RACSAM (2015) 109:65–71
DOI 10.1007/s13398-014-0166-6

ORIGINAL PAPER

# Every $k$-separable Čech-complete space is subcompact

**Jan van Mill · Vladimir V. Tkachuk**

Received: 14 September 2013 / Accepted: 29 January 2014 / Published online: 13 February 2014


**Abstract** We establish that a Čech-complete space $X$ must be subcompact if it has a dense subspace representable as the countable union of closed subcompact subspaces of $X$. In particular, if a Čech-complete space contains a dense $\sigma$-compact subspace then it is subcompact. This result is new even for separable Čech-complete spaces. Furthermore, if $X$ is a compact space of countable tightness then $X \setminus A$ is subcompact for any countable set $A \subset X$. We also show that any $G_\delta$-subset of a dyadic compact space is subcompact and give a comparatively simple proof of the fact that $X \setminus A$ is subcompact for any linearly ordered compact space $X$ and any countable set $A \subset X$.

**Keywords** Subcompact space · Čech-complete space · $k$-separable space · Separable space · Countable tightness · Dyadic compact space

**Mathematics Subject Classification (2010)** Primary: 54D65 · 54D70; Secondary: 54B05 · 54E50

## 1 Introduction

It is a classical theorem that the topology of a metrizable space $X$ can be generated by a complete metric if and only if $X$ is Čech-complete. In the last century it was a motivation

Vladimir V. Tkachuk: Research supported by Promep 912011, Proyecto 12611768 (Mexico). Research supported by CONACyT grant CB-2012-01-178103 (Mexico).

J. van Mill
Department of Mathematics, Vrije Universiteit, De Boelelaan 1081,
1081 HV Amsterdam, The Netherlands
e-mail: j.van.mill@vu.nl

V. V. Tkachuk (✉)
Departamento de Matemáticas, Universidad Autónoma Metropolitana,
Av. San Rafael Atlixco, 186, Col. Vicentina, 09340 Mexico, DF, Mexico
e-mail: vova@xanum.uam.mx

for quite a few attempts to generalize completeness of a metric space by weakening Čech-completenss or considering analogous properties.

For example, pseudocompleteness defined by Oxtoby [8] is weaker than Čech-completeness and for metric spaces it is equivalent to the existence of a dense Čech-complete subspace. The class of pseudocomplete spaces has many nice properties and contains the class of pseudocompact spaces. There are several famous open problems about pseudocompleteness: it is unknown whether it is preserved by open mappings and dense $G_\delta$-subspaces (see [1]). However, it is not difficult to prove that every Čech-complete space is pseudocomplete.

Subcompactness, the weakest of so called Amsterdam properties defined by de Groot (see [6]) is another example of a generalization of completeness of metric spaces. A metrizable space is subcompact if and only if it is Čech-complete. It is known that subcompactness is preserved by open subspaces, free unions and arbitrary products but it is still an open question whether it is preserved by dense $G_\delta$-subspaces (see [2]). In particular, it is not known whether every Čech-complete space is subcompact.

It is also an open question (see [5]) whether $K \backslash A$ is subcompact if $K$ is a compact space and $A \subset X$ is countable. Fleissner, Tkachuk and Yengulalp proved in [5] that $K \backslash A$ is subcompact if $K$ is a compact linearly ordered space and $A \subset K$ is countable. The same conclusion is true (see [5, Corollary 2.8]) if $K$ is an $\omega$-monolithic compact space.

In this paper we establish that a Čech-complete space $X$ must be subcompact provided that it has a dense subspace representable as the countable union of closed subcompact subspaces of $X$. In particular, if $X$ has a dense subspace which is the countable union of closed locally compact subspaces of $X$ then $X$ is subcompact. This conclusion is new even if $X$ is $k$-separable, i.e., has a dense $\sigma$-compact subspace.

Therefore every $k$-separable (and hence every separable) Čech-complete space is subcompact. These results help to solve three open questions formulated in [5]. We show that $X \backslash A$ is subcompact if $X$ is a compact space of countable tightness and $A \subset X$ is countable. This gives a positive answer to questions 3.5 and 3.6 from the paper [5]. If $X$ is a dyadic compact space then any $G_\delta$-subset of $X$ is subcompact: this solves [5, Problem 3.13]. Our methods also give a much simpler proof than in [5] of the fact that $K \backslash A$ is subcompact if $K$ is a compact linearly ordered space and $A \subset K$ is countable.

## 2 Notation and terminology

All spaces are assumed to be Tychonoff. Given a space $X$, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \backslash \{\emptyset\}$. Let $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any $x \in X$; if $A \subset X$ then $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. All ordinals are identified with the set of their predecessors and are assumed to carry the interval topology. A space is Čech-complete if it is homeomorphic to a $G_\delta$-subset of a compact space. We denote by $\mathbb{D}$ the set $\{0, 1\}$ with the discrete topology.

Given a space $Y$, a non-empty family $\mathcal{U} \subset \tau^*(Y)$ is called *a regular filterbase* if, for any $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $\overline{W} \subset U \cap V$. The space $Y$ is *subcompact* if it has a base $\mathcal{B} \subset \tau^*(Y)$ such that every regular filterbase $\mathcal{U} \subset \mathcal{B}$ has non-empty intersection; such a base is also called *subcompact*. A space $X$ has countable tightness (this is denoted by $t(X) \leq \omega$) if $\overline{A} = \bigcup\{\overline{B} : B \subset A$ and $|B| \leq \omega\}$ for any set $A \subset X$.

A space $X$ is *k-separable* if it has a dense $\sigma$-compact subspace. A compact space is called *dyadic* if it is a continuous image of the Cantor discontinuum $\mathbb{D}^\kappa$ for some cardinal $\kappa$.

The rest of our terminolofy is standard and follows [4]; the survey of Hodel [7] can be consulted for definitions of cardinal invariants.

## 3 Subcompactness in Čech-complete spaces

Our purpose is to find nice classes in which every Čech-complete space is subcompact. We will show that $k$-separable spaces form such a class; this fact implies that every $G_\delta$-subset of a dyadic compact space is subcompact.

The proof of the following two lemmas is an easy exercise.

**Lemma 3.1** *Suppose that $X$ is a space and $\mathcal{B} \subset \tau^*(X)$ is a regular filterbase. If there exists a compact set $K \subset X$ such that $\overline{U} \cap K \neq \emptyset$ for any $U \in \mathcal{B}$ then $\bigcap \mathcal{B} \neq \emptyset$.*

**Lemma 3.2** *If $X$ is a space and $\mathcal{U}$ is a regular filterbase in $X$ then*

*(a) for any set $U \in \mathcal{U}$ the family $\mathcal{G}_U = \{G \in \mathcal{U} : \overline{G} \subset U\}$ is a regular filterbase in $X$;*
*(b) for every $U \in \mathcal{U}$, we have $\bigcap \mathcal{U} \neq \emptyset$ if and only if $\bigcap \mathcal{G}_U \neq \emptyset$.*

The next statement formalizes the idea of the proof of the main result of this paper.

**Lemma 3.3** *Suppose that $K$ is a compact space and we have an increasing sequence $\{L_n : n \in \omega\}$ of closed subspaces of $K$. Let $X = K \setminus \left(\bigcup_{n\in\omega} L_n\right)$ and assume that for every $n \in \omega$ we have a family $\mathcal{U}_n$ of open subsets of $X$ such that $\mathrm{cl}_K(U) \cap L_n = \emptyset$ for each $U \in \mathcal{U}_n$. Assume additionally that we have a regular filterbase $\mathcal{B} \subset \mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ with the following property:*

*$(*)$ for any $U \in \mathcal{B}$ and $m \in \omega$ there exists a set $V \in \mathcal{B}$ such that $\overline{V} \subset U$ and $V \notin \bigcup_{i\le m} \mathcal{U}_i$.*

*Then $\bigcap \mathcal{B} \neq \emptyset$.*

*Proof* Using $(*)$ it is easy to construct a sequence $\{U_n : n \in \omega\} \subset \mathcal{B}$ such that $\overline{U}_{n+1} \subset U_n$ and $U_{n+1} \notin \bigcup_{i\le n} \mathcal{U}_i$ for each $n \in \omega$. Therefore $U_{n+1} \in \mathcal{U}_k$ for some $k > n$ and hence $\mathrm{cl}_K(U_{n+1}) \cap L_n = \emptyset$ for each $n \in \omega$. This shows that the non-empty compact set $F = \bigcap\{\mathrm{cl}_K(U_n) : n \in \omega\}$ is contained in $X$. Therefore

$$F = F \cap X = \bigcap\{\mathrm{cl}_K(U_n) \cap X : n \in \omega\} = \bigcap\{\overline{U}_n : n \in \omega\} = \bigcap_{n\in\omega} U_n.$$

If $\mathrm{cl}_K(U) \cap F = \emptyset$ for some $U \in \mathcal{B}$ then there exists $n \in \omega$ such that $\mathrm{cl}_K(U) \cap \mathrm{cl}_K(U_n) = \emptyset$ and hence $U \cap U_n = \emptyset$ which is a contradiction. This shows that $\mathrm{cl}_K(U) \cap F = \mathrm{cl}_K(U) \cap X \cap F = \overline{U} \cap F \neq \emptyset$ for any $U \in \mathcal{B}$ and hence we can apply Lemma 3.1 to conclude that $\bigcap \mathcal{B} \neq \emptyset$. □

The proof of the following two statements is straightforward and can be left to the reader.

**Proposition 3.4** *Suppose that $X$ is a space and $\mathcal{U}$ is a regular filterbase in $X$. If $\mathcal{U} = \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_k$ for some $k \in \omega$ then there exists $i \le k$ such that $\mathcal{U}_i$ is a regular filterbase.*

**Proposition 3.5** *If $X$ is a space and $Y \subset X$ suppose that $\mathcal{U}$ is a regular filterbase in $X$ such that $U \cap Y \neq \emptyset$ for any $U \in \mathcal{U}$. Then $\mathcal{U}_Y = \{U \cap Y : U \in \mathcal{U}\}$ is a regular filterbase in $Y$.*

**Theorem 3.6** *Suppose that $X$ is a Čech-complete space and $\{Y_i : i \in \omega\}$ is a family of subcompact subspaces of $X$.*

*(a) If every $Y_i$ is closed in $X$ and $Y = \bigcup_{i\in\omega} Y_i$ is dense in $X$ then $X$ is subcompact.*
*(b) If $X = \bigcup_{i\in\omega} Y_i$ then $X$ is subcompact.*

*Proof* We will give a simultaneous proof for both statements (a) and (b). Choose first a sequence $\{K_i : i \in \omega\}$ of increasing compact subsets of $\beta X \setminus X$ such that $\beta X \setminus X = \bigcup_{i\in\omega} K_i$. Fix a subcompact base $\mathcal{C}_i$ in the space $Y_i$ and consider the family $\mathcal{U}_i = \{U \in \tau(X) : U \cap X_i \in \mathcal{C}_i$ and $\mathrm{cl}_{\beta X}(U) \cap K_i = \emptyset\}$ for any $i \in \omega$.

To see that the family $\mathcal{U} = \bigcup_{i\in\omega} \mathcal{U}_i$ is a base in $X$ take any point $x \in X$ and $U \in \tau(x, X)$. If we are proving (a) then we pick a number $n \in \omega$ with $Y_n \cap U \neq \emptyset$. In the proof of (b) we observe that there exists $n \in \omega$ such that $x \in Y_n$. Choose a set $V \in \tau(x, X)$ such that $V \subset U$ and $\mathrm{cl}_{\beta X}(V) \cap K_n = \emptyset$. We have two cases:

(i) $x \in Y_n$. The proof of this case goes for both (a) and (b). Find a set $H \in \mathcal{C}_n$ with $x \in H \subset V$; it is easy to find $G \in \tau(X)$ such that $G \subset V$ and $G \cap Y_n = H$. It is immediate that $G \in \mathcal{U}_n$ and $x \in G \subset U$.

(ii) $x \notin Y_n$. This can occur only if we prove (a). Take a point $y \in U \cap Y_n$ and a set $W \in \tau(y, X)$ such that $W \subset U$ and $\mathrm{cl}_{\beta X}(W) \cap K_n = \emptyset$. It is easy to find a set $H \in \tau(X)$ such that $H \subset W$ and $y \in H \cap Y_n \in \mathcal{C}_n$. Then $G = (V \backslash Y_n) \cup H \in \mathcal{U}_n$ and $x \in G \subset U$ so $\mathcal{U}$ is, indeed, a base in $X$.

The proof that $\mathcal{U}$ is subcompact goes for both (a) and (b).

Suppose that $\mathcal{B} \subset \mathcal{U}$ is a regular filterbase and $\bigcap \mathcal{B} = \emptyset$. If the property $(*)$ from Lemma 3.3 holds for $\mathcal{B}$ then $\bigcap \mathcal{B} \neq \emptyset$ which is a contradiction so we can assume, without loss of generality, that there exist $U \in \mathcal{B}$ and $m \in \omega$ for which we have the inclusion $\mathcal{G}_U = \{V \in \mathcal{B} : \overline{V} \subset U\} \subset \bigcup_{i\le m} \mathcal{U}_i$.

Since $\mathcal{G}_U \subset \mathcal{B}$ is a regular filterbase and $\bigcap \mathcal{G}_U = \emptyset$ by Lemma 3.2, it follows from $\mathcal{G}_U \subset \bigcup_{i\le m} \mathcal{U}_i$ that we can forget about the set $U$ and consider that $\mathcal{B} \subset \bigcup\{\mathcal{U}_i : i \in A\}$ where $A \subset \omega$ is a finite set and $n = |A|$ is the minimal number for which there exists a regular filterbase $\mathcal{V} \subset \mathcal{B}$ such that $\bigcap \mathcal{V} = \emptyset$ and $\mathcal{V}$ is contained in the union of $n$-many families $\mathcal{U}_i$.

If $n = 1$ then $A = \{i\}$ for some number $i \in \omega$ and hence we can apply Proposition 3.5 to see that the family $\mathcal{W} = \{V \cap X_i : V \in \mathcal{B}\} \subset \mathcal{C}_i$ is a regular filterbase and hence $\bigcap \mathcal{W} \neq \emptyset$ which implies $\bigcap \mathcal{B} \neq \emptyset$. Therefore $n > 0$; let $\mathcal{B}_i = \mathcal{B} \cap \mathcal{U}_i$ for all $i \in A$. By Proposition 3.4, there exists $i \in A$ such that $\mathcal{B}_i$ is a regular filterbase. Proposition 3.5 shows that $\mathcal{B}'_i = \{U \cap X_i : U \in \mathcal{B}_i\} \subset \mathcal{C}_i$ is a regular filterbase so $P = \bigcap \mathcal{B}'_i \neq \emptyset$.

If $P \subset U$ for any $U \in \mathcal{B}$ then $\bigcap \mathcal{B} \neq \emptyset$ so there exists $U \in \mathcal{B}$ such that $P$ is not contained in $U$. Since no element of $\mathcal{G}_U$ contains $P$ and $P$ is contained in every element of $\mathcal{B}_i$, we have the inclusion $\mathcal{G}_U \subset \bigcup\{\mathcal{U}_j : j \in A\backslash\{i\}\}$ which, together with $\bigcap \mathcal{G}_U = \emptyset$ (Lemma 3.2) gives a contradiction with the choice of the number $n$. □

**Corollary 3.7** *Suppose that a Čech-complete space $X$ has a dense subset which is a countable union of closed locally compact subsets of $X$. Then $X$ is subcompact.*

*Proof* This is because every locally compact space is subcompact so Theorem 3.6(a) can be applied. □

**Corollary 3.8** *If $X$ is a Čech-complete space which has a dense subspace representable as the countable union of closed discrete subsets of $X$ then $X$ is subcompact.*

The conclusion of Corollary 3.7 is new even if the locally compact summands are compact. Recall that a space is called *k-separable* if it has a dense $\sigma$-compact subspace.

**Corollary 3.9** *Any Čech-complete $k$-separable space is subcompact.*

*Proof* If $X$ is Čech-complete and $\{K_n : n \in \omega\}$ is a family of compact subspaces of $X$ such that $\bigcup_{n\in\omega} K_n$ is dense in $X$ then every $K_n$ is locally compact and closed in $X$ so Corollary 3.7 does the rest. □

**Corollary 3.10** *Every separable Čech-complete space is subcompact.*

The following lemma might be known but we could not find the respective reference.

**Lemma 3.11** *If $X$ is a dyadic compact space then every non-empty $G_\delta$-subset of $X$ is $k$-separable.*

*Proof* Take any sequence $\{K_n : n \in \omega\}$ of compact subsets of $X$ such that $Y = X \setminus \bigcup_{n\in\omega} K_n \neq \emptyset$; we must prove that the space $Y$ is $k$-separable. For some cardinal $\kappa$ there exists a continuous onto map $\varphi : \mathbb{D}^\kappa \to X$. Let $F_n = \varphi_n^{-1}(K_n)$ for any $n \in \omega$. Then $Y = \varphi(Y')$ where $Y' = \mathbb{D}^\kappa \setminus (\bigcup_{n\in\omega} F_n)$.

Call a subset $F \subset \mathbb{D}^\kappa$ *standard* if there exists a countable set $A \subset \kappa$ and $f \in \mathbb{D}^A$ such that $F = \{f\} \times \mathbb{D}^{\kappa\setminus A}$. It is clear that every standard set is compact and, for any $f \in Y'$ there exists a standard set $S_f$ such that $f \in S_f \subset Y'$. We will use the following property of the space $\mathbb{D}^\kappa$:

$(T)$ if $\mathcal{G}$ is a family of $G_\delta$-subsets of $\mathbb{D}^\kappa$, then there is a countable $\mathcal{G}' \subset \mathcal{G}$ such that $\bigcup \mathcal{G}'$ is dense in $\bigcup \mathcal{G}$.

The property $(T)$ was established directly for $\mathbb{D}^\kappa$ in Statement (8) of the proof of Theorem 14 of the paper [3]. Since the paper [3] is in Russian, it is worth mentioning that a stronger theorem was proved in [9, Corollary 1.8]. The paper [9] shows that the property $(T)$ holds for any product of Lindelöf $\Sigma$-groups. It relies on the paper [10] where the property $(T)$ was established for one Lindelöf $\Sigma$-group. Both articles [9] and [10] are in English and, taken together, they contain a complete proof of the property $(T)$ in a much more general situation.

It follows from $Y' = \bigcup\{S_f : f \in Y'\}$ that we can apply the property $(T)$ to conclude that there exists a countable set $A \subset Y'$ such that $\bigcup\{S_f : f \in A\}$ is dense in $Y'$, which shows that $Y'$ is $k$-separable. Thus $Y$ is $k$-separable being a continuous image of $Y'$. □

The following corollary gives a positive answer to Problem 3.13 from [5].

**Corollary 3.12** *If $X$ is a dyadic compact space then any non-empty $G_\delta$-subset of $X$ is subcompact.*

*Proof* If $Y$ is a non-empty $G_\delta$-subset of $X$ then $Y$ must be Čech-complete and $k$-separable by Lemma 3.11 so $Y$ is subcompact by Corollary 3.9. □

**Corollary 3.13** *If $G$ is a compact topological group then every non-empty $G_\delta$-subset of $G$ is subcompact.*

**Corollary 3.14** *If $X$ is a compact space and $\pi w(X) \leq \omega$, then every dense $G_\delta$-subset of $X$ is subcompact.*

*Proof* If $Y$ is a dense $G_\delta$-subset of $X$ then $\pi w(Y) = \pi w(X) = \omega$ and hence $Y$ is a Čech-complete separable space. Corollary 3.10 does the rest. □

**Proposition 3.15** *Suppose that $X$ is a compact space of countable tightness. If $A \subset X$ is countable then $X\setminus A = S \cup L$ where $S$ is a separable Čech-complete space and $L$ is locally compact.*

*Proof* Observe that $X\setminus A = \overline{X\setminus A}\setminus(A \cap \overline{X\setminus A})$ so passing to $\overline{X\setminus A}$ and $A \cap \overline{X\setminus A}$ if necessary, we can assume, without loss of generality, that $Y = X\setminus A$ is dense in $X$. It follows from $t(X) \leq \omega$ that there is a countable set $B \subset Y$ such that $\overline{A} \subset \overline{B}$. As a consequence, $Y = (\overline{B} \cap Y) \cup (X\setminus\overline{A})$, i.e., the sets $S = \overline{B} \cap Y$ and $L = X\setminus\overline{A}$ are as promised. □

**Theorem 3.16** *Suppose that $X$ is a compact space such that $\overline{A}$ has countable tightness for any countable set $A \subset X$. Then $X\setminus A$ is subcompact for any countable $A \subset X$.*

*Proof* Fix an arbitrary countable set $A \subset X$ and observe that we have the equality $X \backslash A = (X \backslash \overline{A}) \cup (\overline{A} \backslash A)$. Recall that subcompactness is finitely additive by [5, Theorem 2.5]; the set $X \backslash \overline{A}$ is locally compact and hence subcompact so it suffices to prove that $\overline{A} \backslash A$ is subcompact.

Since $t(\overline{A}) = \omega$, we can apply proposition 3.15 to find a separable Čech-complete set $S \subset \overline{A}$ and a locally compact set $L \subset \overline{A}$ such that $\overline{A} \backslash A = S \cup L$. The space $L$ is subcompact being locally compact and the space $S$ is subcompact by Corollary 3.10; apply Theorem 2.5 of [5] once again to see that $\overline{A} \backslash A$ is subcompact. □

The following fact gives a positive answer to Questions 3.5 and 3.6 of the paper [5].

**Corollary 3.17** *If $X$ is a compact space of countable tightness then $X \backslash A$ is subcompact for any countable $A \subset X$.*

In the paper [5] a very complicated proof was given that $X \backslash A$ is subcompact if $X$ is a linearly ordered compact space and $A \subset X$ is countable. Our methods make it possible to give a much simpler proof.

**Corollary 3.18** *([5]) If $X$ is a linearly ordered compact space then $X \backslash A$ is subcompact for any countable $A \subset X$.*

*Proof* It suffices to observe that $\overline{A}$ is perfectly normal and hence has countable tightness for any countable set $A \subset X$; Theorem 3.16 does the rest. □

It was proved in [5, Theorem 2.1] that every scattered space is subcompact. Since every locally compact space is also subcompact, it is natural to try to find a class $\mathcal{P}$ such that all scattered spaces and all locally compact spaces belong to $\mathcal{P}$ and every element of $\mathcal{P}$ is subcompact.

**Definition 3.19** Say that a space $X$ is locally-compact-scattered (or simply $lc$-scattered) if every non-empty closed subspace of $X$ has a point of local compactness.

It is clear that locally compact spaces and scattered spaces are $lc$-scattered so the following fact generalizes Theorem 2.1 of [5].

**Theorem 3.20** *If $X$ is an $lc$-scattered space then every closed subspace of $X$ is subcompact.*

*Proof* The $lc$-property is, evidently, preserved by closed subspaces so it suffices to show that $X$ is subcompact. Let $X_0$ be the set of all points of local compactness of $X$. The set $X_0$ is open in $X$ because $X_0 = \bigcup\{U : U \in \tau^*(X) \text{ and } \overline{U} \text{ is compact}\}$. Proceeding inductively, if $\beta$ is an ordinal and we have disjoint sets $\{X_\alpha : \alpha < \beta\}$ then let $X_\beta$ be the set of points of local compactness of $X \backslash \bigcup\{X_\alpha : \alpha < \beta\}$.

Since the space $X$ is $lc$-scattered, there is a minimal ordinal $\mu$ such that $X = \bigcup\{X_\alpha : \alpha < \mu\}$. Say that a set $U \in \tau(X)$ is *adequate* if there exists $\alpha < \mu$ such that $U \subset \bigcup\{X_\beta : \beta \leq \alpha\}$ while the set $U \cap X_\alpha$ is non-empty and has compact closure; let $id(U) = \alpha$.

It is straightforward that the family $\mathcal{U}$ of all adequate subsets of $X$ is a base of $X$. To see that $\mathcal{U}$ is subcompact, suppose that $\mathcal{B} \subset \mathcal{U}$ is a regular filterbase and consider the ordinal $\gamma = \min\{id(U) : U \in \mathcal{B}\}$. Take any $U \in \mathcal{B}$ with $id(U) = \gamma$. The set $K = \overline{U \cap X_\gamma}$ is compact; assume that $V \in \mathcal{B}$ and $V \cap K = \emptyset$. Pick a set $W \in \mathcal{B}$ such that $W \subset U \cap V$; then $W \cap X_\gamma = \emptyset$ and hence $W \subset \bigcup\{X_\alpha : \alpha < \gamma\}$, i.e., $id(W) < \gamma$ which is a contradiction. Therefore $V \cap K \neq \emptyset$ for any $V \in \mathcal{B}$ so we can apply Lemma 3.1 to conclude that $\bigcap \mathcal{B} \neq \emptyset$. □

## 4 Open problems

The current paper and [5] contain a lot of information about subcompactness of Čech-complete spaces. However, some natural questions remain open.

**Question 4.1** *Suppose that a Čech-complete space has a dense set of isolated points. Must it be subcompact?*

**Question 4.2** *Suppose that $X$ is a Čech-complete space and $c(X) \leq \omega$. Must $X$ be subcompact?*

**Question 4.3** *Suppose that $X$ is a Čech-complete space and $\omega_1$ is a caliber of $X$. Must $X$ be subcompact?*

**Question 4.4** *Suppose that $X$ is a Lindelöf Čech-complete space. Must $X$ be subcompact?*

**Question 4.5** *Suppose that $X$ is a hereditarily Lindelöf Čech-complete space. Must $X$ be subcompact?*

**Question 4.6** *Suppose that $X$ is a first countable Čech-complete space. Must $X$ be subcompact?*

**Question 4.7** *Suppose that $X$ is a hereditarily Lindelöf Čech-complete space. Must $X$ have a dense $\sigma$-compact subspace?*

**Question 4.8** *(D.J. Lutzer) Suppose that $X$ is a Čech-complete space with a dense $\sigma$-discrete subspace. Must $X$ be subcompact?*

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