http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

**TOPOLOGY PROCEEDINGS**
Volume 45 (2015)
Pages 1-4

E-Published on January 29, 2014

# A NOTE ON AN UNUSUAL CHARACTERIZATION OF THE PSEUDO-ARC

JAN VAN MILL

Abstract. Lewis showed that the pseudo-arc is the unique non-degenerate continuum having the property that any two copies of it that are setwise near each other in terms of the Hausdorff distance are homeomorphically near each other. We present a new proof of this fact based on a well-known result of Bing, standard facts from infinite-dimensional topology and the Effros Theorem.

## 1. Introduction

*All spaces under discussion are separable metric.* For all undefined notions, see Nadler [7] and van Mill [5].

A compactum $X$ is said to have *property* $\mathsf{HN}$ (for 'homeomorphically near'), Lewis [4], if for any copy $X_0$ of $X$ in the Hilbert cube $Q$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any copy $X_1$ of $X$ in $Q$ such that $d_H(X_0, X_1) < \delta$ there exists a homeomorphism $h\colon X_0 \to X_1$ such that $d\big(x, h(x)\big) < \varepsilon$ for each $x \in X_0$. In [4], Lewis proved the following:

**Theorem 1.1.** *The pseudo-arc is the only non-degenerate continuum with property* $\mathsf{HN}$.

The aim of this note is to present a new proof of this fact, based on Bing's Theorem from [1] that the space of pseudo-arcs is a dense $G_\delta$-subset of $C(Q)$, standard facts from infinite-dimensional topology and the Effros Theorem from [2] (see also [6]). In fact, besides Bing's result, we need no specifics in our proof about the pseudo-arc. This is rather curious and it may make our method applicable in different situations.

---

2010 *Mathematics Subject Classification.* Primary 54F65; Secondary 54F50, 54C25.

*Key words and phrases.* Pseudo-arc, infinite-dimensional topology, Effros Theorem.

## 2. Proofs

For a finite collection $\mathscr{U}$ of open subsets of $Q$ we put

$$N(\mathscr{U}) = \{B \in C(Q) : (B \subseteq \bigcup \mathscr{U}) \,\&\, (\forall U \in \mathscr{U})(B \cap U \neq \emptyset)\}.$$

Let $\mathscr{Z}(Q)$ denote the collection of all $Z$-sets in $Q$. For a non-degenerate continuum $X$, put

$$C(Q, X) = \{A \in C(Q) : A \text{ is homeomorphic to } X\},$$

and

$$C_{\mathscr{Z}}(Q, X) = C(Q, X) \cap \mathscr{Z}(Q),$$

respectively.

**Proposition 2.1.** *If $X$ is a non-degenerate continuum with property* HN*, then $C_{\mathscr{Z}}(Q, X)$ is a dense $G_\delta$-subset of $C(Q)$.*

*Proof.* We will first show that $C_{\mathscr{Z}}(Q, X)$ is dense. To this end, take arbitrary $B \in C(Q)$ and $\varepsilon > 0$. There is a finite collection of open subsets $\mathscr{U}$ of $Q$ such that $B \in N(\mathscr{U})$ while $d_H(B, C) < \varepsilon$ for each $C \in N(\mathscr{U})$. By [7, Theorem 19.2] we may pick $I \in N(\mathscr{U})$ such that $I \approx [0, 1]$. Let $r \colon Q \to I$ be a retraction. By the Mapping Replacement Theorem [5, Theorem 6.4.8], $r$ can be approximated arbitrarily closely by a $Z$-imbedding. Hence we may assume that there is an element $Y \in \mathscr{Z}(Q) \cap N(\mathscr{U})$ such that $Y \approx Q$. There is a topological copy $Z$ of $X$ which is contained in $Y$. By the Homeomorphism Extension Theorem [5, Theorem 6.4.6], homeomorphisms between finite subsets of $Q$ can be extended to homeomorphisms of $Q$. Hence we may assume that $Z \cap U \neq \emptyset$ for every $U \in \mathscr{U}$, i.e., $Z \in N(\mathscr{U})$.

We will next show that $C_{\mathscr{Z}}(Q, X)$ is a second category subset of $C(Q)$. Indeed, assume that for every $i$, $\mathscr{N}_i$ is a closed and nowhere dense subset of $C(Q)$. Pick any element $Z \in C_{\mathscr{Z}}(Q, X)$.

*Claim* 1. Fix $i \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists a homeomorphism $f \colon Q \to Q$ such that $d(f, 1_Q) < \varepsilon$ and $f(Z) \notin \mathscr{N}_i$.

Using our assumptions, pick $\delta > 0$ such that for any copy $X_0$ of $X$ in $Q$ such that $d_H(X_0, Z) < \delta$ there exists a homeomorphism $h \colon Z \to X_0$ such that $d(h, 1_Z) < \varepsilon$. Since $C_{\mathscr{Z}}(Q, X)$ is dense in $C(Q)$ we may pick $X_0 \in C_{\mathscr{Z}}(Q, X) \setminus \mathscr{N}_i$ such that $d_H(Z, X_0) < \delta$. Hence there exists a homeomorphism $h \colon Z \to X_0$ such that $d(h, 1_Z) < \varepsilon$. By the Homeomorphism Extension Theorem [5, Theorem 6.4.6], we may extend this homeomorphism to a homeomorphism $f \colon Q \to Q$ such that $d(f, 1_Q) < \varepsilon$.

Hence we can 'free' $Z$ from $\mathscr{N}_i$ by an arbitrarily small move. This means that in an inductive process we can free $Z$ from all the $\mathscr{N}_i$. This has to be done with a little care so that once $Z$ is free from some $\mathscr{N}_i$, the limit homeomorphism does not carry it back to that $\mathscr{N}_i$. But this can easily be achieved by the Claim and a standard application of the Inductive Convergence Criterion [5, Theorem 6.1.2] (cf., [5, the proof of Theorem 6.4.5]).

Let $\mathscr{H}(Q)$ denote the group of homeomorphisms of $Q$ endowed with the standard compact-open topology. Then $\mathscr{H}(Q)$ is Polish, and the Homeomorphism Extension Theorem [5, Theorem 6.4.6], shows that it acts transitively on the second category space $C_{\mathscr{Z}}(Q,X)$. By the Effros Theorem from [2] (see also [6]), it follows that $C_{\mathscr{Z}}(Q,X)$ is Polish and hence a $G_\delta$-subset of $C(Q)$. □

This leads us to a proof of Lewis' result from [4].

**Theorem 2.2.** *Let $X$ be a non-degenerate continuum. Then the following statements are equivalent:*

1. *$X$ has property* HN*,*
2. *$C_{\mathscr{Z}}(Q,X)$ is a dense $G_\delta$-subset of $C(Q)$,*
3. *$C(Q,X)$ is a dense $G_\delta$-subset of $C(Q)$,*
4. *$C(Q,X)$ contains a dense $G_\delta$-subset of $C(Q)$,*
5. *$X$ is homeomorphic to the pseudo-arc.*

*Proof.* For (1) ⇒ (2), we simply apply Proposition 2.1. For (2) ⇒ (5), recall Bing's Theorem [1] quoted above that the collection of pseudo-arcs is a dense $G_\delta$-subset of $C(Q)$. Since by the Baire Category Theorem any two dense $G_\delta$-subsets of $C(Q)$ intersect, we conclude that $X$ is homeomorphic to the pseudo-arc. We achieve (5) ⇒ (1) by another application of the Effros Theorem. Indeed, first note that the connected $Z$-sets form a dense $G_\delta$-subset of $C(Q)$ (Kroonenberg [3, Lemma 2.1(b)]). Hence if $P$ denotes the pseudo-arc, then by Bing's Theorem just quoted and the Baire Category Theorem we obtain that $C_{\mathscr{Z}}(Q,P)$ is a dense $G_\delta$ in $C(Q)$. Now observe that $\mathscr{H}(Q)$ acts transitively on $C_{\mathscr{Z}}(Q,P)$. By the Effros Theorem from [2] (see also [6]), $\mathscr{H}(Q)$ acts micro-transitively on $C_{\mathscr{Z}}(Q,P)$. Pick an arbitrary element $S\in C_{\mathscr{Z}}(Q,P)$, and let $\varepsilon>0$. The evaluation function $\gamma_S\colon \mathscr{H}(Q)\to C_{\mathscr{Z}}(Q,P)$ defined by $\gamma_S(h)=h(S)$ is a continuous, open surjection. By continuity of $\gamma_S$ there exists $\theta>0$ such that

$$\gamma_S(\{g\in\mathscr{H}(Q): d(g,1_Q)<\theta\})\subseteq\{A\in C_{\mathscr{Z}}(Q,P): d_H(S,A)<\varepsilon\}.$$

Since $\gamma_S$ is open, there exists $\delta>0$ such that

$$\{A\in C_{\mathscr{Z}}(Q,P): d_H(A,S)<\delta\}\subseteq\gamma_S(\{g\in\mathscr{H}(Q): d(g,1_Q)<\theta\}).$$

Hence this $\delta$ has the following property: if $T\in C_{\mathscr{Z}}(Q,P)$ and $d_H(S,T)<\delta$, then there is a homeomorphism $f\colon Q\to Q$ such that $f(S)=T$ and $d(f,1_Q)<\varepsilon$.

To prove that $P$ has property $\mathsf{HN}$, take arbitrary $P_0 \in C(Q,P)$ and $\varepsilon > 0$. We assume without loss of generality that $\varepsilon < 1$. Define $f\colon Q \to Q$ by $f(x) = (1 - {}^1\!/_3\varepsilon)x$. Then $f$ is a $Z$-imbedding and $d_H(f(A), f(B)) \leq d_H(A,B)$ for all $A, B \in C(Q)$. Put $S = f(P_0)$ and let $\delta > 0$ be as above for $S$ and ${}^1\!/_3\varepsilon$. Now take an arbitrary $P_0 \in C(Q,P)$ such that $d_H(P_0, P_1) < \delta$. Then $d_H(S, f(P_1)) < \delta$. Hence there is a homeomorphism $\alpha\colon Q \to Q$ such that $d(\alpha, 1_Q) < {}^1\!/_3\varepsilon$ and $\alpha(S) = f(P_1)$. Hence the function $\beta\colon P_0 \to P_1$ defined by $\beta(x) = f^{-1}(\alpha(f(x)))$ is a homeomorphism such that for every $x \in P_0$, $d(x, \beta(x)) < \varepsilon$.

The statements (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are a direct consequence of Bing's Theorem. □

## References


[1] R. H. Bing, *Concerning hereditarily indecomposable continua*, Pac. J. Math. **1** (1951), 43–51.

[2] E. G. Effros, *Transformation groups and $C^*$-algebras*, Annals of Math. **81** (1965), 38–55.

[3] N. Kroonenberg, *Pseudo-interiors of hyperspaces*, Compositio Math. **32** (1976), 113–131.

[4] W. Lewis, *Another characterization of the pseudo-arc*, Proceedings of the 1998 Topology and Dynamics Conference (Fairfax, VA), Top. Proc., vol. 23, pp. 235–244.

[5] J. van Mill, *Infinite-dimensional topology: prerequisites and introduction*, North-Holland Publishing Co., Amsterdam, 1989.

[6] J. van Mill, *A note on the Effros Theorem*, Amer. Math. Monthly. **111** (2004), 801–806.

[7] S. B. Nadler, *Hyperspaces of sets*, Marcel Dekker, New York and Basel, 1978.



Faculty of Sciences, Department of Mathematics, VU University Amsterdam, De Boelelaan $1081^a$, 1081 HV Amsterdam, the Netherlands and Faculty of Electrical Engineering, Mathematics and Computer Science, TU Delft, Postbus 5031, 2600 GA Delft, the Netherlands

*E-mail address*: `j.van.mill@vu.nl`