence relation $\sim$: $h_{s,t}(x) \sim x$ for all $x \in [s]$.

Next we see that there are many saturated λ-sets and λ'-sets.

LEMMA 2.3.

(1) *If X is a λ'-set then X^* is a λ'-set.*

(2) *For every λ-set $X \subseteq 2^\omega$ there is a saturated λ-set Y of the same cardinality.*

Proof. The first clause follows directly from Sierpiński's observation that a union of countably many λ'-sets is a λ'-set.

To prove the second clause, fix first an embedding $\varphi : 2^\omega \to 2^\omega$ such that for distinct $x, y \in 2^\omega$, $\varphi(x) \not\sim \varphi(y)$. Such an embedding exists by a theorem of Silver (see [35]). Next let $Y = \varphi[X]^*$. Then Y is a λ-set by the previous lemma. ∎

3. Knaster–Reichbach covers. Let X and Y be zero-dimensional spaces, let $A \subseteq X$ be closed and nowhere dense in X and let $B \subseteq Y$ be closed and nowhere dense in Y. Moreover, let $h : A \to B$ be a homeomorphism. A triple $\langle \mathscr{U}, \mathscr{V}, \alpha \rangle$ is called a *Knaster–Reichbach cover*, or *KR*-cover, for $\langle X \setminus A, Y \setminus B, h \rangle$ if the following conditions are satisfied:

(1) $\mathscr{U}$ is a partition of $X \setminus A$ into non-empty clopen subsets of X,
(2) $\mathscr{V}$ is a partition of $Y \setminus B$ into non-empty clopen subsets of Y,
(3) $\alpha : \mathscr{U} \to \mathscr{V}$ is a bijection,
(4) if for every $U \in \mathscr{U}$, $g_U : U \to \alpha(U)$ is a bijection, then the combination mapping $\tilde{h} = h \cup \bigcup_{U \in \mathscr{U}} g_U$ is continuous at all points of A, and its inverse $\tilde{h}^{-1}$ is continuous at all points of B.

KR-covers were used by Knaster and Reichbach [18] to prove homeomorphism extension results in the class of all zero-dimensional spaces. The term KR-cover was first used by van Engelen [7] who proved their existence in a general setting.

Lemma 3.1 ([7, Lemma 3.2.2]). *Let X and Y be zero-dimensional separable metrizable spaces, and let A and B be non-empty closed nowhere dense subspaces of X and Y, respectively. If $h : A \to B$ is a homeomorphism, then there exists a KR-cover for $\langle X \setminus A, Y \setminus B, h \rangle$.*

4. CDH metric spaces from λ-sets. Perhaps the main result of this paper is the following:

Theorem 4.1. *Let $X \subseteq 2^\omega$ be an uncountable saturated λ-set. Then X is a* relatively CDH *subspace of 2^ω, i.e. for any countable dense subsets D_0, D_1 of X there is a homeomorphism h of 2^ω such that*

(1) *$h[D_0] = D_1$, and*
(2) *$h[X] = X$.*

Proof. Start by fixing a metric on 2^ω. Observe that X, being saturated, is dense in 2^ω. Since X is uncountable, there exists a saturated set $E_i \subset X$ such that $D_i \subset E_i$ and $E_i \setminus D_i$ is countable dense for each $i \in 2$. Let $E = E_0 \cup E_1$

and $Y = X \setminus E$, notice that these sets are saturated as well. Since X is a λ-set, there is a collection $\{K_n : n < \omega\}$ of closed nowhere dense subsets of 2^ω such that $Y = X \cap \bigcup\{K_n : n < \omega\}$. Enumerate E as $\{d_k : k < \omega\}$.

We will construct the homeomorphism h recursively. In step $n < \omega$, we will find a homeomorphism $h_n : 2^\omega \to 2^\omega$ and a pair of closed nowhere dense sets G_n^0 and G_n^1 such that:

(a) For $i \in 2$, $G_n^i \cup K_n \subset G_{n+1}^i$.
(b) For $i \in 2$, $E \cap G_n^i$ is finite and contains $\{d_k : k < n\}$.
(c) $h_n[G_n^0] = G_n^1$, $h_n[D_0 \cap G_n^0] = D_1 \cap G_n^1$, $h_n[E \cap G_n^0] = E \cap G_n^1$ and $h_n[Y] = Y$.
(d) For every $k < n$, $h_n{\restriction}_{G_k^0} = h_k$.

We will also need a KR-cover $\langle \mathscr{I}_n^0, \mathscr{I}_n^1, \alpha_n\rangle$ for $\langle 2^\omega \setminus G_n^0, 2^\omega \setminus G_n^1, h_n{\restriction}_{G_n^0}\rangle$ satisfying the following:

(e) Given $i \in 2$, if $I \in \mathscr{I}_n^i$, then I has diameter $< 1/(n+1)$.
(f) For each $i \in 2$, $\mathscr{I}_{n+1}^i$ refines $\mathscr{I}_n^i$, that is, every element of $\mathscr{I}_{n+1}^i$ is contained in an element of $\mathscr{I}_n^i$.
(g) If $I \in \mathscr{I}_n^0$, then $h_n[I] = \alpha_n(I)$ and $h_n[I \cap Y] = \alpha_n(I) \cap Y$.
(h) Given $m < n$, if $I \in \mathscr{I}_m^0$ and $J \in \mathscr{I}_n^0$ then $J \subset I$ if and only if $\alpha_n(J) \subset \alpha_m(I)$.

In the first step of the construction, we let $G_0^0 = G_0^1 = \emptyset$, $\mathscr{I}_0^0 = \mathscr{I}_0^1 = \{2^\omega\}$, $\alpha_0 = \{\langle 2^\omega, 2^\omega\rangle\}$ and we set h_0 to be the identity function.

So assume we are in step $n+1$ for some $n \in \omega$.

First, let $G = G_n^0 \cup K_n \cup \{d_n\}$. We may assume that $d_n \notin G_n^0$ and let $I_0 \in \mathscr{I}_n^0$ be such that $d_n \in I_0$. Now, choose $e \in E_1 \cap \alpha_n(I_0)$ in such a way that $e \in D_1$ if and only if $d_n \in D_0$; this is possible by the choice of E_0 and E_1. Let $\{I_0^0(m) : m < \omega\}$ be a partition of $I_0 \setminus \{d_n\}$ into infinitely many clopen subsets. Similarly, let $\{I_0^1(m) : m < \omega\}$ be a partition of $\alpha_n(I_0) \setminus \{e\}$ into infinitely many clopen subsets. Define $\mathscr{I}' = (\mathscr{I}_n^0 \setminus \{I_0\}) \cup \{I_0^0(m) : m < \omega\}$ and $\beta = \alpha_n \cup \{\langle I_0^0(m), I_0^1(m)\rangle : m < \omega\}$.

Let $U \in \mathscr{I}'$ and $V = \beta(U)$. Every clopen subset of 2^ω is a finite union of pairwise disjoint sets of the form $[s]$ for $s \in 2^{<\omega}$. Thus we may assume that $U = [s_0] \cup \cdots \cup [s_k]$ and $V = [t_0] \cup \cdots \cup [t_k]$ where these sets are pairwise disjoint. Then the function $f_U : U \to V$ defined by $f_U(x) = h_{s_j,t_j}(x)$ if $x \in [s_j]$ is a homeomorphism such that $f_U[U \cap Y] = V \cap Y$.

Next we define

$$H(x) = \begin{cases} h_n(x) & \text{if } x \in G_n^0, \\ e & \text{if } x = d_n, \\ f_U(x) & \text{if } x \in U \in \mathscr{I}'. \end{cases}$$

Then $H : 2^\omega \to 2^\omega$ is a homeomorphism. By Lemma 3.1, for every $I \in \mathscr{I}'$ there is a KR-cover $\langle \mathscr{I}^0(I), \mathscr{I}^1(I), \alpha_I\rangle$ for $\langle I \setminus G, \beta(I) \setminus H[G], H{\restriction}_{G\cap I}\rangle$. Define

$\mathscr{I}^0 = \bigcup\{\mathscr{I}^0(I) : I \in \mathscr{I}'\}$, $\mathscr{I}^1 = \bigcup\{\mathscr{I}^1(I) : I \in \mathscr{I}'\}$ and $\alpha = \bigcup\{\alpha_I : I \in \mathscr{I}'\}$. Notice that $\langle \mathscr{I}^0, \mathscr{I}^1, \alpha\rangle$ is a KR-cover for $\langle 2^\omega \setminus G, 2^\omega \setminus H[G], H{\restriction}_G\rangle$. If necessary, this KR-cover may be refined to a KR-cover where all clopen subsets involved have diameter $< 1/(n+2)$.

It is not hard to see that G, α, $\mathscr{I}^0$, $\mathscr{I}^1$ and H have the desired properties (a)–(h) when $i = 0$. In order to finish the induction step, we have to do another refinement to both the homeomorphism and the covers so that (a)–(h) hold also when $i = 1$. This is entirely analogous to what we have done, so we leave the details to the reader.

Having carried out the recursive construction, observe that the sequences $(h_n)_n$ and $(h_n^{-1})_n$ are Cauchy (in the complete space of homeomorphisms of 2^ω endowed with the topology of uniform convergence), hence $h = \lim_{n\to\infty} h_n$ exists and is a homeomorphism with the desired properties. ∎

In particular, if X is a saturated λ-set then X is CDH, and the following corollaries easily follow.

Corollary 4.2. *For every uncountable cardinal $\kappa \le \mathfrak{c}$, the following statements are equivalent:*

1. *There is a meager in itself CDH metric space of size κ.*
2. *There is a λ-set of size κ.*

Corollary 4.3. *For every cardinal κ such that $\omega_1 \le \kappa \le \mathfrak{b}$ there exists a meager in itself CDH metric space of size κ.*

Given the latter, one has to wonder whether there is (in ZFC) a CDH space of any cardinality below $\mathfrak{c}$. We will show that this is not the case.

Theorem 4.4. *It is consistent with ZFC that the continuum is arbitrarily large and every CDH metric space has size either ω_1 or $\mathfrak{c}$, and moreover*

1. *all metric CDH spaces of size ω_1 are λ-sets, and*
2. *all metric CDH spaces of size $\mathfrak{c}$ are non-meager.*

Proof. The model for this is the so-called *Cohen model*. That is, start with a model of CH and force with $\mathbb{C}_\kappa$, the partial order for adding κ-many Cohen reals, where $\kappa \ge \omega_2$. As shown by A. Miller [28, Theorem 22], all λ-sets in the Cohen model have size ω_1.

Claim. *Let X be a crowded separable metric space. If X is CDH after adding some number of Cohen reals then X is meager in itself (hence a λ-set).*

Let D be a countable dense subset of X and let $\dot{C}$ be a name for a Cohen subset of D. By genericity $\Vdash$ "$\dot{C}$ is a dense subset of D". Let $\dot{h}$ be a $\mathbb{C}_\kappa$-name for a homeomorphism of X which sends D onto $\dot{C}$. Then there is a countable set J of ordinals such that $\dot{h}$ is (equivalent to) a $\mathbb{C}_J$-name. Now, as $\mathbb{C}_J$ is countable,

$$X = \bigcup_{p\in\mathbb{C}_J} N_p, \quad \text{where} \quad N_p = \{x \in X : p \text{ decides } \dot{h}(x)\}.$$

Now, it is easy to see that the set N_p is closed and nowhere dense in X (it has only finite intersection with D), hence X is meager in itself. This completes the proof of the Claim.

It follows that if X is a CDH metric space of size less than $\mathfrak{c}$ then X is a λ-set, hence of size ω_1 by Miller's result. ∎

Another corollary of Theorem 4.1 (or rather of its proof) is the following:

THEOREM 4.5. *Let $X \subseteq 2^\omega$ be an uncountable saturated λ'-set. Then for any countable dense subsets D_0, D_1 of X and countable dense subsets E_0, E_1 of $2^\omega \setminus X$ there is a homeomorphism h of 2^ω such that*

(1) $h[D_0] = D_1$,
(2) $h[E_0] = E_1$, *and*
(3) $h[X] = X$.

This has the following interesting consequence (1):

COROLLARY 4.6. *If $X \subseteq 2^\omega$ is an uncountable saturated λ'-set then $2^\omega\setminus X$ is a* (*completely*) *Baire* CDH *space.*

Proof. The space $2^\omega \setminus X$ is clearly a CDH space. To see that it is completely Baire it suffices, by a result of Hurewicz [16] (see [27, pp. 78–79]), to show that $2^\omega \setminus X$ does not contain a closed copy of the rationals. Indeed, if Q were such a copy then Q would be relatively G_δ both in $2^\omega \setminus X$ and in $Q \cup X$, hence it would be G_δ in 2^ω, which is a contradiction. ∎

It should be noted that this result provides the first ZFC example of a metric Baire CDH space which is not completely metrizable. In [14] it is shown (extending a somewhat weaker result by Medini and Milovich [24]) that all non-meager P-ideals (seen as subspaces of 2^ω) are CDH, and it is an older result of Marciszewski [22] that non-meager P-ideals are completely Baire. In fact, it has been recently proved by Kunen, Medini and Zdomskyy [19] that an ideal $\mathscr{I}$ is CDH if and only if $\mathscr{I}$ is a non-meager P-ideal. It is, however, an open problem whether non-meager P-ideals exist in ZFC (see [17]).

Unlike the case of spaces which are meager in themselves, we do not know how to manipulate cardinalities of Baire CDH spaces.

(1) Recall that a space X is *completely Baire* if all of its non-empty closed subspaces are Baire.

QUESTION 4.7. Is it consistent with ZFC to have a metric Baire CDH space without isolated points of size less than $\mathfrak{c}$?

The last comment on the construction deals with products. In a recent paper, Medini [23] constructed a zero-dimensional metric CDH space whose square is not CDH under Martin's Axiom, and asked whether similar examples exist for higher dimensions. Our example answers one of his questions:

THEOREM 4.8. *Let $X \subseteq 2^\omega$ be an uncountable saturated λ-set. Then X^n is* CDH *for every $n \in \omega$ while X^ω is not* CDH.

Proof. First we note that X^ω is not CDH. This follows from a recent result of Hernández-Gutiérrez [13], who showed that if X^ω is CDH and X is crowded then X contains a copy of the Cantor set, which of course a λ-set does not contain.

To finish the proof it suffices to show that X^n is homeomorphic to a saturated λ-set for every $n \in \omega$. The space X^n is a λ-set by Lemma 2.1. Now consider the function $\psi_n : (2^\omega)^n \to 2^\omega$ defined by

$$\psi_n(x_0, \dots, x_{n-1})(nk + j) = x_j(k).$$

The function ψ_n is a homeomorphism between $(2^\omega)^n$ and 2^ω, and it should be obvious that if X is a saturated subset of 2^ω then $\psi_n[X^n]$ is also saturated. ∎

The property of being saturated describes how a set is situated inside 2^ω. It would be interesting to characterize meager in themselves CDH spaces internally. Recall that a zero-dimensional space X is *strongly homogeneous* if all non-empty clopen subsets of X are mutually homeomorphic. The methods of the proof of Theorem 4.1 can be used to show the following:

PROPOSITION 4.9. *Let X be a λ-set such that*

(1) *X is strongly homogeneous, and*
(2) *if A and B are subsets of X with A countable and B nowhere dense, then there is a nowhere dense set $B' \subseteq X$ homeomorphic to B with $A \cap B' = \emptyset$.*

Then X is CDH.

Note that every CDH space satisfies condition (2) while, as we have already seen, a meager in itself CDH metric space is a λ-set.

QUESTION 4.10. Let X be a zero-dimensional homogeneous meager in itself CDH metric space. Is X strongly homogeneous?

A similar question was raised by S. V. Medvedev [25, Question 1]. A related question is the following:

QUESTION 4.11. Is there a metric meager in itself CDH space X containing a countable dense set D such that X cannot be embedded in $X \setminus D$?

5. Compact CDH spaces from λ'-sets. The literature on non-metrizable CDH spaces is rather scarce. Arhangel'skii and van Mill in [2] showed that a CDH space has size at most $\mathfrak{c}$. They also realized that no ZFC example of a compact non-metrizable CDH space was known. Consistent examples were known; Steprāns and Zhou [36] observed that 2^κ is CDH assuming $\kappa < \mathfrak{p}$. A non-metrizable hereditarily separable and hereditarily Lindelöf compact CDH space was constructed in [2] assuming CH. In [2] it was also shown that, assuming $2^\omega < 2^{\omega_1}$, every compact CDH space is first countable.

A natural candidate for a compact non-metrizable CDH space seemed to be Aleksandrov's *double arrow* (or *split interval*) space $\mathbb{A}$ [1]. It turned out that the space is not CDH [2]; in fact, it has $\mathfrak{c}$ types of countable dense sets [13]. It was conjectured that a slight modification of the double arrow should provide a ZFC example. The first attempts were unsuccessful: Hernández-Gutiérrez [13] showed that both $\mathbb{A} \times 2^\omega$ and $\mathbb{A}^\omega$ are non-CDH. These results follow directly from the following:

Proposition 5.1 ([13]). *Let X and Y be crowded spaces with countable π-bases. If the product $X \times Y$ is* CDH*, then either both X and Y contain a copy of 2^ω or neither does.*

Here we will show that the *double arrow space over a saturated λ'-set* is CDH, hence there are compact non-metrizable CDH spaces in ZFC, indeed.

First let us fix some notation for linearly ordered spaces. Given a linearly ordered topological space $(X,<)$, a function $f : X \to X$ and $x \in X$ which is neither the least nor the largest element of X, we will say that f is *monotone at x* if there exist $a, b \in X$ with $a < x < b$ such that either $f[(a,x)] \subset (\leftarrow, f(x))$ and $f[(x,b)] \subset (f(x), \rightarrow)$, or $f[(a,x)] \subset (f(x), \rightarrow)$ and $f[(x,b)] \subset (\leftarrow, f(x))$.

We consider here 2^ω as a linearly ordered topological space ordered lexicographically, or (equivalently) consider it as a subspace of $\mathbb{R}$ with the induced order. Let $Q_i = \{x \in 2^\omega : \exists n < \omega\ \forall m \geq n\ (x(m) = i)\}$ for $i \in 2$ and $Q = Q_0 \cup Q_1$. The set Q_0 consists of the least element of 2^ω and all those points that have an immediate predecessor. Similarly, Q_1 consists of the greatest element of 2^ω and all those points that have an immediate successor.

Definition 5.2. For each $X \subset 2^\omega \setminus Q$, let $\mathbb{A}(X)$ be the set $((2^\omega \setminus Q_0) \times \{0\}) \cup ((Q_0 \cup X) \times \{1\})$ with the order topology given by the lexicographic order.

It is easy to see that $\mathbb{A}(X)$ is a 0-dimensional separable compact Hausdorff space of weight $|X|$. Note that $\mathbb{A}(C)$ is homeomorphic to the Cantor set whenever $C \subset 2^\omega \setminus Q$ is countable. In fact, using a result of Ostaszewski's [30] it can be easily shown that every compact 0-dimensional separable linearly ordered space without isolated points is homeomorphic to $\mathbb{A}(X)$ for some $X \subset 2^\omega \setminus Q$.

If $X \subset Y \subset 2^\omega \setminus Q$, let $\pi_X^Y : \mathbb{A}(Y) \to \mathbb{A}(X)$ be the natural projection defined by

$$\pi_X^Y(\langle x, t\rangle) = \begin{cases} \langle x, 0\rangle & \text{if } x \in Y \setminus X, \\ \langle x, t\rangle & \text{otherwise.} \end{cases}$$

The following is not hard to see.

LEMMA 5.3. *Let $X \subset Y \subset 2^\omega \setminus Q$ and let $h : \mathbb{A}(X) \to \mathbb{A}(X)$ be a homeomorphism. Assume that h is monotone at each point of the form $\langle x, 0\rangle$ with $x \in Y \setminus X$ and $h[(Y \setminus X) \times \{0\}] = (Y \setminus X) \times \{0\}$. Then there exists a homeomorphism $H : \mathbb{A}(Y) \to \mathbb{A}(Y)$ such that $\pi_X^Y \circ H = h \circ \pi_X^Y$.*

PROPOSITION 5.4. *Let D_0 and D_1 be countable dense subsets of 2^ω and let $W \subset 2^\omega$ be such that $W \cap (D_0 \cup D_1 \cup Q) = \emptyset$ and $W \cup D_0 \cup D_1$ is a λ-set. Furthermore, assume that the following condition holds:*

- ($*$) *For any two non-empty clopen intervals I, J of 2^ω, there is an order isomorphism $f : I \to J$ such that $f[I \cap W] = J \cap W$.*

Then there is a homeomorphism $h : 2^\omega \to 2^\omega$ such that $h[D_0] = D_1$, $h[W] = W$ and h is monotone at each point of W.

Proof. Start by fixing some metric for 2^ω. Since $W \cup D_0 \cup D_1$ is a λ-set, there exists a collection $\{K_n : n < \omega\}$ of closed nowhere dense subsets of 2^ω such that $W \subset \bigcup\{K_n : n < \omega\}$ and $(D_0 \cup D_1) \cap K_n = \emptyset$ for each $n < \omega$. Let $D_0 = \{d_n^0 : n < \omega\}$, $D_1 = \{d_n^1 : n < \omega\}$.

We will construct the homeomorphism we are looking for by defining a sequence of autohomeomorphisms of 2^ω. In step $n < \omega$, we will construct a homeomorphism $h_n : 2^\omega \to 2^\omega$ and a pair of closed nowhere dense sets G_n^0 and G_n^1 that have the following properties:

- (a) For each $i \in 2$, $G_n^i \cup K_n \subset G_{n+1}^i$.
- (b) For each $i \in 2$, $D_i \cap G_n^i$ is finite and contains $\{d_k^i : k < n\}$.
- (c) $h_n[G_n^0] = G_n^1$, $h_n[D_0 \cap G_n^0] = D_1 \cap G_n^1$ and $h_n[W] = W$.
- (d) For each $k < n$, $h_n\restriction_{G_k^0} = h_k$.

We will also need a KR-cover $\langle \mathscr{I}_n^0, \mathscr{I}_n^1, \alpha_n\rangle$ for $\langle 2^\omega \setminus G_n^0, 2^\omega \setminus G_n^1, h_n\restriction_{G_n^0}\rangle$. In order to ensure the monotonicity of h at all points of W, we will construct these KR-covers with the following additional properties:

- (e) Given $i \in 2$, if $I \in \mathscr{I}_n^i$, then I is a clopen interval of the form $[q_0, q_1]$ where $q_j \in Q_j$ for $j \in 2$ and I has diameter $< 1/(n+1)$.
- (f) For all $i \in 2$, $\mathscr{I}_{n+1}^i$ refines $\mathscr{I}_n^i$, that is, every element of $\mathscr{I}_{n+1}^i$ is contained in an element of $\mathscr{I}_n^i$.
- (g) If $I \in \mathscr{I}_n^0$, then $h_n[I] = \alpha_n(I)$, $h_n[I \cap W] = \alpha_n(I) \cap W$ and $h_n\restriction_I$ is an order isomorphism.

(h) Given $m < n$, if $I \in \mathscr{I}^0_m$ and $J \in \mathscr{I}^0_n$ then $J \subset I$ if and only if $\alpha_n(J) \subset \alpha_m(I)$.

(i) Given $x \in W$, let $k = \min\{m < \omega : x \in G^0_m\}$. Then there are $a, b \in Q$ with $a < x < b$ such that if $n \geq k$, $I, J \in \mathscr{I}^0_n$, $I \subset [a, x]$ and $J \subset [x, b]$ then $\alpha_n(I) \subset (\leftarrow, h_m(x)]$ and $\alpha_n(J) \subset [h_m(x), \rightarrow)$.

To start with the construction, let $G^0_0 = G^1_0 = \emptyset$, $\mathscr{I}^0_0 = \mathscr{I}^1_0 = \{2^\omega\}$, $\alpha_0 = \{\langle 2^\omega, 2^\omega\rangle\}$ and let h_0 be the identity function.

In step $n+1$ of the construction, first notice that the homeomorphism $h_n : 2^\omega \to 2^\omega$ fixes W and is monotone at every point of W by (g) and (i). Now, let $G = G^0_n \cup K_n \cup \{d^0_n\}$. The point d^0_n may or may not be already contained in G_n. We will describe the construction in the case that $d^0_n \notin G_n$ and d^0_n does not have immediate predecessors or successors; the other cases can be treated in an analogous way. Let $I_0 \in \mathscr{I}^0_n$ be such that $d^0_n \in I_0$.

If $J \in \mathscr{I}^0_n \setminus \{I_0\}$ then partition J into finitely many clopen intervals with endpoints contained in $Q \setminus G$. This is indeed possible because Q is dense in the dense open subset $J \setminus G$ of J and the endpoints of J are already in Q. Call this collection $\mathscr{I}(J, 0)$. Let $\mathscr{I}(J, 1) = \{h_n[K] : K \in \mathscr{I}(J, 0)\}$ and let $\alpha_J : \mathscr{I}(J, 0) \to \mathscr{I}(J, 1)$ be such that $\alpha_J(K) = h_n[K]$ for all $K \in \mathscr{I}(J, 0)$. Notice that by (g), $\mathscr{I}(J, 1)$ will also consist of clopen intervals.

Since $G^0_n \cup K_n$ intersects I_0 in a closed subset that does not contain d^0_n, and d^0_n does not have immediate predecessors or successors, there are $a_j \in Q_j \cap I_0$ for $j \in 2$ such that $d^0_n \in (a_0, a_1)$ and $[a_0, a_1] \cap (G^0_n \cup K_n) = \emptyset$. Let $b_j = h_n(a_j)$ for $j \in 2$; notice that $b_j \in Q_j$ for $j \in 2$ because h_n is an order isomorphism. Notice that this guarantees that $[a_0, a_1]$ and $[b_0, b_1]$ are clopen intervals. Choose an $e \in (b_0, b_1) \cap D_1$ that has no immediate predecessors or successors.

Partition $[a_0, a_1] \setminus \{d^0_n\} = \bigcup\{U^i_m : m < \omega,\ i \in 2\}$ and $[b_0, b_1] \setminus \{e\} = \bigcup\{V^i_m : m < \omega,\ i \in 2\}$ so that

(1) if $k < \omega$ and $i \in 2$, then U^i_m and V^i_m are non-empty clopen intervals of diameter $< 1/(n+2)$,

(2) if $k < \omega$, $p \in U^0_m$, $q \in U^0_{m+1}$, $r \in U^1_m$ and $s \in U^1_{m+1}$, then $p < q < d^0_n < s < r$, and

(3) if $k < \omega$, $p \in V^0_m$, $q \in V^0_{m+1}$, $r \in V^1_m$ and $s \in V^1_{m+1}$, then $p < q < e < s < r$.

Let

$$\mathscr{I}(I_0, 0, 0) = \{U^i_m : m < \omega,\ i \in 2\}, \qquad \mathscr{I}(I_0, 1, 0) = \{V^i_m : m < \omega,\ i \in 2\}.$$

Also, the set $I_0 \setminus [a_0, a_1]$ can be partitioned into finitely many clopen intervals with diameter $< 1/(n+2)$; call this collection $\mathscr{I}(I_0, 0, 1)$. Let $\mathscr{I}(I_0, 1, 1) = \{h_n[K] : K \in \mathscr{I}(I_0, 0, 1)\}$ and let $\mathscr{I}(I_0, i) = \mathscr{I}(I_0, i, 0) \cup \mathscr{I}(I_0, i, 1)$ for $i \in 2$.

Next we define a bijection $\beta : \mathscr{I}(I_0,0) \to \mathscr{I}(I_0,1)$ as follows: For $m < \omega$ and $i \in 2$ let $\beta(U^i_m) = V^i_m$, and for $K \in \mathscr{I}(I_0,0,1)$ let $\beta(K) = h_n[K]$.

Put

$$\mathscr{I}^j = \bigcup\{\mathscr{I}(J,j) : J \in \mathscr{I}^j_n\} \quad \text{for } j \in 2,$$
$$\alpha = \beta \cup \bigcup\{\alpha_J : J \in \mathscr{I}^0_n \setminus \{I_0\}\},$$

so that $\alpha : \mathscr{I}^0 \to \mathscr{I}^1$ is a bijection. We also define a homeomorphism $H : 2^\omega \to 2^\omega$. For each $m < \omega$ and $i \in 2$ we use property $(*)$ to find an order isomorphism $f^i_m : U^i_m \to V^i_m$. We let

$$H(x) = \begin{cases} h_n(x) & \text{if } x \in 2^\omega \setminus [a_0, a_1], \\ e & \text{if } x = d^0_n, \\ f^i_m(x) & \text{if } x \in U^i_m, \text{ for some } m < \omega,\ i \in 2. \end{cases}$$

Then $H : 2^\omega \to 2^\omega$ is a homeomorphism. Notice that $(\mathscr{I}^0, \mathscr{I}^1, \alpha)$ is a KR-cover for $(2^\omega \setminus G, 2^\omega \setminus H[G], H{\restriction}_G)$. Moreover, it is not hard to see that G, α, $\mathscr{I}^0$, $\mathscr{I}^1$ and H have the desired properties from the list (a)–(i) when $i = 0$, but perhaps these properties do not hold for $i = 1$. In order to finish step $n + 1$, we have to do another refinement to both the homeomorphism and the covers so that (a)–(i) hold when $i = 1$ too. This is entirely analogous to what we have done, so we leave the details to the reader.

Finally, we will show that there is a homeomorphism $h : 2^\omega \to 2^\omega$ that extends $h_n{\restriction}_{G^0_n}$ for every $n < \omega$. Assume that $x \notin \bigcup\{G^0_n : n < \omega\}$. For every $n < \omega$, let I^x_n be the unique element of $\mathscr{I}^0_n$ that contains x. It is easy to check that $\bigcap\{\alpha_n(I^x_n) : n < \omega\} = \{y\}$ for some $y \in 2^\omega$. Let $h(x) = y$. We leave the verification that this indeed defines a homeomorphism to the reader. In this way, by properties (a)–(c), we immediately deduce that $h[W] = W$ and $h[D_0] = D_1$. ∎

THEOREM 5.5. *Let $Y \subset 2^\omega \setminus Q$ be a saturated λ'-set. Then $X = \mathbb{A}(Y)$ is a compact linearly ordered* CDH *space of weight $|Y|$.*

Proof. To show that X is CDH, let D and E be countable dense subsets of $\mathbb{A}(Y)$.

Define $G = \{x \in Y : \langle x, i\rangle \in D \cup E \text{ for some } i \in 2\}$, $F = G^*$ and $Z = Y \setminus F$. Notice that Z is a saturated λ'-set in 2^ω and $Z \cap Q = \emptyset$. Consider the space $\mathbb{A}(F)$. Note that $\pi^Z_F : X \to \mathbb{A}(F)$ is the identity when restricted to $D \cup E$. Let $D' = \pi^Z_F[D]$ and $E' = \pi^Z_F[E]$; these are countable dense subsets of $\mathbb{A}(F)$.

We would like to find a homeomorphism of $\mathbb{A}(F)$ that can be lifted to a homeomorphism of X using Lemma 5.3. Thus, let us argue that we can use

Proposition 5.4 to find an appropriate autohomeomorphism of the Cantor set $\mathbb{A}(F)$. Let $W = Z \times 2$.

Clearly, Z is a λ'-set in 2^ω and from this it is easy to prove that W is a λ'-set in $\mathbb{A}(F)$. Notice that the set $(F \times 2) \cup (Q_0 \times \{1\}) \cup (Q_1 \times \{0\})$ is the set of all the points of $\mathbb{A}(F)$ with no immediate successor or no immediate predecessor along with the least and greatest elements of $\mathbb{A}(F)$, hence W does not contain any of these points. Also, notice that D' and E' do not intersect W.

Then it remains to prove that the condition $(*)$ in Proposition 5.4 holds. Let I and J be two clopen intervals of $\mathbb{A}(F)$. Then there are $a, b, c, d \in 2^\omega$ with $a < b$, $c < d$, $I = [\langle a, 1\rangle, \langle b, 0\rangle]$ and $J = [\langle c, 1\rangle, \langle d, 0\rangle]$. Notice that a and c have no immediate successor, and b and d have no immediate predecessor. Therefore, there are $\{s_n : n \in \mathbb{Z}\} \cup \{t_n : n \in \mathbb{Z}\} \subset 2^{<\omega}$ such that

(i) $(a, b) = \bigcup\{[s_n] : n \in \mathbb{Z}\}$,
(ii) $(c, d) = \bigcup\{[t_n] : n \in \mathbb{Z}\}$,
(iii) if $n, m \in \mathbb{Z}$, $n < m$, $x \in [s_n]$ and $y \in [s_m]$, then $x < y$, and
(iv) if $n, m \in \mathbb{Z}$, $n < m$, $x \in [t_n]$ and $y \in [t_m]$, then $x < y$.

Define $g : [a, b] \to [c, d]$ by

$$g(x) = \begin{cases} c & \text{if } x = a, \\ d & \text{if } x = b, \\ h_{s_n,t_n}(x) & \text{if } n \in \mathbb{Z} \text{ and } x \in [s_n], \end{cases}$$

for all $x \in [a, b]$. Then g is an order isomorphism and, since F and Z are saturated, $g[F \cap [a, b]] = F \cap [c, d]$ and $g[Z \cap [a, b]] = Z \cap [c, d]$. Define $f : I \to J$ as $f(\langle x, t\rangle) = \langle g(x), t\rangle$ for all $\langle x, t\rangle \in I$; this function is well-defined, and is, in fact, an order isomorphism such that $f[I \cap W] = J \cap W$.

Thus, by Proposition 5.4, there is a homeomorphism $h : \mathbb{A}(F) \to \mathbb{A}(F)$ such that $h[D'] = E'$ and h is monotone at each point of W. By Lemma 5.3, there is a homeomorphism $H : X \to X$ such that $\pi^Z_F \circ H = h \circ \pi^Z_F$, so $H[D] = E$. ∎

Recall that every set of size less than $\mathfrak{b}$ is a λ'-set. So:

Corollary 5.6. *There exists a linearly ordered, compact,* 0*-dimensional* CDH *space of weight κ for any κ of size less than $\mathfrak{b}$.*

More importantly, there is a λ'-set of size $\aleph_1$ (see [29]). Hence:

Theorem 5.7. *There exists a linearly ordered, compact,* 0*-dimensional* CDH *space of weight ω_1.*

The use of λ'-sets here is necessary. If $X \subset 2^\omega \setminus Q$ is a Baire space, by using the same method as in [2] it is possible to prove that the space $\mathbb{A}(X)$ is not CDH. Also, since in the Cohen model all λ-sets have cardinality ω_1 ([28, Theorem 22]) we do not have examples of compact CDH spaces of weight $\mathfrak{c}$ in ZFC.

QUESTION 5.8. Is there a compact CDH space of weight $\mathfrak{c}$ in ZFC?

QUESTION 5.9. Is there a non-metrizable CDH continuum?

Acknowledgements. We wish to thank S. V. Medvedev and the anonymous referee for commenting on the paper. They pointed out several inaccuracies that appeared in an earlier version of the paper and helped us to improve the paper in general.

The first-listed author would like to thank the Centro de Ciencias Matemáticas at Morelia for its support during his PhD studies there.

The second-listed author was supported by a PAPIIT grant IN 102311 and CONACyT grant 177758.

The third-listed author is pleased to thank the Centro de Ciencias Matemáticas at Morelia for generous hospitality and support.

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Received 7 September 2013;
in revised form 28 February 2014