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NONHOMOGENEITY OF REMAINDERS, III

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Abstract. We present a cardinal inequality on the number of homeomorphisms of remainders of nowhere locally compact spaces. We also discuss the question when the complement of a Σ-product in an arbitrary Cantor cube is homogeneous, or a topological group.

1. Introduction

All topological spaces under discussion are Tychonoff.

A space X is *homogeneous* if for any two points $x, y \in X$ there is a homeomorphism h from X onto itself such that $h(x) = y$. If bX is a compactification of a space X, then $bX \setminus X$ is called its *remainder*.

In this note we continue our study begun in [1, 2] concerning the (non)homogeneity of arbitrary remainders of topological spaces. We present a variation of a recent cardinal inequality in [1] on the number of homeomorphisms of remainders of nowhere locally compact spaces. By examples we demonstrate that both inequalities are independent. We also discuss the question when the complement of a Σ-product in an arbitrary Cantor cube is homogeneous, or a topological group.

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2. Preliminaries

For a space X, we let $\mathscr{H}(X)$ denote its group of homeomorphisms. If $A \subseteq X$, then $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ denote its closure and interior, respectively. Similarly, $\overline{A}$ denotes the closure of A if no confusion can arise.

We let $\mathsf{RO}(X)$ denote the complete Boolean algebra of all *regular open* subsets of X, where a set is *regular open* if it is the interior of its own closure. It is easy to see (and well-known) that for every space X we have $|\mathsf{RO}(X)| \le 2^{d(X)}$, where $d(X)$ denotes the density of X.

Let βX denote the Čech-Stone compactification of X.

The *Hausdorff separating weight* of a space X, abbreviated $\mathrm{Hsw}(X)$, is the least infinite cardinal κ for which there exists a family $\mathscr{U}$ of open subsets of X such that the cardinality of $\mathscr{U}$ does not exceed κ while moreover for all distinct $x, y \in X$ there exist disjoint $U, V \in \mathscr{U}$ such that $x \in U$ and $y \in V$. Observe that $\mathrm{Hsw}(X)$ is rather 'small' since it is obviously bounded by $|X|$.

If $f\colon X \to Y$ is a multivalued function, and $A \subseteq Y$, then $f^{-1}(A) = \{x \in X : f(x) \subseteq A\}$. We say that f is *upper semi-continuous* provided that $f^{-1}(U)$ is open in X for every open subset U of Y.

We refer to Juhász [6] for undefined terminology on cardinal functions.

3. Another bound on the number of homeomorphisms of remainders

In [1], we proved the following: *let X be a nowhere locally compact space with a compactification bX, then*

$$(\dagger) \qquad |\mathscr{H}(bX \setminus X)| \le |\mathsf{RO}(X)|^{\mathrm{Hsw}(X)} \le 2^{d(X)\mathrm{Hsw}(X)} \le 2^{\mathrm{nw}(X)} \le 2^{|X|}.$$

Here $\mathrm{nw}(X)$ denotes the *netweight* of X. This implies that if X in addition is countable, then $|\mathscr{H}(bX \setminus X)| \le \mathfrak{c}$. The aim of this section is to present a variation of this inequality and to show that it is independent of the previous one.

Theorem 3.1. *Let X be a nowhere locally compact space with a compactification bX. Then*

$$(\ddagger) \qquad |\mathscr{H}(bX \setminus X)| \le |X|^{d(X)} \le 2^{|X|}.$$

Proof. First observe that both X and the remainder $Y = bX \setminus X$ are dense in bX. If $x \in X$, then $\mathscr{U}_x$ denotes the family of all neighborhoods of x in bX. Now let $f \in \mathscr{H}(Y)$ be arbitrary, and define for every $x \in X$, the set $f^{\#}(x)$, as follows:

$$f^{\#}(x) = \bigcap_{U \in \mathscr{U}_x} \overline{f(U \cap Y)}$$

(here 'closure' denotes closure in bX.) It was shown in the proof of [1, Theorem 3.1] that the following statements hold:

(1) For every $x \in X$, $f^{\#}(x)$ is a nonempty compact subset of X.
(2) $f^{\#}$ is upper semi-continuous.
(3) If $f, g \in \mathscr{H}(Y)$ and $f \neq g$, then there exists $x \in X$ such that $f^{\#}(x) \cap g^{\#}(x) = \emptyset$.

Let $D \subseteq X$ be dense.

Claim 1. If $f, g \in \mathscr{H}(Y)$ and $f \neq g$, then there exists $d \in D$ such that $f^{\#}(d) \cap g^{\#}(d) = \emptyset$.

By (3), there exists $x \in X$ such that $f^{\#}(x) \cap g^{\#}(x) = \emptyset$. Hence by (1) we may pick disjoint open subsets U and V of X such that $f^{\#}(x) \subseteq U$ and $g^{\#}(x) \subseteq V$. By (2), there is an open neighborhood E of x such that for each $y \in E$ we have $f^{\#}(y) \subseteq U$ and $g^{\#}(y) \subseteq V$. Hence any $d \in E$ is as desired.

Let $\prec$ be a well-ordering on X. Now for $f \in \mathscr{H}(bX \setminus X)$ we define $\tilde{f} \colon D \to X$ by $\tilde{f}(d) = \min f^{\#}(d)$. Here the minimum of course refers to the well-ordering $\prec$. It follows by Claim 1, that the assignment $f \mapsto \tilde{f}$ is one-to-one. Hence $|\mathscr{H}(bX \setminus X)| \leq |X|^{|D|}$, and so we are done. □

We will now show that the two bounds (†) and (‡) are independent. That is, for a nowhere locally compact space X, there is in general no relation between $|\mathsf{RO}(X)|^{\mathrm{Hsw}(X)}$ and $|X|^{d(X)}$.

Example 3.2. There is a nowhere locally compact space X such that

$$|\mathsf{RO}(X)|^{\mathrm{Hsw}(X)} < |X|^{d(X)}.$$

Proof. Indeed, let $X = \mathscr{F}[\mathbb{R}]$ denote the Pixley-Roy hyperspace of the real numbers $\mathbb{R}$ ([7]; see also [5]). Hence $\mathscr{F}[\mathbb{R}]$ has the set of all nonempty finite subsets of $\mathbb{R}$ as its underlying set. For $F, U \subseteq \mathbb{R}$, where $F \in \mathscr{F}[\mathbb{R}]$ and U is an open neighborhood of F in the euclidean topology on $\mathbb{R}$, put

$$[F, U] = \{G \in \mathscr{F}[\mathbb{R}] : F \subseteq G \subseteq U\}.$$

The topology on $\mathscr{F}[\mathbb{R}]$ is generated by the base of all such $[F, U]$'s.

Observe that $\mathscr{F}[\mathbb{R}]$ is nowhere locally compact.

It is known that the topology on X is finer than the Vietoris topology on the the set of all nonempty subsets of $\mathbb{R}$ ([5, Proposition 2.1]). This topology has countable weight, hence $\mathrm{Hsw}(X) = \omega$. Moreover, X satisfies the countable chain condition, [7], and clearly, $\mathrm{w}(X) = \mathfrak{c}$. As a consequence,

$$|\mathsf{RO}(X)| \leq \mathrm{w}(X)^{c(X)} = \mathfrak{c}^{\omega} = \mathfrak{c},$$

and so

$$|\mathsf{RO}(X)|^{\mathrm{Hsw}(X)} = \mathfrak{c}^\omega = \mathfrak{c}. \tag{1}$$

On the other hand, $d(X) = \mathfrak{c}$. To see this, assume that $\mathscr{G}$ is any subset of X of size less than $\mathfrak{c}$. Pick $x \in \mathbb{R} \setminus \bigcup \mathscr{G}$. Then the nonempty open subset $[\{x\}, (x-1, x+1)]$ of X misses $\mathscr{G}$, i.e., $\mathscr{G}$ is not dense (this argument is definately well-known). Hence

$$|X|^{d(X)} = \mathfrak{c}^\mathfrak{c} = 2^\mathfrak{c} > \mathfrak{c},$$

as required. □

Question 3.3. Does $\mathscr{F}[\mathbb{R}]$ have a homogeneous remainder? Is a homogeneous remainder of $\mathscr{F}[\mathbb{R}]$ first-countable (what if CH)?

Observe that by inequality (†) in §3 and (1) above it follows that if R is a homogeneous remainder of $\mathscr{F}[\mathbb{R}]$, then $|R| \leq \mathfrak{c}$.

Example 3.4. There is a nowhere locally compact space X such that

$$|X|^{d(X)} < |\mathsf{RO}(X)|^{\mathrm{Hsw}(X)}.$$

Proof. We may assume that the ordinal space $Y = W(\mathfrak{c})$ is a subspace of $2^\mathfrak{c}$. Let D be a countable dense subspace of $2^\mathfrak{c}$, and put $X = Y \cup D$. Then $|X| = \mathfrak{c}$ and $d(X) = \omega$, hence $|X|^{d(X)} = \mathfrak{c}$.

We claim that $\mathrm{Hsw}(X) \geq \mathrm{cf}(\mathfrak{c})$, the cofinality of $\mathfrak{c}$. Striving for a contradiction, let $\mathscr{U}$ be an open collection in X of size less than $\mathrm{cf}(\mathfrak{c})$. We may assume that $\mathscr{U}$ is closed under finite intersections and unions. Let $\mathscr{V} = \{U \in \mathscr{U} : U \cap W(\mathfrak{c}) \neq \emptyset\}$. For every $V \in \mathscr{V}$, pick an arbitrary element $\alpha(V) \in V \cap W(\mathfrak{c})$. The set $S = \{\alpha(V) : V \in \mathscr{V}\}$ has size less than $\mathrm{cf}(\mathfrak{c})$, hence has compact closure in $W(\mathfrak{c})$. Let $\overline{S}$ denote that closure. Pick an element $p \in W(\mathfrak{c}) \setminus \overline{S}$. Then $\{p\}$ and $\overline{S}$ cannot be separated by disjoint elements of $\mathscr{U}$ since the element of $\mathscr{U}$ that would contain p would miss S. Hence indeed, $\mathrm{Hsw}(X) \geq \mathrm{cf}(\mathfrak{c})$, from which it follows that

$$|\mathsf{RO}(X)|^{\mathrm{Hsw}(X)} \geq 2^{\mathrm{cf}(c)} > \mathfrak{c},$$

as required. □

4. Applications

Inequalities such as (†) and (‡) in the previous section allow one to conclude that many spaces are not homogeneous. Consider e.g., the case of a Cantor cube 2^κ and its dense subset X_κ. Here κ is an infinite cardinal number. We assume throughout that X_κ is nowhere locally compact so that its complement $Y_\kappa = 2^\kappa \setminus X$ is dense as well.

Let us first assume that X_κ is countable. Then Y_κ is not homogeneous if $2^\kappa > \mathfrak{c}$. This is easy. Indeed, by Theorem 3.1 it follows that $|\mathscr{H}(Y_\kappa)| \le \mathfrak{c}$, while $|Y_\kappa| = 2^\kappa > \mathfrak{c}$.

As a consequence, if $\kappa \ge \omega_1$, then Y_κ is not homogeneous under $2^{\omega_1} > \mathfrak{c}$. Interestingly, $2^{\omega_1} \setminus X_{\omega_1}$ *is* homogeneous under $\mathsf{MA}{+}\neg\mathsf{CH}$ by [1, Theorem 4.2]. Also observe that $2^\omega \setminus X_\omega$ is homogeneous, being homeomorphic to the space of irrational numbers.

Let us now pass to the potentially more complicated case where X_κ is uncountable. A particularly interesting case is when X_κ is the standard Σ-product Σ_κ in 2^κ, i.e,

$$\Sigma_\kappa = \{f \in 2^\kappa : |\{\alpha < \kappa : f(\alpha) = 1\}| \le \omega\}.$$

Observe that each permutation of κ induces a homeomorphism of 2^κ under which Σ_κ is invariant. This implies that for $X_\kappa = \Sigma_\kappa$ we have $|\mathscr{H}(X_\kappa)| = |\mathscr{H}(Y_\kappa)| = 2^\kappa$. Hence Y_κ cannot be shown to be nonhomogeneous by having too few homeomorphisms. Observe that X_κ is homogeneous being a subgroup of 2^κ.

Proposition 4.1. *Assume that $\kappa > \omega_1$. Then $2^\kappa \setminus \Sigma_\kappa$ is not homogeneous.*

Proof. It will be convenient to adopt the above notation $X_\kappa = \Sigma_\kappa$ and $Y_\kappa = 2^\kappa \setminus X_\kappa$. Consider the points $f, g \in 2^\kappa$ defined by $f(\alpha) = 1 \Leftrightarrow \alpha \le \omega_1$ and g is the constant function 1. It is clear that there is a subset $A \subseteq X_\kappa$ such that $|A| = \omega_1$ and $f \in \overline{A}$. Striving for a contradiction, assume that there is a homeomorphism $\xi\colon Y_\kappa \to Y_\kappa$ sending f onto g. Since Y_κ contains the Σ-product in 2^κ with base point g and κ is uncountable, it follows that $\beta Y_\kappa = 2^\kappa$. Hence ξ can be extended to a homeomorphism $\beta\xi\colon 2^\kappa \to 2^\kappa$. So g is in the closure of $\beta\xi(A)$ which is a subset of X_κ. But this is clearly impossible since $|\beta\xi(A)| = \omega_1$ and $\omega_1 < \kappa$; just observe that any limit point of $\beta\xi(A)$ will have many coordinates equal to 0. $\square$

Hence the situation for complements of Σ-products is completely settled for $\kappa \ne \omega_1$: $2^\kappa \setminus \Sigma_\kappa$ is homogeneous if $\kappa = \omega$ (for it is empty) and not if $\kappa > \omega_1$. For $\kappa = \omega_1$, we have no idea. But we do know that $2^{\omega_1} \setminus \Sigma_{\omega_1}$ is not (homeomorphic to) a topological group under $2^\omega < 2^{\omega_1}$, which is a consequence of our final result.

Theorem 4.2. *Suppose that κ is of uncountable cofinality such that $\kappa^\omega < 2^\kappa$ and $Z = 2^\kappa$. Then, for any $X \subseteq Z$ such that X is dense in Z and $|X| < |Z|$, the complement $Z \setminus X$ is not (homeomorphic to) a topological group.*

Proof. Put $Y = Z \setminus X$. Clearly, X and Y are both dense in Z.

For each $a \in Z$, let Z_a be the Σ-product in Z with the base-point a.

We have: $|Z_a| = \kappa^\omega < 2^\kappa = |Z|$. For distinct a and b in Z, the sets Z_a and Z_b are either disjoint or coincide. Therefore, there exists a subset A of Z such that the family $\gamma = \{Z_a : a \in A\}$ is disjoint and covers Z. Clearly, $|A| = 2^\kappa$.

Since $|\gamma| = 2^\kappa$ and $|X| < 2^\kappa$, there exists $b \in A$ such that $Z_b \subseteq Y$. This shows that Y is pseudocompact since X_b is countably compact and dense in Z (and hence in Y), and κ has uncountable cofinality. And also $Z = \beta Y$ since $Z = \beta Z_b$.

Let us now assume that Y is homeomorphic to a topological group. We will bring this assumption to a contradiction. By pseudocompactness of Y and the fact that $\beta Y = Z$, there exists a group operation on the space Z which turns Z into a topological group and Y into a subgroup of Z (Comfort and Ross [4]). Hence, the space $X = Z \setminus Y$ contains a dense topological copy of Y. As a consequence, $|X| \geq |Y| = 2^\kappa$, a contradiction. □

Notice that there are plenty of subspaces X of Z such that X is dense in Z and $|X| < |Z|$.

It was shown in [1, Theorem 5.2] that there are pseudocompact topological groups no remainder of which is homogeneous. From Proposition 4.1 we conclude that there are many pseudocompact (even countably compact) subgroups of Cantor cubes whose complements are not homogeneous. This prompts an obvious question, which is answered by the following result.

Example 4.3. There is a pseudocompact noncompact group G with a compactification bG such that $bG \setminus G$ is homogeneous.

Proof. Let $A \subseteq 2^{\omega_1}$ be the standard Σ-product. Pick a point $p \in 2^{\omega_1} \setminus A$, and let M be a maximal subgroup of 2^{ω_1} containing A but not containing p. It is easy to see that M has index 2. □

Can there be a countably compact such topological group? This is so if there is an Ulam-measurable cardinal κ. Let p be a κ-complete nonprincipal ultrafilter on κ. Consider p to be a subset of the Cantor group 2^κ. Then its complement, the dual ideal of p, is a countably compact subgroup of 2^κ of index 2. This observation can be found e.g. in Comfort and Remus [3].

5. Questions

In this section we formulate some open problems.

Question 5.1. Suppose that $X = 2^{\mathfrak{c}}$, is the Cantor group with weight $\mathfrak{c}$, and $Z \subseteq X$ be dense in X with $|Z| < 2^{\mathfrak{c}}$. Then can the complement $X \setminus Z$ be homogeneous?

Theorem 4.2 and Question 5.1 motivate the next concrete question:

Question 5.2. Suppose that $X = 2^{\omega_1}$, and Z the standard Σ-product contained in X. Then can the complement $X \setminus Z$ be homogeneous?

For $x \in 2^{\omega_1}$ we put $A(x) = x^{-1}(\{0\})$ and $B(x) = \omega_1 \setminus A(x)$. We claim that $X \setminus Z$ has only two types of points (which might be the same types). Indeed, put

$$S = \{x \in X \setminus Z : |A(x)| \leq \omega\}, \text{and } T = \{x \in X \setminus Z : |A(x)| = \omega_1\}.$$

We will show that all points of S are of the same type in $X \setminus Z$ and, similarly, for T.

Observe that for every $x \in X \setminus Z$ we have that $B(x)$ has size ω_1.

Now if x and y in $X \setminus Z$ are such that both $A(x)$ and $A(y)$ have the same cardinality, then there is a permutation of ω_1 which sends $B(x)$ onto $B(y)$ and $A(x)$ onto $A(y)$. The homeomorphism of X that is induced by this permutation then takes x onto y and $X \setminus Z$ is invariant under this homeomorphism.

This proves that all points of T are of the same type in $X \setminus Z$.

Now take arbitrary $x, y \in S$. We will show that we may assume without loss of generality that both $A(x)$ and $A(y)$ are infinite. Assume that $A(x)$ is finite, and let $T \subseteq B(x)$ be countably infinite. Define a homeomorphism f of 2^{ω_1} by

$$f(p)(\alpha) = \begin{cases} p(\alpha) & (\alpha \notin T), \\ 1 + p(\alpha) (\text{mod } 2) & (\alpha \in T). \end{cases}$$

Then $X \setminus Z$ is invariant under f, and $f(x)$ has the property that $|A(f(x))| = \omega$. There similarly is a homeomorphism g of 2^{ω_1} such that $X \setminus Z$ is invariant under g, and $g(y)$ has the property that $|A(g(y))| = \omega$. Now by the above, there is a homeomorphism h of 2^{ω_1} such that $X \setminus Z$ is invariant under h, and $h(f(x)) = g(y)$. Hence all points of S are of the same type as well.

Observe that under $2^\omega < 2^{\omega_1}$ we have by Theorem 4.2 that $X \setminus Z$ is not a topological group.

A similar question for Cantor cubes 2^κ with κ greater than ω_1 was answered in the negative by Proposition 4.1.

Question 5.3. Is there a noncompact countably compact topological group G with a compactification bG such that $bG \setminus G$ is homogeneous?

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