Topology and its Applications 225 (2017) 27–33

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Proceedings on the International Conference on Set-Theoretic Topology and its Applications, Yokohama 2015

Nonnormality of Čech–Stone-remainders of topological groups

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ARTICLE INFO

Article history:
Received 9 November 2015
Received in revised form 8 May 2016
Accepted 8 May 2016
Available online 27 April 2017

MSC:
54D35
54D40
54A25

Keywords:
Remainder
Compactification
Topological group
Normal space
C^*-embedded
Uniform ultrafilter
F-space
Free sequence

ABSTRACT

It is known that every remainder of a topological group is Lindelöf or pseudocompact. Motivated by this result, we study in this paper the question when a topological group G has a normal remainder. Under mild conditions on G we show that under the Continuum Hypothesis, if the Čech–Stone remainder G^* of G is normal, then it is Lindelöf.

1. Introduction

All topological spaces under discussion are Tychonoff.

By a *remainder* of a space X we mean the subspace $bX \setminus X$ of a compactification bX of X. Among the best known remainders are the Čech–Stone remainders $X^* = \beta X \setminus X$ for arbitrary spaces X and the 1-point remainders $\alpha Y \setminus Y$ for locally compact spaces Y.

Remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. A nice example demonstrating this, is Arhangel'skii's

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[1] The work of the first-named author is supported by RFBR, project 15-01-05369.

http://dx.doi.org/10.1016/j.topol.2017.04.008
0166-8641/

Theorem from [4]: every remainder of a topological group is Lindelöf or pseudocompact. All remainders of locally compact groups are compact, hence both Lindelöf and pseudocompact. For non-locally compact groups there is a dichotomy: every remainder is either Lindelöf or pseudocompact.

Observe that if X is a separable and metrizable topological space, then it has a separable metrizable compactification. The remainder of this compactification is separable metrizable as well, and hence Lindelöf. This implies that the Čech–Stone remainder $X^* = \beta X \setminus X$ of X is Lindelöf, being a perfect preimage of a Lindelöf space. Hence all remainders of X are Lindelöf since every remainder is a continuous image of X^*. Similarly, if a space X has at least one Lindelöf remainder, then all remainders are Lindelöf. (This is folklore.)

In this paper we are interested in the question when the normality of a remainder of a topological group forces that remainder to be Lindelöf, or forces other remainders to be normal.

As we saw above, this is always the case for separable metrizable groups. But not always so, as can be demonstrated by an example that was brought to our attention by Buzyakova (for a different reason). Supply $G = \{0,1\}^{\omega_1}$ with the topology generated by all boxes that are determined by countably many coordinates. Then G is a topological group, is linearly ordered and hence has a linearly ordered compactification. Hence the remainder of G in this compactification is monotonically normal and therefore, hereditarily normal. But that remainder is not Lindelöf, simply observe that G is a P-space and that any P-space with a Lindelöf remainder is discrete. We will show that the Čech–Stone remainder of this topological group G is not normal. Hence it is not true that the normality of a specific remainder implies that all remainders are normal. Hence normal remainders behave differently compared to Lindelöf remainders.

There are many results in the literature on so called points of nonnormality in Čech–Stone remainders. A point x of a space X is said to be a point of nonnormality, if $X \setminus \{x\}$ is not normal. It was shown by Gillman (see [9]) that under CH, every non-P-point is a point of nonnormality of ω^*. This also holds for P-points, as was shown independently by Warren [23] and Rajagopalan [20]. Hence under CH, all points of ω^* are nonnormality points. The first nonnormality points in ω^* in ZFC, were constructed by Błaszczyk and Szymański [8]. The question whether every point of ω^* is a nonnormality point in ZFC remains unsolved and is a classical problem by now. More recent results on nonnormality points in Čech–Stone compactifications can be found e.g. in Bešlagić and van Douwen [7], Logunov [15], Terasawa [21], and Fleissner and Yengulalp [11].

So there is quite an extensive literature on nonnormality points in Čech–Stone remainders, and history tells us that these results and the remaining problems are complicated. It is only recently that the question of when a remainder of a topological group is normal was asked for the first time in Arhangel'skii [5] (Section 3). It was shown there that no Dowker space can be a remainder of a topological group (Theorem 3.1).

Surprisingly, we are unaware of any other question in the literature that asks for conditions on X that imply that X^* is normal, or some remainder of X is normal, whereas for Lindelöf remainders such conditions are well-known (Henriksen and Isbell [13]). We will show that for G a nowhere locally compact topological group that contains a nonempty compact G_δ-subset, if the character of G is at most $\mathfrak{c}$, and G^* is normal, then G^* is Lindelöf under CH. We will formulate several applications of this result, for example to Moscow topological groups which constitute a rather large class of topological groups.

2. Preliminaries

We abbreviate Čech–Stone remainder as *CS-remainder*.

A subspace Y of a space X is said to be C^*-*embedded* in X if every bounded continuous function $f\colon Y \to \mathbb{R}$ can be extended to a bounded continuous function $\bar{f}\colon X \to \mathbb{R}$. Here $\mathbb{R}$ denotes the space of real numbers.

A *cardinal* is an initial ordinal, and an *ordinal* is the set of smaller ordinals. We use κ, λ and μ to denote cardinals; we always assume $\kappa \geq \omega$. Cardinals are endowed with the discrete topology.

By $U(\omega_1)$ we mean the subspace of *uniform* ultrafilters on ω_1, i.e.,

$$U(\omega_1) = \{p \in \beta\omega_1 : (\forall\, P \in p)(|P| = \omega_1)\}.$$

A point $p \in \beta\omega_1 \setminus U(\omega_1)$ is called *subuniform*.

An F-space is a space in which cozero-sets are C^*-embedded. Observe that Tietze's Theorem implies that every closed subspace of a normal F-space is again an F-space. Also, every countable subspace of an F-space is C^*-embedded (Walker [22, p. 37]). It follows from Parovičenko's characterization of ω^*, [19], that under CH, the space one gets from ω_1^* by collapsing $U(\omega_1)$ to a single point, is homeomorphic to ω^*. For details, see Comfort and Negrepontis [9] and [16, §1.4]. Observe that the collapsed set $U(\omega_1)$ is a P-point in the quotient space.

It is known that for every locally compact, noncompact and σ-compact space X, the CS-remainder X^* is an F-space. The proof is trivial, by a nontrivial observation of Negrepontis [18, Proof of Theorem 3.2]. If $A \subseteq X^*$ is an arbitrary F_σ-subset of X^*, then $Y = X \cup A$ is σ-compact, hence normal, hence A is C^*-embedded in Y being closed in Y, hence in βX since $\beta Y = \beta X$. A *normal* space X is an F-space iff any two disjoint open F_σ-subsets of X have disjoint closures in X ([16, 1.2.2(b)]).

A nonempty space is *small* if βX has weight at most $\mathfrak{c}$. This is equivalent to the statement that $|C^*(X)| = 2^\omega$.

Woods [24] proved that under CH, each small countably compact F-space is compact.

For all undefined notions, see Engelking [10].

3. The main result

We begin by identifying some subspaces in X^* that are (close to being) F-spaces.

Lemma 3.1. *Let X be a space. If S is a compact G_δ-subset of βX such that $S \cap X$ is compact, then every open F_σ-subset of $S \setminus X$ is C^*-embedded in its closure (in $S \setminus X$, which coincides with its closure in X^*).*

Proof. Let U be an open F_σ-subset of $Y = S \setminus X$, and let $y \in \mathrm{cl}_{X^*}(U) \setminus U$. Observe that $y \in Y$. Moreover, let $f : U \to \mathbb{R}$ be a bounded continuous function. Since $y \notin S \cap X$, there is a compact neighborhood T of y in S which misses the compact set $S \cap X$ and which is a G_δ-subset of S. Clearly, T misses X and is a compact G_δ-subset of βX. Moreover, y is in the closure of $U \cap T$. Put $Z = \beta X \setminus T$. Then Z contains X, hence $\beta Z = \beta X$ and $Z^* = T$. Since Z is locally compact and σ-compact, it follows that T is a compact F-space (see §2). Hence every open F_σ-subset of T is a cozero-set, and so C^*-embedded. We conclude that we can extend $f \restriction T$ to a continuous function $f'_y : (U \cap T) \cup \{y\} \to \mathbb{R}$. Since T is a neighborhood of y in $S \setminus X$, this means that we can extend f to a continuous function $f_y : U \cup \{y\} \to \mathbb{R}$. The union of the f_y's is the desired extension of f. $\square$

Corollary 3.2 (CH). *Let X be a space such that X^* is normal and pseudocompact. If S is a compact G_δ-subset of βX such that $S \cap X$ is compact, then $S \setminus X$ is an F-space and has the property that each countable subset is C^*-embedded in βX and has compact closure in $S \setminus X$.*

Proof. Let $A \subseteq S \setminus X$ be countable, and let B denote the closure of A in $S \setminus X$. Then B is a small F-space, being a separable closed subspace of the normal F-space $S \setminus X$ (Lemma 3.1). Clearly, $S \setminus X$ is countably compact. Hence B is a small countably compact F-space, and so is compact by Woods' result quoted in §2. Since every countable subspace of an F-space is C^*-embedded in that F-space, we conclude that A is C^*-embedded in B, which is a compact subset of βX, and hence in βX. $\square$

Corollary 3.3 (CH). *Let X be a nowhere locally compact space of character at most $\mathfrak{c}$ such that X^* is normal and pseudocompact. Then no nonempty compact subset of X is a G_δ-subset of X.*

Proof. Striving for a contradiction, assume that K is a nonempty compact G_δ-subspace of X. Write $X \setminus K$ as $\bigcup_{n<\omega} A_n$, where each A_n is closed in X. Pick an arbitrary $p \in K$. For each $n < \omega$, let Z_n be a compact G_δ-subset of βX that contains p but misses $\mathrm{cl}_{\beta X}(A_n)$. Then $S_0 = \bigcap_{n<\omega} Z_n$ is a compact G_δ-subset of βX containing p and $L = S_0 \cap X$ is compact, being a closed subspace of K.

This defines S_0. Since X^* is pseudocompact, we may pick a point $x_0 \in S_0 \setminus X$. Let $\{U_\alpha : \alpha < \omega_1\}$ be a neighborhood base at p in S_0 such that $U_0 = S_0$.

Suppose that we defined for some $\alpha < \omega_1$, a decreasing sequence $\{S_\beta : \beta < \alpha\}$ of compact G_δ-subsets of S_0 such that $p \in S_\beta \subseteq U_\beta$, and points $x_\beta \in (S_\beta \setminus X) \setminus \overline{\{x_\gamma : \gamma < \beta\}}$. By Corollary 3.2, $\{x_\beta : \beta < \alpha\}$ has compact closure in S_0. Hence we let S_α be a compact G_δ-subset of $\bigcap_{\gamma<\alpha} S_\gamma$ which contains p, is contained in U_α, and misses the closure of $\{x_\beta : \beta < \alpha\}$. Since X^* is pseudocompact, we may pick a point $x_\alpha \in S_\alpha \setminus X$. This completes the transfinite construction; put $F = \{x_\alpha : \alpha < \omega_1\}$. Then F is clearly a free sequence and hence is discrete; it moreover converges to p by construction. The map $\alpha \mapsto x_\alpha$ can be extended to a continuous map $f\colon \beta\omega_1 \to E$ (here ω_1 has the discrete topology), where E is the closure of F in βX.

Claim 1. *If $q \in E \setminus X$, then $f^{-1}(q)$ is a single point which is contained in $\beta\omega_1 \setminus U(\omega_1)$.*

Indeed, let C be a closed neighborhood of q in βX that misses the compact set $S_0 \cap X$. Observe that $C \cap F$ is countable, hence $C \cap F$ has compact closure in $S_0 \setminus X$ and is C^*-embedded in βX (Corollary 3.2). Now, $f^{-1}(C)$ is a closed neighborhood of $f^{-1}(q)$ in $\beta\omega_1$ and $f^{-1}(C) \cap \omega_1$ is countable. Hence $f^{-1}(q)$ consists entirely of subuniform ultrafilters of ω_1. Striving for a contradiction, assume that $f^{-1}(q)$ contains two distinct points, say a and b. Then there are disjoint subsets A and B of $f^{-1}(C) \cap \omega_1$ such that a is in the closure of A and b is in the closure of B. Then $f(A)$ and $f(B)$ are disjoint subsets of $C \cap F$, and consequently have disjoint closures in βX since $C \cap F$ is discrete and C^*-embedded in βX. This is a contradiction, since both closures contain q.

Since F converges to p in βX, it is now clear that $E \setminus F$ is homeomorphic to $\beta\omega_1/U(\omega_1)$. Hence CH implies that X^* contains a closed copy of the space $\omega^* \setminus \{t\}$, where t is a P-point of ω^* (Comfort and Negrepontis [9]). But that space is not normal under CH, as was shown independently by Rajagopalan [20] and Warren [23] (see also [17]). (Actually, we do not need CH here, since Kunen and Parsons [14] proved that $\beta\omega_1 \setminus U(\omega_1)$ is nonnormal in ZFC.) This is a contradiction. $\square$

We now come to our main result.

Theorem 3.4 (CH). *Let G be a topological group of character at most $\mathfrak{c}$ which contains a nonempty compact G_δ-subset. Then if G^* is normal, it is Lindelöf.*

Proof. It is clear that we may assume that G is not locally compact. By Arhangel'skii [4], G^* is either pseudocompact or Lindelöf. But G^* is not pseudocompact by Corollary 3.3. Hence it is Lindelöf. $\square$

Corollary 3.5 (CH). *Let G be a countable topological group such that G^* is normal. Then G^* is Lindelöf and G is metrizable.*

Proof. Since by Theorem 3.4 G^* is Lindelöf, G contains a compact G_δ-subset of βG (Henriksen and Isbell [13]). This set has an isolated point since G is countable. But then G is first-countable, and hence metrizable by the Birkhoff–Kakutani Theorem. $\square$

Corollary 3.6 *(CH). Let G be a topological group of character at most $\mathfrak{c}$ which is a union of a countable pairwise disjoint family of compacta. Then if G^* is normal, it is Lindelöf.*

Let G denote Buzyakova's topological group that was discussed in §1. It is a P-space and so βG is basically disconnected, [12, 6M.1], and hence an F-space, [12, 14N.4]. Countable subspaces of βG are consequently C^*-embedded in βG. Observe that G has character ω_1, and that since G is a P-space, every countable subset of G^* has compact closure in G^*. Hence by using the same arguments as in the proof of Corollary 3.3, it follows that G^* is not normal in ZFC (by the result of Kunen and Parsons [14], we do not need CH here).

4. Moscow spaces and groups

Recall that a topological space X is said to be *Moscow* if the closure of any open subset U of X is the union of some family of G_δ-subsets of X (Arhangel'skii [2]).

Quite a few large classes of Moscow spaces and groups are productive. For example, the product of every family of first countable spaces is a Moscow space. The class of Moscow spaces also contains all spaces of countable pseudocharacter and all extremally disconnected spaces. Under mild restrictions, topological groups are Moscow. For example, all locally bounded groups, every subgroup of a σ-compact group, groups with countable Souslin number and Fréchet–Uryson groups are Moscow. The notion of a Moscow space was shown to interact especially well with homogeneity (see [2]), so no wonder that it works especially well in the class of topological groups.

For details and references, see Arhangel'skii [1] and [3] and the applications of the main result, obtained in the preceding section, to Moscow topological groups given below.

Theorem 4.1 *(CH). Let G be a Moscow topological group of character at most $\mathfrak{c}$. Then if G^* is normal, it is either Lindelöf or C^*-embedded in βG.*

We need the following result that is of independent interest.

Proposition 4.2. *Suppose that G is a Moscow non-locally compact topological group, and $Y = bG \setminus G$ is the remainder of G in a compactification bG of G. Furthermore, suppose that A and B are open subsets of Y that have disjoint closures in Y. Then either 1) the closures of A and B in bG are disjoint, or 2) there exists a nonempty compact G_δ-subspace of G.*

Proof. Assume the closures of A and B in bG are not disjoint, and fix $x \in bG$ such that $x \in \overline{A} \cap \overline{B}$.

Fix open subsets U, V of bG such that $A = U \cap Y$, and $B = V \cap Y$. Clearly, Y is dense in bG. Therefore, A and B are dense in U and V, respectively.

Claim 1. *The intersection of the closures in bG of U and V is a compact subset F of G, that is, $\overline{U} \cap \overline{V} = F \subseteq G$.*

This is so, since the closures of A and B in Y are disjoint.

It follows from Claim 1 that $x \in G$. Therefore, x belongs to the closures in G of the open subsets $U \cap G$ and $V \cap G$ of G. Since G is Moscow, it follows that there are closed in G G_δ-subsets K_1 and K_2 of G such that $x \in K_1 \subseteq \overline{U}$ and $x \in K_2 \subseteq \overline{V}$. Then $K = K_1 \cap K_2 \subseteq \overline{U} \cap \overline{V} = F \subseteq G$. Therefore, K is closed in F, and hence, K is compact. Clearly, K is a G_δ-subset of G, which is as desired. □

Observe that the last statement can be reformulated as follows:

Theorem 4.3. *Suppose that G is a Moscow topological group such that no nonempty compact subset of G is a G_δ in it. Then G^* is C^*-embedded in the Čech–Stone compactification βG of G, that is, G is the CS-remainder of its CS-remainder G^*.*

This theorem reduces Theorem 4.1 to Theorem 3.4. Simply observe that if the second alternative in Proposition 4.2 holds, then Theorem 4.1 indeed follows from Theorem 3.4. If the first alternative in Proposition 4.2 holds, then, clearly, G^* is C^*-embedded in βG and hence, Theorem 4.1 holds in this case as well.

Corollary 4.4 *(CH). Let G be a topological group of character at most $\mathfrak{c}$. Suppose also that G satisfies at least one of the following conditions:*

(1) *The Souslin number of G is countable.*
(2) *The tightness of G is countable.*
(3) *G is σ-compact.*
(4) *G is submetrizable.*
(5) *G is a Lindelöf Σ-space.*
(6) *G is extremally disconnected.*
(7) *The κ-tightness $t\kappa(G)$ of G is countable.*

Then if G^* is normal, it is either Lindelöf or C^*-embedded in βG.

Recall that the κ-tightness $t\kappa(G)$ of G is countable, if for every point x in the closure of an open subset U of G belongs to the closure of some countable subset of U.

Problem 4.5. Let G be a precompact topological group of uncountable weight. Is G^* nonnormal?

For partial answers to this question, see Arhangel'skii and van Mill [6].

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