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A theorem on remainders of topological groups

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It has been established in [7–9] that a non-locally compact topological group G with a first-countable remainder can fail to be metrizable. On the other hand, it was shown in [6] that if some remainder of a topological group G is perfect, then this remainder is first-countable. We improve considerably this result below: it is proved that in the main case, when G is not locally compact, the space G is separable and metrizable. Some corollaries of this theorem are given, and an example is presented showing that the theorem is sharp.

1. Preliminaries

By 'a space' we understand a Tychonoff topological space. By a *remainder* of a space X we mean the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. We follow the terminology and notation in [11]. In particular, a space X has *countable type* if every compact subspace S of X is contained in a compact subspace T of X which has a countable base of open neighbourhoods in X. A space X is *perfect* if every closed subset of X is a G_δ-set in X. A space Y is *ω-bounded* if the closure in Y of every countable subset of Y is compact.

The *spread* of a space X is uncountable if there exists an uncountable discrete in itself subspace of X. For the definition and main properties of free sequences, see [2], Chapter II.

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Recall that *paracompact p-spaces* [1] can be characterized as preimages of metrizable spaces under perfect mappings. *A Lindelöf p-space* is a preimage of a separable metrizable space under a perfect mapping. For the concept of a p-space and the properties of p-spaces see [1]. Every p-space is of countable type. For information on topological groups see [10].

2. Introduction

A series of results on remainders of topological groups have been obtained in [3–9]. They show that the remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. We continue this line of investigation in this paper and consider the following question: *when is some remainder of a topological group G hereditarily Lindelöf?* Some partial results in this direction were obtained in [6]. In particular, it has been established there that *if G is a topological group with a remainder Y which is a perfect space, then Y is first-countable* (see Corollary 2.15 in [6]). The main result below is a considerable strengthening of this statement in the case when G is assumed to be not locally compact: the group G turns out to be metrizable! Since there exists a non-metrizable non-locally compact topological group G with a first-countable remainder (see [7–9]), this conclusion cannot be derived directly from the theorem in [6]. This result answers question 3.9 from [6] in the affirmative.

3. The theorem

Theorem 3.1. *Suppose that G is a non-locally compact topological group with a remainder $Y = bG \setminus G$ in a compactification bG of G such that Y is perfect. Then G is separable and metrizable, and Y is a hereditarily Lindelöf, first-countable p-space with a countable π-base, and every compact subspace of Y has a countable base of open neighbourhoods in Y.*

Proof. *Fact 1.* The remainder Y is first-countable.

Fact 1 follows from Corollary 2.15 in [6]. For the sake of completeness, we present its proof.

Notice that Y is dense in bG, since G is nowhere locally compact. The remainder Y is either pseudocompact or Lindelöf, since G is a topological group [5]. If Y is pseudocompact, then the conclusion holds, since Y is perfect.

Thus, it remains to consider the case when Y is Lindelöf. Then Y is hereditarily Lindelöf, since Y is perfect. Therefore, the Souslin number $c(Y)$ is countable. Since Y is dense in bG, we conclude that the Souslin number $c(bG)$ is also countable. Hence, $c(G) \leq \omega$, since G is dense in bG. However, G is paracompact, since G is a topological group and the remainder Y of G is Lindelöf (see [6] or [5]). Therefore, the space G is Lindelöf, since every paracompact space with countable Souslin number is Lindelöf. Hence, Y is a space of countable type, by the Henriksen–Isbell Theorem. Since Y is also perfect, the conclusion that Y is first-countable follows and Fact 1 is established.

Fact 2. Either G is separable and metrizable, or Y is ω-bounded.

This fact is also known (see [6]), but again we prove it for the sake of completeness. Indeed, suppose that A is a countable subset of Y such that the closure of A in Y is not compact. Then, clearly, there exists a point $p \in G$ such that $p \in \overline{A}$. Since Y is first-countable and dense in bG, and A is countable, we can easily define a countable π-base η of bG at p. Since G is dense in bG, it follows that $\xi = \{V \cap G : V \in \eta\}$ is a countable π-base of G at p. Taking into account that G is a topological group, we conclude that G is first-countable and hence, metrizable. Therefore, Y is Lindelöf (see [6]). Hence, as we saw in the proof of Fact 1, Y is hereditarily Lindelöf of which it follows that the Souslin number of G is countable. Since G is metrizable, we conclude that G is also separable and Fact 2 is established.

Fact 3. If the closure in Y of every countable subset of Y is compact, then there exists a free sequence in Y of uncountable length and hence, the spread of Y is uncountable.

Indeed, Y is dense in bG, but doesn't coincide with bG. Also, Y is countably compact, since the closure in Y of every countable subset of Y is compact. Hence we can construct by transfinite recursion a free sequence in Y of uncountable length by the same standard argument as this is done in the proof of Theorem 2.2.13 in [2]. Since every free sequence is a discrete subspace, it follows that the spread of Y is uncountable. Here is the construction of the free sequence.

Fix a point $e \in G$. Clearly, $e \notin Y$ but $e \in \overline{Y}$. Suppose that $\beta < \omega_1$ and, for every $\alpha < \beta$, an open neighbourhood V_α of e in bG and a point $y_\alpha \in Y$ have been defined so that $\overline{V_\alpha} \cap \overline{\{y_\gamma : \gamma < \alpha\}} = \varnothing$ and $y_\alpha \in \bigcap\{V_\gamma : \gamma \leq \alpha\}$. Then the set $A_\beta = \{y_\alpha : \alpha < \beta\}$ is a countable subset of Y. Hence, e is not in the closure of A_β in bG. Therefore, there exists an open neighbourhood V_β of e in bG such that $\overline{V_\beta} \cap \overline{\{y_\alpha : \alpha < \beta\}} = \varnothing$. Put $H_\beta = \bigcap\{V_\alpha : \alpha \leq \beta\}$. Since $e \in \overline{Y}$ and H_β is a G_δ-subset of bG such that $e \in H_\beta$, it follows from countable compactness of Y that $H_\beta \cap Y \neq \varnothing$. Now we define y_β to be any member of the set $H_\beta \cap Y$ we like. The construction is complete. In this way we have obtained a transfinite sequence $\xi = \{y_\alpha : \alpha < \omega_1\}$ of points in Y which satisfies the following condition:

$$\overline{\{y_\alpha : \alpha < \beta\}} \cap \overline{\{y_\alpha : \beta \leq \alpha < \omega_1\}} = \varnothing,$$

for every $\beta < \omega_1$. This follows from the obvious inclusion $\{y_\alpha : \beta \leq \alpha < \omega_1\} \subset V_\beta$ and the definition of V_β. Thus, we see that ξ is a free sequence in Y and in bG of the length ω_1. Fact 3 is established.

Using Facts 1, 2, and 3, we get:

Fact 4. G is separable and metrizable.

Assume the contrary. Then, by Fact 2, the closure in Y of every countable subset of Y is compact. Notice that this implies that Y is countably compact. It also follows by Fact 3 that the spread of Y is uncountable. Since Y is perfect, there exists an uncountable discrete subset of Y which is closed in Y. However, this is clearly impossible, since Y is countably compact.

Now we can easily complete the proof of the theorem, as follows. The remainder Y is a Lindelöf p-space by Fact 4 since every remainder of a separable metrizable space is a Lindelöf p-space [3]. Since G is separable and first-countable in bG, it easily follows that Y is separable. Since Y is clearly first-countable (in Y and in bG), it follows that Y has a countable π-base (in Y and in bG). We also observe that every compact subspace of Y has a countable base of open neighbourhoods in Y, since Y is a perfect p-space (recall that every p-space is a space of countable type). □

4. Comments, questions, and another theorem

The next statement answers a question in [6] (Problem 3.9).

Corollary 4.1. *A remainder Y of a topological group G is a perfect space if and only if Y is hereditarily Lindelöf.*

When G is not locally compact, the last assertion follows from Theorem 3.1. If G is locally compact, then the remainder Y is compact and the above statement also holds. However, we cannot claim that G in Corollary 4.1 has a countable base. To see this, just take the Alexandroff double of the closed unit interval $[0, 1]$. Thus, the assumption in Theorem 3.1 that G is not locally compact cannot be dropped.

Corollary 4.2. *A non-locally compact topological group G is separable and metrizable if and only if some remainder of it is a perfect space.*

The next example demonstrates that Theorem 3.1 is sharp from yet another point of view.

Example 4.3. There exists a compactification bQ of the space Q of rational numbers (taken with the usual topology) such that the remainder $Y = bQ \setminus Q$ is nowhere locally compact, hereditarily Lindelöf, hereditarily separable, Čech-complete non-metrizable space. Indeed, let B be the "two arrows" compactum. Recall that B is hereditarily Lindelöf, hereditarily separable, first-countable, dense-in-itself, but not metrizable. Fix an arbitrary countable dense subspace S of B. Clearly, S has a countable base and hence, is metrizable. Since S is countable and has no isolated points, the space S is homeomorphic to the space Q of rational numbers, and B is a compactification bS of S. The remainder $Y = bS \setminus S$ is hereditarily Lindelöf, hereditarily separable, Čech-complete non-metrizable space. This last property of Y easily follows from the non-metrizability of B. Hence this example shows that in Theorem 3.1 we cannot claim that the remainder Y has to be metrizable.

Clearly, the proof of Theorem 3.1 can be obviously transformed into a proof of the following statement:

Theorem 4.4. *Suppose that G is a non-locally compact topological group with a remainder $Y = bG \setminus G$ in a compactification bG of G such that Y is first-countable and the spread of Y is countable. Then G is separable and metrizable, and Y is a Lindelöf p-space.*

The next question posed in [6] remains open.

Problem 4.5. Suppose that some remainder of a topological group G is a normal space. Then does it follow that every remainder of G is Lindelöf?

Problem 4.6. Suppose that some remainder of a topological group G is a normal first-countable space. Then does it follow that G is metrizable?

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References

[1] A.V. Arhangel'skii, On a class of spaces containing all metric and all locally compact spaces, Mat. Sb. 67 (109) (1965) 55–88, English translation: Transl. Am. Math. Soc. 92 (1970) 1–39.
[2] A.V. Arhangelskii, Structure and classification of topological spaces and cardinal invariants, Usp. Mat. Nauk 33 (6) (1978) 29–84 (in Russian). English translation: Russ. Math. Surv. 33 (6) (1978) 33–96.
[3] A.V. Arhangel'skii, Remainders in compactifications and generalized metrizability properties, Topol. Appl. 150 (2005) 79–90.
[4] A.V. Arhangel'skii, First-countability, tightness, and other cardinal invariants in remainders of topological groups, Topol. Appl. 154 (2007) 2950–2961.
[5] A.V. Arhangel'skii, Two types of remainders of topological groups, Comment. Math. Univ. Carol. 49 (1) (2008) 119–126.
[6] A.V. Arhangel'skii, A study of remainders of topological groups, Fundam. Math. 203 (2009) 165–178.
[7] A.V. Arhangel'skii, J. van Mill, On topological groups with a first-countable remainder, Topol. Proc. 42 (2013) 157–163.
[8] A.V. Arhangel'skii, J. van Mill, On topological groups with a first-countable remainder, II, Topol. Appl. 195 (2015) 143–150.
[9] A.V. Arhangel'skii, J. van Mill, On topological groups with a first-countable remainder, III, Indag. Math. 25 (1) (2014) 35–43.
[10] A.V. Arhangel'skii, M.G. Tkachenko, Topological Groups and Related Structures, Atlantis Stud. Math., vol. 1, Atlantis Press/World Scientific, Paris, 2008.
[11] R. Engelking, General Topology, second ed., Heldermann Verlag, Berlin, 1989.