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Countable Dense Homogeneous Rimcompact Spaces and Local Connectivity

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Abstract. We prove that every nonmeager connected Countable Dense Homogeneous space is locally connected under some additional mild connectivity assumption. As a corollary we obtain that every Countable Dense Homogeneous connected rimcompact space is locally connected.

1. Introduction

All spaces under discussion are separable metric.

A space X is *Countable Dense Homogeneous* (abbreviated: CDH) provided that for all countable dense subsets D and E of X there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$. For more information on this concept, see Arhangel'skii and van Mill [2]. Bennett [3] proved that every connected CDH-space is homogeneous.

In 1972, Fitzpatrick [6] proved that every locally compact, connected CDH-space is locally connected. Fitzpatrick and Zhou [7] asked in 1992 whether every Polish, connected CDH-space is locally connected. This problem is one of the few problems in [7] that is still open and was the motivation for the current investigations.

For a space X and $x \in X$ we let $Q(x, X)$ denote the *quasi-component* of x in X. That is, $Q(x, X)$ is the intersection of all clopen subsets of X that contain x. Observe that if $x \in X$, and X is a subspace of Y, then $Q(x, X) \subseteq Q(x, Y)$.

Theorem 1.1. *Let X be a nonmeager connected* CDH*-space and assume that for some point x in X we have that for every open neighborhood W of x, $Q(x, W) \setminus \{x\}$ is nonempty. Then X is locally connected.*

Corollary 1.2. *Every rimcompact connected* CDH*-space is locally connected.*

This corollary generalizes the result of Fitzpatrick just quoted. Observe that we do not require our space to be nonmeager.

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2. Preliminaries

As usual, for a subset U of a space X, we put $\operatorname{Fr} U = \overline{U} \setminus \operatorname{Int} U$; it is called the *boundary* of U.

A space X is *meager* if it can be expressed as a countable union of nowhere dense sets. Clearly, every Baire space (see below) is nonmeager.

A space X is called *rimcompact* if there exists an open base $\mathcal{B}$ for X such that $\operatorname{Fr} B$ is compact for each $B \in \mathcal{B}$. For more information on this concept, see Aarts and Nishiura [1].

For a space X we let $\mathcal{H}(X)$ denote its group of homeomorphisms. If $A \subseteq X$, then $\mathcal{H}(X; A)$ denotes $\{f \in \mathcal{H}(X) : h \text{ restricts to the identity on } A\}$.

We will need the following result.

Proposition 2.1 (van Mill [11, Proposition 3.1]). *Let X be* CDH. *If $F \subseteq X$ is finite and $D, E \subseteq X \setminus F$ are countable and dense in X, then there is an element $f \in \mathcal{H}(X; F)$ such that $f(D) \subseteq E$.*

A space X is a *λ-set* if every countable subspace is G_δ. It was shown by Fitzpatrick and Zhou [7, Theorem 3.4] that every meager CDH-space is a λ-set. There are such CDH-spaces, see [5] and [8].

A space is *Polish* if it has an admissible complete metric. A space is *Baire* if the intersection of any countable family of dense open sets in the space is dense. A space is *analytic* if it is a continuous image of the space of irrational numbers.

3. Proof of Theorem 1.1

Let X be any nonmeager CDH-space which is connected and contains a point x such that for every open neighborhood W of X, $Q(x, W) \setminus \{x\}$ is nonempty. By Bennett [3], X is homogeneous. Hence this property of the point x is shared by all points.

Lemma 3.1. *For every open neighborhood V of a point x in X we have that the interior of $Q(x, V)$ is nonempty.*

Proof. Striving for a contradiction, assume that for some open V in X containing x we have that $Q(x, V)$ has empty interior in X. Since V is open in X, and $Q(x, V)$ is closed in V, this clearly implies that $Q(x, V)$ is nowhere dense in X.

For every n pick an open neighborhood U_n of x such that $\operatorname{diam} U_n < 2^{-n}$. The assumptions imply that for every n, there exists $y_n \in Q(x, U_n) \setminus \{x\}$.

Since $Q(x, V)$ is nowhere dense in X, we may pick a countable dense subset $E \subseteq X \setminus Q(x, V)$. Put $D = E \cup \{y_n : n \in \mathbb{N}\}$. By Proposition 2.1, there exists $f \in \mathcal{H}(X)$ such that $f(x) = x$ and $f(D) \subseteq E$. Pick n so large that $f(U_n) \subseteq V$. Since $y_n \in Q(x, U_n) \setminus \{x\}$ we have that $f(y_n) \in Q(f(x), f(U_n)) \setminus \{f(x)\} = Q(x, f(U_n)) \setminus \{x\} \subseteq Q(x, V) \setminus \{x\}$. Since $f(y_n) \in E$ and $E \cap Q(x, V) = \emptyset$, this is a contradiction. $\square$

Corollary 3.2. *For every open subset V of X and $x \in V$, we have that the interior of $Q(x, V)$ is dense in $Q(x, V)$.*

Proof. Assume that the interior W of $Q(x, V)$ is not dense in $Q(x, V)$. Then there are $y \in Q(x, V)$ and an open subset U of x such that $y \in U \subseteq V$ and $U \cap W = \emptyset$. By Lemma 3.1, the interior P of $Q(y, U)$ is nonempty. However, $Q(y, U) \subseteq Q(y, V) = Q(x, V)$, hence $P \subseteq Q(x, V)$ and hence $P \subseteq W$. This is a contradiction since $\emptyset \neq P \subseteq U \cap W = \emptyset$. $\square$

Lemma 3.3. *There is a point $x \in X$ with the following property: for every open neighborhood V of x, the quasi-component $Q(x, V)$ is a neighborhood of x.*

Proof. Let $\mathcal{U}_1$ be a maximal pairwise disjoint collection of nonempty open subsets of X each of diameter less than 2^{-1}. Clearly, $\bigcup \mathcal{U}_1$ is dense. Fix $U \in \mathcal{U}_1$. Each quasi-component of U has dense interior by Corollary 3.2. Hence the interiors of all the quasi-components of elements of $\mathcal{U}_1$ form a pairwise disjoint open (and hence countable) collection with dense union. Let $\mathcal{U}_2$ be a maximal pairwise disjoint collection of nonempty open subsets of X each of diameter less than 2^{-2} and having the property that every element $V \in \mathcal{U}_2$ is contained in some quasi-component of some member from $\mathcal{U}_1$. It is clear that $\mathcal{U}_2$ has dense union. Hence we can continue the same construction with all the quasi-components of members from $\mathcal{U}_2$, thus creating the family $\mathcal{U}_3$. Etc. At the end of the construction, we have a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of subfamilies of pairwise disjoint nonempty open subsets of X such that for every n,

1. $\bigcup \mathcal{U}_n$ is dense in X,
2. if $V \in \mathcal{U}_{n+1}$, then there exist $U \in \mathcal{U}_n$ and $p \in U$ such that $V \subseteq Q(p, U)$,
3. $\operatorname{mesh} \mathcal{U}_n < 2^{-n}$.

Since X is nonmeager, the collection $\{X \setminus \bigcup \mathcal{U}_n : n \in \mathbb{N}\}$ does not cover X. Hence there is a point $x \in X$ for which there exists for every $n \in \mathbb{N}$ an element $U_n \in \mathcal{U}_n$ such that $x \in U_n$. We claim that x is as required. To this end, let V be any open neighborhood of x. By (3), there exists n such that $x \in U_n \subseteq V$. Since by (2), $x \in U_{n+1} \subseteq Q(p, U_n)$ for some $p \in U_n$, we have $x \in U_{n+1} \subseteq Q(x, U_n)$. But $Q(x, U_n) \subseteq Q(x, V)$, and so $Q(x, V)$ is a neighborhood of x. □

Again by homogeneity, the property of the point x in Lemma 3.3 is shared by all points.

Corollary 3.4. *Every quasi-component of an arbitrary open subset of X is open.*

Now let V be a nonempty open subset of X, and let W be a quasi-component of V. Observe that W is a clopen subset of V since the quasi-components of V form a pairwise disjoint family. If W is not connected, then we can write W as $A \cup B$, where A and B are disjoint nonempty open subsets of W. But then A and B are clearly clopen in V, which implies that W is not a quasi-component. Hence quasi-components of open subsets of X are both open and connected. So we arrive at the conclusion that X is locally connected. This completes the proof of Theorem 1.1.

Let us return to the question whether every connected Polish CDH-space is locally connected. Theorem 1.1 implies that a counterexample is very tricky. It is connected, yet its properties resemble those of complete Erdős space in [4].

4. Proof of Theorem 1.2

To begin with, let us prove the following simple but curious fact.

Proposition 4.1. *Every meager CDH-space which has an open base $\mathcal{U}$ such that $\operatorname{Fr} U$ is analytic for every $U \in \mathcal{U}$, is zero-dimensional.*

Proof. By the result of Fitzpatrick and Zhou quoted in §2, it follows that X is a λ-set. Observe that by the Baire Category Theorem, a countable dense subspace of a Cantor set K is not a G_δ-subset of K. This implies that X does not contain a copy of the Cantor set. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open basis for X such that $\operatorname{Fr} U_n$ is analytic for every n. Clearly, every $\operatorname{Fr} U_n$ is countable since every uncountable analytic space contains a copy of the Cantor set, [10, Corollary 1.5.13]. Let $D = \bigcup_n \operatorname{Fr} U_n$. Then D is countable and hence G_δ and so $X \setminus D$ can be written as $\bigcup_n F_n$, where every F_n is closed in X. Since $F_n \cap \overline{U}_m = F_n \cap U_m$ for all n and m, it follows that each F_n is zero-dimensional. So the cover

$$\{\{d\} : d \in D\} \cup \{F_n : n \in \mathbb{N}\}$$

of X consists of countably many closed and zero-dimensional subsets. Hence X is zero-dimensional by the Countable Closed Sum Theorem [10, Theorem 3.2.8]. □

Let X be any CDH-space which is connected and rimcompact. Then X is nonmeager by the previous result.

Pick an arbitrary $x \in X$.

Lemma 4.2. *For every open neighborhood V of x we have that $Q(x, V) \setminus \{x\} \neq \emptyset$.*

Proof. Pick an open set A such that $x \in A \subseteq \overline{A} \subseteq V$ while moreover $\operatorname{Fr} A$ is compact. We claim that $Q(x, V)$ meets $\operatorname{Fr} A$. Indeed, pick an arbitrary (relatively) clopen $E \subseteq V$ that contains x. Then $E \cap \overline{A}$ is clopen in $\overline{A}$, hence closed in X, and contains x. Suppose that $(E \cap \overline{A}) \cap \operatorname{Fr} A = \emptyset$. Then $E \cap \overline{A} = E \cap A$ is nonempty and clopen in X which contradicts connectivity. Hence the collection

$$\{E \cap \operatorname{Fr} A : E \text{ is a (relatively) clopen subset of } V \text{ that contains } x\}$$

is a family of closed subsets of $\operatorname{Fr} A$ with the finite intersection property. By compactness of $\operatorname{Fr} A$, the set $Q(x, V)$ consequently meets $\operatorname{Fr} A$. □

So X is as in Theorem 1.1, and we are done.

It was noted by Lyubomyr Zdomskyy that if a connected, CDH, rim-σ-compact space X has dimension greater than 1, then it is locally connected. Striving for a contradiction, assume that X is not locally connected. From Theorem 1.1 it follows that there is a base $\mathcal{U}$ at a point x in X such that $Q(x, U) = \{x\}$ for all $U \in \mathcal{U}$. Hence for every $U \in \mathcal{U}$, $\{x\}$ is a countable intersection of clopen subsets of U. This together with the homogeneity of X easily implies that every compact subspace of U is zero-dimensional. As a result, every compact subspace of X must be zero-dimensional. Then the rim-σ-compactness yields that there is a base with zero-dimensional boundaries, and hence the space X must have dimension 1.

In the light of Proposition 4.1, the question whether every rimcompact connected CDH-space is Polish, is natural. It was shown by Hrušák and Zamora Avilés [9] that every Borel CDH-space is Polish. As a consequence, a counterexample to this question is not Borel. The answer is in the negative, at least consistently. Let X be an $\aleph_1$-dense subset of the 2-sphere $\mathbb{S}^2$. The proof of the main theorem in Steprāns and Watson [12] shows that $Y = \mathbb{S}^2 \setminus X$ is CDH under $\mathsf{MA}_{\aleph_1}$ for σ-centered posets. It is clear that Y is connected and locally connected. It is also clear that Y not Polish since $\aleph_1 < \mathfrak{c}$. Moreover, every $y \in Y$ has a neighborhood base the boundary of every element of which misses X so that Y is rimcompact.

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