Countable Dense Homogeneous Rimcompact Spaces and Local Connectivity

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**Abstract.** We prove that every nonmeager connected Countable Dense Homogeneous space is locally connected under some additional mild connectivity assumption. As a corollary we obtain that every Countable Dense Homogeneous connected rimcompact space is locally connected.

1. Introduction

All spaces under discussion are separable metric.

A space $X$ is Countable Dense Homogeneous (abbreviated: CDH) provided that for all countable dense subsets $D$ and $E$ of $X$ there is a homeomorphism $f: X \to X$ such that $f(D) = E$. For more information on this concept, see Arhangel’skii and van Mill [2]. Bennett [3] proved that every connected CDH-space is homogeneous.

In 1972, Fitzpatrick [6] proved that every locally compact, connected CDH-space is locally connected. Fitzpatrick and Zhou [7] asked in 1992 whether every Polish, connected CDH-space is locally connected. This problem is one of the few problems in [7] that is still open and was the motivation for the current investigations.

For a space $X$ and $x \in X$ we let $Q(x,X)$ denote the quasi-component of $x$ in $X$. That is, $Q(x,X)$ is the intersection of all clopen subsets of $X$ that contain $x$. Observe that if $x \in X$, and $X$ is a subspace of $Y$, then $Q(x,X) \subseteq Q(x,Y)$.

**Theorem 1.1.** Let $X$ be a nonmeager connected CDH-space and assume that for some point $x$ in $X$ we have that for every open neighborhood $W$ of $x$, $Q(x,W) \setminus \{x\}$ is nonempty. Then $X$ is locally connected.

**Corollary 1.2.** Every rimcompact connected CDH-space is locally connected.

This corollary generalizes the result of Fitzpatrick just quoted. Observe that we do not require our space to be nonmeager.
2. Preliminaries

As usual, for a subset $U$ of a space $X$, we put $\text{Fr } U = \overline{U} \setminus \text{Int } U$; it is called the boundary of $U$.

A space $X$ is meager if it can be expressed as a countable union of nowhere dense sets. Clearly, every Baire space (see below) is nonmeager.

A space $X$ is called rimcompact if there exists an open base $\mathcal{B}$ for $X$ such that Fr $B$ is compact for each $B \in \mathcal{B}$. For more information on this concept, see Aarts and Nishiura [1].

For a space $X$ we let $\mathcal{H}(X)$ denote its group of homeomorphisms. If $A \subseteq X$, then $\mathcal{H}(X; A)$ denotes $\{f \in \mathcal{H}(X) : h \text{ restricts to the identity on } A\}$.

We will need the following result.

Proposition 2.1 (van Mill [11, Proposition 3.1]). Let $X$ be CDH. If $F \subseteq X$ is finite and $D, E \subseteq X \setminus F$ are countable and dense in $X$, then there is an element $f \in \mathcal{H}(X; F)$ such that $f(D) \subseteq E$.

A space $X$ is a $\lambda$-set if every countable subspace is $G_\delta$. It was shown by Fitzpatrick and Zhou [7, Theorem 3.4] that every meager CDH-space is a $\lambda$-set. There are such CDH-spaces, see [5] and [8].

A space is Polish if it has an admissible complete metric. A space is Baire if the intersection of any countable family of dense open sets in the space is dense. A space is analytic if it is a continuous image of the space of irrational numbers.

3. Proof of Theorem 1.1

Let $X$ be any nonmeager CDH-space which is connected and contains a point $x$ such that for every open neighborhood $W$ of $x$, $Q(x, W) \setminus \{x\}$ is nonempty. By Bennett [3], $X$ is homogeneous. Hence this property of the point $x$ is shared by all points.

Lemma 3.1. For every open neighborhood $V$ of a point $x$ in $X$ we have that the interior of $Q(x, V)$ is nonempty.

Proof. Striving for a contradiction, assume that for some open $V$ in $X$ containing $x$ we have that $Q(x, V)$ has empty interior in $X$. Since $V$ is open in $X$, and $Q(x, V)$ is closed in $V$, this clearly implies that $Q(x, V)$ is nowhere dense in $X$.

For every $n$ pick an open neighborhood $U_n$ of $x$ such that $\text{diam } U_n < 2^{-n}$. The assumptions imply that for every $n$, there exists $y_n \in Q(x, U_n) \setminus \{x\}$.

Since $Q(x, V)$ is nowhere dense in $X$, we may pick a countable dense subset $E \subseteq X \setminus Q(x, V)$. Put $D = E \cup \{y_n : n \in \mathbb{N}\}$. By Proposition 2.1, there exists $f \in \mathcal{H}(X)$ such that $f(x) = x$ and $f(D) \subseteq E$. Pick $n$ so large that $f(U_n) \subseteq V$. Since $y_n \in Q(x, U_n) \setminus \{x\}$ we have that $f(y_n) \not\in f(U_n) \setminus \{x\} = Q(x; f(U_n)) \setminus \{x\} \subseteq Q(x, V) \setminus \{x\}$. Since $f(y_n) \in E$ and $E \cap Q(x, V) = \emptyset$, this is a contradiction.

Corollary 3.2. For every open subset $V$ of $X$ and $x \in V$, we have that the interior of $Q(x, V)$ is dense in $Q(x, V)$.

Proof. Assume that the interior $W$ of $Q(x, V)$ is not dense in $Q(x, V)$. Then there are $y \in Q(x, V)$ and an open subset $U$ of $x$ such that $y \in U \subseteq V$ and $U \cap W = \emptyset$. By Lemma 3.1, the interior $P$ of $Q(y, U)$ is nonempty. However, $Q(y, U) \subseteq Q(y, V) = Q(x, V)$, hence $P \subseteq Q(x, V)$ and hence $P \subseteq W$. This is a contradiction since $\emptyset \neq P \subseteq U \cap W = \emptyset$.

Lemma 3.3. There is a point $x \in X$ with the following property: for every open neighborhood $V$ of $x$, the quasi-component $Q(x, V)$ is a neighborhood of $x$.

Proof. Let $\mathcal{U}_1$ be a maximal pairwise disjoint collection of nonempty open subsets of $X$ each of diameter less than $2^{-1}$. Clearly, $\bigcup \mathcal{U}_1$ is dense. Fix $U \in \mathcal{U}_1$. Each quasi-component of $U$ has dense interior by Corollary 3.2. Hence the interiors of all the quasi-components of elements of $\mathcal{U}_1$ form a pairwise disjoint open (and hence countable) collection with dense union. Let $\mathcal{U}_2$ be a maximal pairwise disjoint collection of nonempty open subsets of $X$ each of diameter less than $2^{-2}$ and having the property that every element $V \in \mathcal{U}_2$ is contained in some quasi-component of some member from $\mathcal{U}_1$. It is clear that $\mathcal{U}_2$ has dense
union. Hence we can continue the same construction with all the quasi-components of members from \( \mathcal{U}_n \), thus creating the family \( \mathcal{U}_3 \). Etc. At the end of the construction, we have a sequence \( \{ \mathcal{U}_n : n \in \mathbb{N} \} \) of subfamilies of pairwise disjoint nonempty open subsets of \( X \) such that for every \( n \),

1. \( \bigcup \mathcal{U}_n \) is dense in \( X \),
2. if \( V \in \mathcal{U}_{n+1} \), then there exist \( U \in \mathcal{U}_n \) and \( p \in U \) such that \( V \subseteq Q(p, U) \),
3. \( \text{mesh} \mathcal{U}_n < 2^{-n} \).

Since \( X \) is nonmeager, the collection \( \{ X \setminus \bigcup \mathcal{U}_n : n \in \mathbb{N} \} \) does not cover \( X \). Hence there is a point \( x \in X \) for which there exists for every \( n \in \mathbb{N} \) an element \( U_n \in \mathcal{U}_n \) such that \( x \in U_n \). We claim that \( x \) is as required. To this end, let \( V \) be any open neighborhood of \( x \). By (3), there exists \( n \) such that \( x \in U_n \subseteq V \). Since by (2), \( x \in U_{n+1} \subseteq Q(p, U_n) \) for some \( p \in U_n \), we have \( x \in U_{n+1} \subseteq Q(x, U_n) \). But \( Q(x, U_n) \subseteq Q(x, V) \), and so \( Q(x, V) \) is a neighborhood of \( x \). \( \square \)

Again by homogeneity, the property of the point \( x \) in Lemma 3.3 is shared by all points.

**Corollary 3.4.** Every quasi-component of an arbitrary open subset of \( X \) is open.

Now let \( V \) be a nonempty open subset of \( X \), and let \( W \) be a quasi-component of \( V \). Observe that \( W \) is a clopen subset of \( V \) since the quasi-components of \( V \) form a pairwise disjoint family. If \( W \) is not connected, then we can write \( W \) as \( A \cup B \), where \( A \) and \( B \) are disjoint nonempty open subsets of \( W \). But then \( A \) and \( B \) are clearly clopen in \( V \), which implies that \( W \) is not a quasi-component. Hence quasi-components of open subsets of \( X \) are both open and connected. So we arrive at the conclusion that \( X \) is locally connected. This completes the proof of Theorem 1.1.

Let us return to the question whether every connected Polish CDH-space is locally connected. Theorem 1.1 implies that a counterexample is very tricky. It is connected, yet its properties resemble those of complete Erdős space in [4].

**4. Proof of Theorem 1.2**

To begin with, let us prove the following simple but curious fact.

**Proposition 4.1.** Every meager CDH-space which has an open base \( \mathcal{U} \) such that \( \text{Fr} \mathcal{U} \) is analytic for every \( U \in \mathcal{U} \), is zero-dimensional.

**Proof.** By the result of Fitzpatrick and Zhou quoted in §2, it follows that \( X \) is a \( \lambda \)-set. Observe that by the Baire Category Theorem, a countable dense subspace of a Cantor set \( K \) is not a \( G_\delta \)-subset of \( K \). This implies that \( X \) does not contain a copy of the Cantor set. Let \( \mathcal{U} = \{ U_n : n \in \mathbb{N} \} \) be an open basis for \( X \) such that \( \text{Fr} U_n \) is analytic for every \( n \). Clearly, every \( \text{Fr} U_n \) is countable since every uncountable analytic space contains a copy of the Cantor set, [10, Corollary 1.5.13]. Let \( D = \bigcup_n \text{Fr} U_n \). Then \( D \) is countable and hence \( G_\delta \) and so \( X \setminus D \) can be written as \( \bigcup_n F_n \), where every \( F_n \) is closed in \( X \). Since \( F_n \cap \overline{F_m} = F_n \cap U_m \) for all \( n \) and \( m \), it follows that each \( F_n \) is zero-dimensional. So the cover

\[
\{ \{ d : d \in D \} \cup \{ F_n : n \in \mathbb{N} \} \}
\]

of \( X \) consists of countably many closed and zero-dimensional subsets. Hence \( X \) is zero-dimensional by the Countable Closed Sum Theorem [10, Theorem 3.2.8]. \( \square \)

Let \( X \) be any CDH-space which is connected and rimcompact. Then \( X \) is nonmeager by the previous result.

Pick an arbitrary \( x \in X \).

**Lemma 4.2.** For every open neighborhood \( V \) of \( x \) we have that \( Q(x, V) \setminus \{ x \} \neq \emptyset \).
Proof. Pick an open set $A$ such that $x \in A \subseteq \overline{A} \subseteq V$ while moreover $\text{Fr}A$ is compact. We claim that $Q(x, V)$ meets $\text{Fr}A$. Indeed, pick an arbitrary (relatively) clopen $E \subseteq V$ that contains $x$. Then $E \cap \overline{A}$ is clopen in $\overline{A}$, hence closed in $X$, and contains $x$. Suppose that $(E \cap \overline{A}) \cap \text{Fr}A = \emptyset$. Then $E \cap \overline{A} = E \cap A$ is nonempty and clopen in $X$ which contradicts connectivity. Hence the collection

$$\{E \cap \text{Fr}A : E \text{ is a (relatively) clopen subset of } V \text{ that contains } x\}$$

is a family of closed subsets of $\text{Fr}A$ with the finite intersection property. By compactness of $\text{Fr}A$, the set $Q(x, V)$ consequently meets $\text{Fr}A$. \hfill \Box

So $X$ is as in Theorem 1.1, and we are done.

It was noted by Lyubomyr Zdomskyy that if a connected, CDH, rim-$\sigma$-compact space $X$ has dimension greater than 1, then it is locally connected. Striving for a contradiction, assume that $X$ is not locally connected. From Theorem 1.1 it follows that there is a base $\mathcal{U}$ at a point $x$ in $X$ such that $Q(x, U) = \{x\}$ for all $U \in \mathcal{U}$. Hence for every $U \in \mathcal{U}$, $[x]$ is a countable intersection of clopen subsets of $U$. This together with the homogeneity of $X$ easily implies that every compact subspace of $U$ is zero-dimensional. As a result, every compact subspace of $X$ must be zero-dimensional. Then the rim-$\sigma$-compactness yields that there is a base with zero-dimensional boundaries, and hence the space $X$ must have dimension 1.

In the light of Proposition 4.1, the question whether every rimcompact connected CDH-space is Polish, is natural. It was shown by Hrušák and Zamora Avilés [9] that every Borel CDH-space is Polish. As a consequence, a counterexample to this question is not Borel. The answer is in the negative, at least consistently. Let $X$ be an $\aleph_1$-dense subset of the 2-sphere $S^2$. The proof of the main theorem in Steprāns and Watson [12] shows that $Y = S^2 \setminus X$ is CDH under $\text{MA}_{\aleph_1}$ for $\sigma$-centered posets. It is clear that $Y$ is connected and locally connected. It is also clear that $Y$ not Polish since $\aleph_1 < c$. Moreover, every $y \in Y$ has a neighborhood base the boundary of every element of which misses $X$ so that $Y$ is rimcompact.

References