On nowhere first-countable compact spaces with countable \( \pi \)-weight

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Abstract. The minimum weight of a nowhere first-countable compact space of countable \( \pi \)-weight is shown to be \( \kappa_B \), the least cardinal \( \kappa \) for which the real line \( \mathbb{R} \) can be covered by \( \kappa \) many nowhere dense sets.

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1. Introduction

All spaces under discussion are Tychonoff.

In [4], the author showed that there is a (naturally defined) compact space \( X \) which is (topologically) homogeneous under \( \text{MA} + \neg \text{CH} \) but not under \( \text{CH} \). This space has countable \( \pi \)-weight, character \( \omega_1 \) and weight \( c \). It is an open problem whether there can be a compact nowhere first-countable homogeneous space of countable \( \pi \)-weight and weight less than \( c \). This cannot be done by a straightforward modification of the method in [4] since from Juhász [2, Theorem 5] it follows that under \( \text{MA} \), every compact space of countable \( \pi \)-weight and weight less than \( c \) is somewhere first-countable. Hence a homogeneous compactum of countable \( \pi \)-weight and weight less than \( c \) is first-countable under \( \text{MA} \) ([4, Theorem 1.5]). Let \( \lambda \) be the minimum weight of a nowhere first-countable compact space of countable \( \pi \)-weight. Clearly, \( \omega_1 \leq \lambda \leq c \). The aim of this note is to show that \( \lambda \) is equal to \( \kappa_B \), the least cardinal \( \kappa \) for which the real line \( \mathbb{R} \) can be covered by \( \kappa \) many nowhere dense sets. Hence there exists a nowhere first-countable compact space of weight \( \kappa_B \) and countable \( \pi \)-weight. Whether such a space can be homogeneous while \( \kappa_B < c \) remains an open problem.

2. Preliminaries

Our basic references are Miller [5], Juhász [1] and Kunen [3].

For every space \( X \), define \( \kappa_B(X) \) to be the least cardinal \( \kappa \) such that \( X \) can be covered by \( \kappa \) many nowhere dense (in \( X \)) subsets of \( X \). In Miller [5, Lemma 1] it is shown that for every crowded Polish space \( X \) we have \( \kappa_B(X) = \kappa_B \).
Let $\text{MA}_\kappa(\text{countable})$ denote the statement that for any countable partial order $\mathbb{P}$ and family $\mathcal{F}$ of dense subsets of $\mathbb{P}$, if $|\mathcal{F}| < \kappa$, then there exists a $\mathbb{P}$-generic filter $G$ over $\mathcal{F}$. It is well-known, see Miller [5, Lemma 2], that $\kappa_B$ is the greatest $\kappa$ for which $\text{MA}_\kappa(\text{countable})$ holds.

The proof of the following result is standard and is included for the sake of completeness.

**Lemma 2.1 (MA$_{\kappa^+}(\text{countable})$).** Let $X$ be a crowded space of weight at most $\kappa$ and of countable $\pi$-weight. Assume that $D$ is a nowhere dense subset of $X$. Then there exist disjoint open sets $U$ and $V$ in $X$ such that $D \subseteq U \cap V$.

**Proof:** Let $\mathcal{U}$ be a countable $\pi$-base for $X$. Put

$$\mathbb{P} = \{\langle p, q \rangle : (p, q \in [\mathcal{U}]^{<\omega}) \land (\bigcup p \cap \bigcup q = \emptyset) \land (\bigcup p \cup \bigcup q \subseteq X \setminus D)\}.$$  

Order $\mathcal{P}$ in the natural way by $\langle p_0, q_0 \rangle \leq \langle p_1, q_1 \rangle$ iff $\bigcup p_1 \subseteq \bigcup p_0$ and $\bigcup q_1 \subseteq \bigcup q_0$. Let $\mathcal{V}$ be an open base for $X$ such that $|\mathcal{V}| \leq \kappa$. Let $W = \{V \in \mathcal{V} : V \cap D \neq \emptyset\}$. For every $W \in \mathcal{W}$, put

$$W^* = \{\langle p, q \rangle \in \mathcal{P} : (\bigcup p \cap W \neq \emptyset) \land (\bigcup q \cap W \neq \emptyset)\}.$$  

We claim that $W^*$ is dense in $\mathcal{P}$. To prove this, take an arbitrary $\langle p, q \rangle \in \mathcal{P}$. By assumption, $(\bigcup p \cup \bigcup q) \cap D = \emptyset$ and $W \cap D \neq \emptyset$. Since $X$ is crowded, there exist $U, V \in \mathcal{W}$ such that

$$W \cup U \subseteq W \setminus (D \cup p \cup q).$$  

Hence $p' = p \cup U$ and $q' = q \cup V$ belong to $\mathcal{P}$ and, clearly, $\langle p', q' \rangle \leq \langle p, q \rangle$. By our assumptions, there is a filter $F$ in $\mathbb{P}$ such that for every $W \in \mathcal{W}$ we have $W^* \cap F \neq \emptyset$. Put

$$U = \bigcup\{p : (\exists q \in [\mathcal{U}]^{<\omega})(\langle p, q \rangle \in F)\},$$

and

$$V = \bigcup\{q : (\exists p \in [\mathcal{U}]^{<\omega})(\langle p, q \rangle \in F)\},$$

respectively. Then $U$ and $V$ are clearly as required. \qed

It was shown in Miller [5, Theorem 1] that $\kappa_B$ has uncountable cofinality. (Interestingly, Shelah [6] showed that the measure analogue of this may fail.)

### 3. Proofs

Theorem 5 and Lemma 4 in Juhász [2] imply that if $X$ is countably compact, nowhere first-countable, and has a dense set of points of countable $\pi$-character, then $w(X) \geq \kappa_B$. For completeness sake, we include a simple proof of a weaker result which suffices for our purposes.
Lemma 3.1 (Juhász [2]). Let $\kappa$ be a cardinal for which there exists a compact nowhere first-countable space $X$ with countable $\pi$-weight and weight $\kappa$. Then $\kappa_B \leq \kappa$.

**Proof:** Let $\mathcal{B}$ be an open base for $X$ such that $|\mathcal{B}| = \kappa$. Moreover, let $\mathcal{U}$ be a countable $\pi$-base for $X$. For every $B \in \mathcal{B}$, put

$$S(B) = \overline{B} \setminus \bigcup \{U \in \mathcal{U} : U \subseteq B\}.$$ 

Since $\mathcal{U}$ is a $\pi$-base, it is clear that for every $B \in \mathcal{B}$ the set $S(B)$ is a nowhere dense closed subset of $X$.

We claim that $\bigcup_{B \in \mathcal{B}} S(B) = X$. To this end, pick an arbitrary $x \in X$. The collection $\mathcal{V} = \{U \in \mathcal{U} : x \in U\}$ is countable. Since $\chi(x, X) > \omega$, there exists $B \in \mathcal{B}$ which contains no $U \in \mathcal{V}$. Hence for every $U \in \mathcal{U}$ which is contained in $B$ it follows that $x \notin U$, i.e., $x \in S(B)$.

There is an irreducible continuous surjection $f : X \to Y$, where the weight of $Y$ is countable. Hence $Y$ is covered by the collection of nowhere dense closed sets

$$\{f(S(B)) : B \in \mathcal{B}\}.$$ 

Clearly $Y$ is crowded since $X$ is. From this we conclude that $\kappa_B \leq \kappa$, as required. \hfill \Box

If $X$ is a compact space and $A$ and $B$ are closed subsets of $X$ such that $A \cup B = X$, then $X(A, B)$ denotes the topological sum $(\{0\} \times A) \cup (\{1\} \times B)$ of $A$ and $B$ and $\pi_{A, B} : X(A, B) \to X$ is defined by

$$\pi_{A, B}(t) = \begin{cases} a & (t = (0, a), a \in A), \\ b & (t = (1, b), b \in B). \end{cases}$$

Observe that $t \in A \cap B$ if and only if $|\pi_{A, B}^{-1}(\{t\})| \geq 2$ if and only if $|\pi_{A, B}^{-1}(\{t\})| = 2$.

**Lemma 3.2.** $\pi_{A, B} : X(A, B) \to X$ is irreducible if and only if $A \setminus B$ is dense in $A$ and $B \setminus A$ is dense in $B$.

**Proof:** It will be convenient to denote $\{0\} \times A$ and $\{1\} \times B$ by $A'$ and $B'$, respectively. Assume first that $C \subseteq X(A, B)$ is a proper closed set such that $\pi_{A, B}(C) = X$. We may assume without loss of generality that $U = A' \setminus C$ is nonempty. Put $V = \pi_{A, B}(U)$. Then $V$ is a nonempty relatively open subset of $A$. Moreover, if $x \in V$, then there exists $(1, b) \in B'$ such that $B \ni b = \pi_{A, B}((1, b)) = x$. As a consequence, $V \subseteq B$. There is an open subset $W$ in $X$ such that $W \cap A = V$. Since $V \subseteq B$, obviously $W \subseteq B$. Hence $A \setminus B$ is not dense in $A$.

For the converse implication, assume without loss of generality that $A \setminus B$ is not dense in $A$. Then $(\{0\} \times A \setminus B) \cup (\{1\} \times B)$ is a proper closed subset of $X_{A, B}$ which is mapped onto $X$ by $\pi_{A, B}$. \hfill \Box
Lemma 3.3. There is a nowhere first-countable compact space of weight $\kappa_B$ and countable $\pi$-weight.

Proof: Let $\tau : \kappa_B \to \kappa_B$ be a surjection every fiber of which has size $\kappa_B$. Moreover, let $\{D_\alpha : \alpha < \kappa_B\}$ be a family of closed and nowhere dense subsets of $2^\omega$ covering $2^\omega$. Our space will be the inverse limit $X_{\kappa_B}$ of a continuous inverse system $\{X_\alpha, \beta \leq \alpha < \kappa_B, f_\alpha^\beta\}$ such that $X_0 = 2^\omega$ and for every $\alpha < \kappa_B$ and $\beta \leq \alpha$,

1. $X_\alpha$ is a compact space of weight at most $|\alpha|\cdot\omega$,
2. $f_\alpha^\beta : X_\alpha \to X_\beta$ is a continuous, irreducible surjection,
3. there are closed sets $A_\alpha$ and $B_\alpha$ in $X_\alpha$ such that
   a. $A_\alpha \cup B_\alpha = X_\alpha$,
   b. $A_\alpha \cap B_\alpha \supseteq (f_\alpha^0)^{-1}(D_\tau(\alpha))$,
   c. $A_\alpha \setminus B_\alpha$ and $B_\alpha \setminus A_\alpha$ are dense in $A_\alpha$ respectively $B_\alpha$,
   d. $X_{\alpha+1} = X_\alpha(A_\alpha, B_\alpha)$ and $f_\alpha^{\alpha+1} = \pi_{A_\alpha, B_\alpha}$.

The construction of this inverse sequence is a triviality by a repeated application of Lemmas 2.1 and 3.2. The only thing left to verify is that $X_{\kappa_B}$ has weight $\kappa_B$ and is nowhere first-countable.

Striving for a contradiction, assume that $X_{\kappa_B}$ is first-countable at $t$. Since $\kappa_B$ has uncountable cofinality (see §2), there exists $\beta < \kappa_B$ such that

(†) $(f_\beta^{\kappa_B})^{-1}(\{f_\beta^{\kappa_B}(t)\}) = \{t\}.$

Let $\xi < \kappa_B$ be such that $f_0^{\kappa_B}(t) \in D_\xi$. Pick $\alpha > \beta$ so large that $\tau(\alpha) = \xi$. Then clearly

$$\|(f_\alpha^{\alpha+1})^{-1}(\{f_\alpha^{\kappa_B}(t)\})\| = 2,$$

which contradicts (†).

That the weight of $X_{\kappa_B}$ is at most $\kappa_B$ follows by construction. And that it has weight at least $\kappa_B$ is a consequence of Lemma 3.1 and the fact that it is nowhere first-countable. Observe that $X_0$ has countable weight, and that $X_{\kappa_B}$ admits a continuous, irreducible map onto $X_0$. Hence $X_{\kappa_B}$ has countable $\pi$-weight. \[\square\]

4. Questions

(1) Is there in ZFC a homogeneous nowhere first-countable compact space of countable $\pi$-weight and weight $\kappa_B$?

(2) What are the cardinals of the form $w(X)$, where $X$ is a nowhere first-countable compactum of countable $\pi$-weight?

(Let $\Pi$ denote this set of cardinals. We showed that $\kappa_B \in \Pi$. Moreover, $\mathfrak{c} \in \Pi$. To check this, let $X$ be the absolute of the unit interval. Then $X$ has countable $\pi$-weight, is nowhere first-countable, and has weight $\mathfrak{c}$ (since it contains a copy of $\beta\omega$). We do not know whether there can be a cardinal $\kappa \in \Pi \setminus \{\kappa_B, \mathfrak{c}\}$.)
A natural question is whether there can be a $\kappa$ in $\Pi$ of countable cofinality. This question may have a very simple answer. Indeed, assume that there is a sequence

$$\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$$

in $\Pi$. For every $n$ let $X_n$ be a witness of the fact that $\kappa_n \in \Pi$. Then $X = \prod_{n<\omega} X_n$ is a witness that $\kappa = \sup_{n<\omega} \kappa_n \in \Pi$.

**References**


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