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On topological groups with a first-countable remainder, II

A.V. Arhangel'skii [a,1], J. van Mill [b,*]

[a] *MGU and MPGU, Moscow, Russia*
[b] *KdV Institute for Mathematics, University of Amsterdam, Science Park 105-107, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands*

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ABSTRACT

We establish estimates on cardinal invariants of an arbitrary non-locally compact topological group G with a first-countable remainder Y. We show that the weight of G and the cardinality of Y do not exceed 2^ω. Moreover, the cardinality of G does not exceed 2^{ω_1}. These bounds are best possible as witnessed by a single topological group G. We also prove that every precompact topological group with a first-countable remainder is separable and metrizable. It is known that under Martin's Axiom and the negation of the Continuum Hypothesis, every σ-compact topological group with a first-countable remainder is metrizable. We show that under the Continuum Hypothesis, there is an example of a countable topological group which is not metrizable and has a first-countable remainder. Hence for countable groups, the question of whether the existence of a first-countable remainder is equivalent to being metrizable, is undecidable.

1. Introduction

By 'a space' we understand a Tychonoff topological space. By a *remainder* of a space X we mean the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. We follow the terminology and notation in [10].

A series of results on remainders of topological groups have been obtained in [2,4], and in [6]. They show that the remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. Of course, there is an important exception

* Corresponding author.
E-mail addresses: arhangel.alex@gmail.com (A.V. Arhangel'skii), j.vanMill@uva.nl (J. van Mill).

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to this rule: the case of locally compact topological groups. Indeed, every locally compact non-compact topological group has a remainder consisting of exactly one point. Thus, we will be interested only in the case of non-locally compact topological groups.

It was proved in Arhangel'skii [4] that if a non-locally compact topological group G has a remainder with a G_δ-diagonal, then both G and this remainder are separable metrizable spaces. It is a well known Theorem of Birkhoff and Kakutani that every first-countable topological group is metrizable (see, for example, [8]). One may expect that the first-countability of a remainder would also force the metrizability of the group itself, provided, of course, that the remainder is dense in the compactification. There are many partial results. For example, if G^ω has a first-countable remainder, then G is metrizable. See Arhangel'skii [3] for more details and references. The question was answered in the negative recently in Arhangel'skii and van Mill [7] where it was shown that there is a non-locally compact topological group G of size 2^{ω_1} and character ω_1 which has a first-countable remainder. It was also shown there that these cardinal characteristics of G are no surprise: *every* non-locally compact topological group G with a first-countable remainder has size at most 2^{ω_1} and character at most ω_1.

In this paper we continue these investigations. We were motivated by the result in Arhangel'skii [3] that under Martin's Axiom and the negation of the Continuum Hypothesis (abbreviated $\mathsf{MA}+\neg\mathsf{CH}$) the following holds: if G is a σ-compact topological group with a remainder of countable tightness, then either G is locally compact, or G is metrizable. We prove that under the Continuum Hypothesis (abbreviated CH), there is an example of a non-discrete countable topological group of character ω_1 with a first-countable remainder Y. Hence for countable non-discrete topological groups G the question whether G is metrizable if and only if G has a first-countable remainder, is undecidable. This example also shows that Problem 12 from Arhangel'skii [5] is undecidable. We also prove that every precompact non-locally compact group with a first-countable remainder is separable and metrizable (in fact, we prove a stronger result of which this is a corollary). Moreover, we continue our investigations on cardinal characteristics of non-locally compact topological groups G which have a compactification bG such that $Y = bG \setminus G$ is first-countable. We prove that for such G we have that $\chi(G) \le \omega_1$, $|Y| \le 2^\omega$, $w(bG) \le 2^\omega$ (hence $w(G) \le 2^\omega$) and $|bG| \le 2^{\omega_1}$. The example in Arhangel'skii and van Mill [7] demonstrates that these results are best possible.

2. Cardinal characteristics

We denote by $w(X)$ the *weight* of a space X, and $e(X)$ stands for the *extent* of X. Thus, $e(X)$ is the smallest infinite cardinal number τ such that $|A| \le \tau$, for every closed discrete subspace A of X. The *character* of a space X is denoted by $\chi(X)$.

Lemma 2.1. *For any topological group H, we have: $w(H) \le e(H)\chi(H)$.*

Proof. Fix an open neighbourhood U of the neutral e in H. Let us say that $A \subseteq H$ is U-discrete if $aU \cap A = \{a\}$, for each $a \in A$.

Claim 1. Every U-discrete A subset of H is closed and discrete in H.

Indeed, let A be U-discrete and let $p \notin A$. Let V be a symmetric open neighbourhood of e such that $V^2 \subseteq U$. Then $pV \cap A$ contains at most one element of A. For if there exist distinct $a_1, a_2 \in pV \cap A$, then we can pick $v_1, v_2 \in V$ such that $pv_1 = a_1$ and $pv_2 = a_2$. But then $a_1 = a_2(v_2^{-1}v_1) \in a_2U$ which contradicts $a_1 \ne a_2$. This clearly implies that A is closed, and since A is discrete, we are done.

Fix a base $\mathscr{P}$ of the space H at e such that $|\mathscr{P}| = \chi(H)$, and for each $U \in \mathscr{P}$ fix a maximal U-discrete subset A_U of H.

Claim 2. The union of all these A_U (where U runs over $\mathscr{P}$) is a dense subset M of H such that $|M| \leq e(H)\chi(H)$.

We only need to check that M is dense. But this is trivial. For assume it is not. Then there exist $p \in H$ and $U \in \mathscr{P}$ such that $pU \cap M = \emptyset$. But then $A_U \cup \{p\}$ is U-discrete, which contradicts the maximality of A_U.

Since $w(H) = d(H)\chi(H)$, see e.g. [8, Theorem 5.2.5 a)], we are done. □

Theorem 2.2. *Suppose that G is a non-locally compact topological group with a compactification bG such that its remainder Y is first-countable. Then:*

(a) $\chi(G) \leq \omega_1$ *and* $w(G) \leq 2^\omega$,
(b) $|Y| \leq 2^\omega$,
(c) $w(bG) \leq 2^\omega$ *and* $|bG| \leq 2^{\omega_1}$.

Proof. That $\chi(G) \leq \omega_1$ was proved in Arhangel'skii and van Mill [7]. We will next show that $w(G) \leq 2^\omega$.

Claim 3. $e(G) \leq 2^\omega$.

Assume the contrary. Then we can fix a closed discrete subspace A of G such that $|A| > 2^\omega$. Clearly, Y is dense in bG, and therefore, the space bG is first-countable at each $y \in Y$. Hence the closure of A in bG is a first-countable compactum B such that $|B| \geq |A| > 2^\omega$, a contradiction (Arhangel'skii [1]).

Since we already know that $\chi(G) \leq \omega_1$, it follows from Claim 1 and Lemma 2.1 that $w(G) \leq 2^\omega$. Hence (a) holds.

To prove (b), fix a dense subset S of G such that $|S| \leq 2^\omega$. This is possible by what we just proved. As we observed earlier, the space bG is first-countable at each $y \in Y$. Hence, we can reach every $y \in Y$ by a sequence η_y in S converging to y. Since the cardinality of the set of all such sequences does not exceed 2^ω, we conclude that $|Y| \leq 2^\omega$.

It remains to prove (c). Observe that bG has a network $\mathscr{P}$ such that $|\mathscr{P}| \leq 2^\omega$. Therefore, $w(bG) \leq 2^\omega$, since bG is compact.

Since $\chi(G) \leq \omega_1$ by (a) and G is dense in bG, we have: the character of bG at every $x \in G$ does not exceed ω_1. A similar statement holds for the points of Y. Therefore, the character of bG at any point is not greater than ω_1. Since bG is compact, we consequently have: $|bG| \leq 2^{\omega_1}$. □

3. Precompact topological groups

A space is said to be *ω-bounded* (*strongly ω-bounded*) if the closure of every countable (σ-compact, respectively) subset is compact.

Lemma 3.1. *Suppose that X is a nowhere locally compact space with a remainder Y which is not ω-bounded. Then no remainder of X is ω-bounded.*

Proof. Let bX be a compactification of X such that $Y = bX \setminus X$ contains a countable subset A such that $\overline{A} \cap X \neq \emptyset$. Let $f\colon \beta X \to bX$ be a continuous map that restricts to the identity on X. For every $a \in A$ pick $b_a \in \beta X \setminus X$ such that $f(b_a) = a$. Then clearly if $B = \{b_a : a \in A\}$, then $\overline{B} \cap X \neq \emptyset$. Now let b_1X be an arbitrary compactification of X, and let $g\colon \beta X \to b_1X$ be a continuous surjection which restricts to the identity on X. Put $C = g(B)$. Then, clearly, $\overline{C} \cap X \neq \emptyset$. □

Proposition 3.2. *Suppose that bX is a compactification of a nowhere locally compact space X such that the following two conditions are satisfied:*

(1) *bX is separable.*
(2) *The remainder $Y = bX \setminus X$ has a dense subspace homeomorphic to X.*

Then the remainder Y is not ω-bounded.

Proof. Since X is nowhere locally compact, the subspace Y is also dense in bX and is not compact. Fix a countable subset A of bX such that A is dense in bX. Put $A_X = A \cap X$.

Case 1. A_X is dense in X.

Using condition (2), we fix a subspace H of $Y = bX \setminus X$ such that H is dense in Y and H is homeomorphic to X. Then H is separable, so that we can fix a countable dense subspace C of H. The closure of C in Y is Y and hence, is not compact. Thus, in Case 1 the space Y is not ω-bounded.

Case 2. A_X is not dense in X.

Hence, there exists a nonempty open subset U of bX such that the closure F of U in bX does not intersect A_X. Hence, the set $A_U = A \cap U$ is contained in Y. Since A is dense in bX, the set A_U is dense in U. Thus, the closure of the countable subset A_U of Y in bX is the compactum F. However, F is not contained in Y, since otherwise we would have $U \subset Y$. This is a contradiction, since U is open in bX and Y is, clearly, nowhere locally compact. Therefore, the closure of A_U in Y is not compact (since it is dense in F but is not F). Hence Y is not ω-bounded. $\square$

We will need the following result of Efimov which follows from Corollary 1.3 and Theorem 1.4 in [9].

Theorem 3.3. *(B.A. Efimov [9]) Let τ be an infinite cardinal number. Then we have: if the π-character of a dyadic compactum X is $\leq \tau$ for each $x \in M$, where $M \subseteq X$ and M is dense in X, then the weight of X is $\leq \tau$.*

Proposition 3.4. *Suppose that X is a nowhere locally compact space such that the character of X at every point does not exceed 2^ω. Furthermore, suppose that bX is a dyadic compactification of X such that the remainder $Y = bX \setminus X$ contains a dense subspace homeomorphic to X. Then Y is not ω-bounded.*

Proof. Clearly, the character of bX at any $x \in X$ does not exceed 2^ω. Since bX is a dyadic compactum and X is dense in bX, it follows from Theorem 3.3 that the weight of bX is not greater than 2^ω. Since bX is a dyadic compactum, we conclude that bX is separable. It remains to apply Proposition 3.2. $\square$

Theorem 3.5. *Suppose that X is a homogeneous nowhere locally compact space with a remainder of countable π-character and that the character of X at every point does not exceed 2^ω. Furthermore, suppose that bX is a dyadic compactification of X such that the remainder $Y = bX \setminus X$ contains a dense subspace homeomorphic to X. Then X and bX are separable and metrizable.*

Proof. Let b_1X be a compactification of X such that $Y = b_1X \setminus X$. Observe that it follows from Proposition 3.4 that $bX \setminus X$ is not ω-bounded. Hence by Lemma 3.1 this implies that $b_1X \setminus X$ is not ω-bounded. Let A be a countable subset of $b_1X \setminus X$ containing the point $x_0 \in X$ in its closure. Since every $a \in A$ clearly has countable π-character in b_1X and A is countable, it follows that x_0 has countable π-character in b_1X.

Since X is dense in b_1X, it follows that the π-character of X at x_0 is countable. Since X is homogeneous, we conclude that the π-character of X is countable at every $x \in X$. Since X is also dense in the dyadic compactification bX, it follows that the π-character of bX is countable at every $x \in X$. Now Theorem 3.3 implies that bX is metrizable and so we are done. □

Proposition 3.6. *Suppose that G is a topological group such that the Souslin number of G is countable, and let Y be an arbitrary remainder of G. Then the following two conditions are equivalent:*

(a) *Y is first-countable;*
(b) *Every $y \in Y$ is a G_δ-point in Y.*

Proof. Indeed, Y is either pseudocompact or Lindelöf, by the dichotomy theorem in Arhangel'skii [6]. If Y is pseudocompact then, clearly, (a) and (b) are equivalent. It remains to consider the case when Y is Lindelöf. Since the Souslin number of G is countable, G and Y are paracompact p-spaces by Arhangel'skii [2, Theorem 4.17]. Hence, Y is a Lindelöf p-space and so (a) and (b) are equivalent. Thus, we are done. □

This leads us to the main result in this section.

Theorem 3.7. *If a non-locally compact precompact topological group G has a remainder Y such that every $y \in Y$ is a G_δ-point in the space Y, then G is separable and metrizable*

Proof. Let bG be a compactification of G such that the pseudo-character $Y = bG \setminus G$ is countable. Since the Souslin number of G is countable, G being precompact, by Proposition 3.6 it follows that Y is first-countable. We already know that the character of G is not greater than ω_1 (Theorem 2.2(a)). The Rajkov completion of G is a compact topological group b_1G (see e.g. [8]). Obviously, the remainder $b_1G \setminus G$ contains a translate of G and hence a dense topological copy of G. Since every compact topological group is dyadic [8, §4.1], we can apply Theorem 3.5. □

4. The example

All Abelian groups are written additively.

Let G be an Abelian group. We say that G is *Boolean* provided that each element has order at most 2. We write G additively, and denote its neutral element by e. If G is Boolean, and $F \subseteq G$ is finite, then so is $\langle\langle F\rangle\rangle$.

A nonempty subset A of a Boolean group G is *independent* if for every finite nonempty subset F of A we have $\Sigma F \neq e$. Observe that if A and B are disjoint subsets of G such that $A \cup B$ is independent, then $\langle\langle A\rangle\rangle \cap \langle\langle B\rangle\rangle = \{e\}$. Also observe that if $x_n \in G$ for every $n < \omega$ is chosen such that $x_n \notin \langle\langle\{x_i : i < n\}\rangle\rangle$, then $X = \{x_n : n < \omega\}$ is independent.

Let G be a Boolean topological group. We say that G is *linear* provided that the neutral element of G has a neighbourhood base consisting of subgroups of G.

Lemma 4.1. *Let G be a first-countable, countable, dense-in-itself linear Boolean topological group with compactification bG. If $K \subseteq bG \setminus G$ is σ-compact, then there is a dense-in-itself subgroup H of G having the following properties:*

(1) *G/H is infinite,*
(2) *H is not open,*
(3) *for every finite $F \subseteq G$, $\overline{F+H} \cap K = \emptyset$ (here closure means closure in bG).*

Proof. Since G is countable and first-countable, it has a countable base. By observing that the subgroup generated by a finite subset of G is finite, and the fact that G is dense-in-itself, it is easy to construct two disjoint dense subsets D and E such that $D \cup E$ is independent. Our subgroup H will be a subgroup of $\langle\!\langle D \rangle\!\rangle$. Since $\langle\!\langle D \rangle\!\rangle \cap E = \emptyset$ and E is dense, it follows that H is not open and moreover that G/H is infinite.

Let $f\colon \omega \to \omega \times \omega$ be a surjective map. It will be convenient to denote $f(n)$ by $\langle n_0, n_1 \rangle$.

Enumerate the finite nonempty subsets of G as $\{F_n : n < \omega\}$. Write K as $\bigcup_{n<\omega} K_n$, where K_n is compact for every n. Finally, let $(U_n)_n$ be an open neighbourhood base at $e \in G$ consisting of subgroups of G such that $U_{n+1} \subseteq U_n$ for every n. By recursion on n, we pick an open neighbourhood V_n of e in G and point $p_n \in V_n \cap D$, such that:

(1) V_n is a subgroup of G,
(2) $V_n \subseteq U_n \cap \bigcap_{m<n} V_m$,
(3) $\overline{F_{n_0} + \langle\!\langle \{p_0, \ldots, p_{n-1}\} \rangle\!\rangle + V_n} \cap K_{n_1} = \emptyset$ (here closure means closure in bG).

The construction is a triviality. Indeed, assume that we are at step n of the construction. Put $F = F_{n_0} + \langle\!\langle \{p_0, \ldots, p_{n-1}\} \rangle\!\rangle$. Since F is finite, and K_{n_1} is a compact subset of $bG \setminus G$, for every $x \in F$ we may pick an open neighbourhood V_x of e in G such that $\overline{x + V_x} \cap K_{n_1} = \emptyset$ (again, closure means closure in bG). We may assume that for every $x \in F$, V_x is a subgroup of G. Now put $V_n = U_n \cap \bigcap_{x \in F} V_x \cap \bigcap_{m<n} V_m$, and let p_n be an arbitrarily chosen point from $V_n \cap D$. This completes the recursion.

Now put $H = \langle\!\langle \{p_n : n < \omega\} \rangle\!\rangle$; we claim that H is as required. As we observed before, H is not open. Moreover, H is dense-in-itself since the sequence $(p_n)_n$ converges to e and is contained in $G \setminus \{e\}$.

Now consider arbitrary $n, m < \omega$. We will show that $\overline{F_n + H} \cap K_m = \emptyset$. Indeed, let $i < \omega$ be such that $f(i) = \langle n, m \rangle$. Observe that by construction we have that

$$\overline{F_n + \langle\!\langle \{p_0, \ldots, p_{i-1}\} \rangle\!\rangle + V_i} \cap K_m = \emptyset$$

Since $H \subseteq \langle\!\langle \{p_0, \ldots, p_{i-1}\} \rangle\!\rangle + V_i$, this clearly gives us what we want. □

Corollary 4.2. *Let G be a countable dense-in-itself linear Boolean topological group with first-countable compactification bG. If $K \subseteq bG \setminus G$ is σ-compact, then there are a linear dense-in-itself topology τ on G, a first-countable compactification $c(G, \tau)$ of (G, τ) and a continuous surjection $f\colon c(G, \tau) \to bG$ such that:*

(1) *the original topology on G is strictly contained in τ,*
(2) *f restricts to the identity $(G, \tau) \to G$,*
(3) *if $x \in K$, then $f^{-1}(\{x\})$ is a single point.*

Proof. Let H be the subgroup of G from Lemma 4.1. We endow G/H with the discrete topology, and let $\alpha(G/H) = (G/H) \cup \{\infty\}$ denote its Alexandroff one-point compactification. Let $\phi\colon G \to G/H$ denote the canonical homomorphism. Consider the graph

$$\Gamma = \{\langle x, \phi(x) \rangle : x \in G\}$$

of ϕ in $bG \times \alpha(G/H)$. We identify G and Γ. Observe that the subspace topology τ that Γ (=G) inherits from $bG \times \alpha(G/H)$ is simply the topology we get from G by declaring its subgroup H to be open. This is a linear topology on G that strictly contains the original topology. Clearly, τ is dense-in-itself since H is. Let X denote the closure of Γ in the compact space $bG \times \alpha(G/H)$. Then X is a first-countable compactification $c(G, \tau)$ of (G, τ). Let $f\colon X \to bG$ denote the restriction to X of the projection $bG \times \alpha(G/H) \to bG$. By our identifications, f restricts to the identity on (G, τ). It suffices to check that if $x \in K$, then $f^{-1}(\{x\})$

is a single point. We claim that $f^{-1}(\{x\}) = \{\langle x, \infty\rangle\}$. First observe that by compactness, $f^{-1}(\{x\}) \neq \emptyset$. Clearly, $f^{-1}(\{x\}) \subseteq \{x\} \times \alpha(G/H)$. Striving for a contradiction, assume that for some $p \in G$ we have that $\langle x, \phi(p)\rangle \in X$. By construction, $x \notin \overline{p+H}$. Since $bG \times \{\phi(p)\}$ is clopen in $bG \times \alpha(G/H)$, there is a sequence $(g_i)_i$ in G such that the sequence $(\langle g_i, \phi(p)\rangle)_i$ is entirely contained in Γ and converges to $\langle x, \phi(p)\rangle$. Hence $\phi(g_i) = \phi(p)$ for every i and $g_i \to x$. But this is a contradiction since from this we get that $g_i \in p+H$ for every i and $x \notin \overline{p+H}$. $\square$

Assume CH throughout. We let G be a countable dense subgroup of the Cantor group 2^ω. By transfinite induction on $\alpha \leq \omega_1$ we will construct a linear dense-in-itself group topology τ_α on G. We will denote the space with underlying set G and topology τ_α by G_α. We emphasize that we think of all the spaces G_α as having the same underlying set, namely, G. Along the way we will also construct a first-countable compactification $b_\alpha G_\alpha$ of G_α and for every $\beta \leq \alpha$ a continuous function $f^\alpha_\beta : b_\alpha G_\alpha \to b_\beta G_\beta$ such that (among other things) the following conditions are satisfied:

(K_1) for every $\beta < \alpha$, τ_β is strictly contained in τ_α,
(K_2) for every $\beta \leq \alpha$, f^α_β restricts to the identity $G_\alpha \to G_\beta$,
(K_3) if α is a limit ordinal, then τ_α is the supremum of the topologies $\{\tau_\beta : \beta < \alpha\}$ and $b_\alpha G_\alpha = \varprojlim\{b_\beta G_\beta, f^\beta_\xi\}$.

The example we are looking for will be $b_{\omega_1} G_{\omega_1}$. Observe that (K_1) implies that G_{ω_1} is not first-countable. To ensure that $b_{\omega_1} G_{\omega_1} \setminus G_{\omega_1}$ is first countable, we use CH for bookkeeping.

Let $\tau : \omega_1 \to \omega_1 \times \omega_1$ be a surjective map such that $\tau(\alpha) = \langle \beta, \xi\rangle$ implies that $\beta \leq \alpha$. By CH, for every $\alpha \leq \omega_1$ we enumerate $b_\alpha G_\alpha \setminus G_\alpha$ by $\{x^\alpha_\xi : \xi < \omega_1\}$. Along the way we will construct an F_σ-subset S_α of $b_\alpha G_\alpha$ such that

(K_4) $S_\alpha \subseteq b_\alpha G_\alpha \setminus G_\alpha$,
(K_5) $\bigcup_{\beta<\alpha} (f^\alpha_\beta)^{-1}(S_\beta) \subseteq S_\alpha$,
(K_6) if $\tau(\alpha) = \langle \eta, \xi\rangle$, then $(f^\alpha_\eta)^{-1}(\{x^\eta_\xi\}) \subseteq S_\alpha$,
(K_7) if $\beta < \alpha$ and $x \in S_\beta$, then $(f^\alpha_\beta)^{-1}(\{x\})$ is a single point.

Observe that τ_{ω_1} is dense-in-itself by (K_1) and the fact that τ_α is dense-in-itself for every $\alpha < \omega_1$. Hence G_{ω_1} is nowhere locally compact. To check that $R = b_{\omega_1} G_{\omega_1} \setminus G_{\omega_1}$ is first-countable, take an arbitrary $p \in R$. There exists $\eta < \omega_1$ such that $f^{\omega_1}_\eta(p) \notin G_\eta$. Pick $\xi < \omega_1$ such that $f^{\omega_1}_\eta(p) = x^\eta_\xi$. Let $\tau(\alpha) = \langle \eta, \xi\rangle$, and observe that $\eta \leq \alpha$. By (K_6) we have that $(f^\alpha_\eta)^{-1}(\{x^\eta_\xi\}) \subseteq S_\alpha$. Hence by (K_7) we get that $(f^{\omega_1}_\alpha)^{-1}(\{f^{\omega_1}_\alpha(p)\})$ is a single point. Hence the first-countability of $b_\alpha G_\alpha$ gives us that $b_{\omega_1} G_{\omega_1}$ is first-countable at p.

It remains to perform the construction. But this is a triviality. At limit stages, everything is determined. And at successor stages, we simply need to apply Corollary 4.2.

In Arhangel'skii [5, Problem 12] the following was asked: Suppose that X is a Čech-complete space of countable tightness such that ω_1 is a precaliber of X. Is X separable? What if, in addition, X has a countable remainder? The hope was to get a positive answer in ZFC. For compacta the answer is 'yes', and under MA+¬CH the answer is 'yes' as well (Shapirovskiĭ [11]). We claim that the space $Y = b_{\omega_1} G_{\omega_1} \setminus G_{\omega_1}$ that was constructed in this section is a first-countable counterexample under CH. Hence Problem 12 from [5] is undecidable. To see that Y is not separable, simply observe that by construction Y is ω-bounded and not compact, hence not separable. Alternatively, apply Arhangel'skii [3, Theorem 3.12].

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