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A SHORT PROOF OF TORUŃCZYK'S CHARACTERIZATION THEOREMS

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Abstract. We present short proofs of Toruńczyk's well-known characterization theorems of the Hilbert cube and Hilbert space, respectively.

1. Introduction

All spaces under discussion are assumed to be separable and metrizable.

Recall that a compactum (complete space) Y is *strongly universal* if any map $f : X \longrightarrow Y$ from a compactum (complete space) can be approximated arbitrarily closely by a (closed) embedding into Y. In the non-compact case the closeness is measured by open covers of Y.

The aim of this note is to provide a short proof of the following results:

Theorem 1.1.

(1) *A locally compact space is a Hilbert cube manifold if and only if it is an* ANR *which is strongly universal (with respect to compact spaces).*
(2) *A complete space is a Hilbert space manifold if and only if it is an* ANR *which is strongly universal (with respect to complete spaces).*

This is a reformulation of the main results in Toruńczyk [20–22]. He showed in [20] that a locally compact ANR X is a Hilbert cube manifold if and only if any two maps $\mathbb{I}^k \to X$ for $k \in \mathbb{N}$ can be approximated arbitrarily closely by maps having disjoint images (here $\mathbb{I}$ denotes the closed unit interval $\mathbb{I}$). This was subsequently named the "disjoint cubes property". A complete ANR space has the disjoint cubes property if and only if it is strongly universal with respect to compact spaces ([17, Theorem 7.3.5]). Moreover, Toruńczyk showed in [21, 22] that a complete ANR is a Hilbert space manifold if and only if any map $\bigoplus_{n\in\mathbb{N}} \mathbb{I}^n \to X$ is approximable arbitrarily closely by maps sending $\{\mathbb{I}^n : n \in \mathbb{N}\}$ to discrete families. This was subsequently named the "discrete cubes property". If a complete ANR X has the discrete cubes property, then X is strongly universal with respect to complete spaces ([8, Theorem, p. 127]). For more details, historical comments and references on these facts, see the books [17] and [11].

Toruńczyk's [20–22] results are widely known and were applied in diverse settings. They also inspired the characterization results of the universal Menger spaces by Bestvina [6] and the recent work on Nöbeling spaces [1–3], [19], [15, 16].

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A compactum (complete space) is said to be a *Hilbert type compactum* (*Hilbert type space*) if it is strongly universal and an AR. A closed subset F of a space X is said to be a Z-set in X if the identity map of X can be arbitrarily closely approximated by a map $f : X \longrightarrow X$ such that $f(X) \cap F = \emptyset$.

Definition 1.2. We say that a Hilbert type compactum (space) H is a *model space* for Hilbert type compacta (Hilbert type spaces) if it has the following properties:

(i) (stability) $H \approx H \times \mathbb{I}$;

(ii) (Z-set unknotting theorem) given an open cover $\mathscr{U}$ of H, an open subset Ω of H, homeomorphic Z-sets Z_1 and Z_2 of H contained in Ω and a homeomorphism $\phi : Z_1 \longrightarrow Z_2$ homotopic to the identity map of Z_1 by a homotopy controlled by $\mathscr{U}$ and supported by Ω there exists a homeomorphism $\Phi : H \longrightarrow H$ such that Φ extends ϕ and Φ is controlled by $\mathscr{U}$ and supported by Ω.

The goal of this note is to present a complete and self-contained proof of the fact that the existence of a model space for Hilbert type compacta (Hilbert type spaces) implies the characterization theorem which says that every two Hilbert type compacta (spaces) are homeomorphic. We also show how this characterization leads to a proof of Theorem 1.1.

One would probably expect that our abstract approach will make the proofs longer and more complicated and we were surprised to find out that this approach can considerably shorten and simplify the proofs despite our use of already known techniques and ideas. This was mainly achieved by carefully analyzing existing proofs, extracting essential parts, avoiding unnecessary repetitions, splitting the proofs into short parts and sometimes reversing the historical order of the results. For example, we simplified the proof of Miller's cell-like resolution theorem [18] by using techniques introduced later for proving the characterization theorems.

It was known before the characterization theorems were proved that model spaces for Hilbert type compacta and Hilbert type spaces exist (for example Hilbert cube and Hilbert space respectively, see Chigogidze [11, §2,4] for details and references). One of the features of our approach is that we never work with a particular realization of a model space, we don't even assume that the Hilbert cube Q is a model space for Hilbert type compacta. Although in the compact case we are able to detect at a relatively early stage of the proof that if a model space exists it must be homeomorphic to Q, in the non-compact case we can pretend not to know what a model space looks like until the characterization is proved.

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2. Preliminaries

We assume that the reader is familiar with general facts regarding AR's, Hilbert type compacta and Hilbert type spaces, cell-like maps, etc. Most of the necessary information can be found in [17, §§7.1-7.3] and [11]. See also Chapman [9] and Edwards [12]. Earlier simplifications of Toruńczyk's proofs can be found in [7] and [23].

Let $f : X \longrightarrow Y$ be a proper map and let A be a closed subset of Y. By $X \cup_f A$ we denote the quotient space of X obtained by collapsing the fibers over A to singletons. As usual, for a proper surjection $f : X \longrightarrow Y$, we let $M(f)$ denote the mapping cylinder of f, that is, $M(f)$ is obtained from $X \times \mathbb{I}$ by replacing $X \times \{1\}$ by Y. We usually let $\pi_Y : M(f) \longrightarrow Y$ denote the projection. If $f : X \to Y$ and

$A \subset Y$, then we say that f is *one-to-one over* A if the restriction of f to $f^{-1}(A)$ is one-to-one.

The following result displays a technique for proving the existence of a homeomorphism between an $M(f)$ and the range of f that is known as 'the Edwards trick'. This result is very similar to Lemma 42.1 of [10] which Chapman attributes to Edwards in Note §42 on page 107.

Proposition 2.1. *Let $f : X \to Y$ be a proper surjection between complete metric spaces and let $\pi_Y : M(f) \to Y$ be the natural projection. Assume that for every open cover $\mathscr{V}$ of Y, there is a proper near homeomorphism $\alpha : M(f) \to M(f)$ such that $\pi_Y \circ \alpha$ is $\mathscr{V}$-close to π_Y and for every y either $\alpha(\pi_Y^{-1}(y))$ is a singleton or $\alpha(\pi_Y^{-1}(y))$ does not meet $X \times \{0\}$. Then π_Y is a near homeomorphism.*

Proof. Observe that α is surjective because it is a proper near homeomorphism.

Identify $M(f) \setminus Y$ with $X \times [0, 1)$ and let $\pi_X : M(f) \setminus Y \longrightarrow X$ be the natural projection. With the aim of using Bing shrinking consider open covers $\mathscr{U}_M$ of $M(f)$ and $\mathscr{U}_Y$ of Y respectively. Replacing $\mathscr{U}_Y$ by a finer open cover of Y we may assume that the closures of the sets in $\mathscr{U}_Y$ refine $\mathscr{U}_M$ restricted to Y. Then there are an open cover $\mathscr{U}_X$ of X refining $f^{-1}(\mathscr{U}_Y)$ and an infinite sequence of continuous functions $t_i : X \longrightarrow (0, 1)$ $(i \in \mathbb{N})$ with $t_{i+1}(a) < t_i(a)$ and $\lim_{i\to\infty} t_i(a) = 0$ for every $a \in X$ such that the collection $\mathscr{U}$ of open sets in $M(f)$ defined below refines $\mathscr{U}_M$. Assume that $\mathbb{I}$ is located on the vertical axis as usual so that the points of Y are above the graph of every function t_i in $M(f)$. The collection $\mathscr{U}$ consists of the portions of the sets of $\pi_Y^{-1}(\mathscr{U}_Y)$ above the graph of the function t_3 and the portions of the sets of $\pi_X^{-1}(\mathscr{U}_X)$ between the graphs of t_{i+2} and t_i for $i \in \mathbb{N}$.

Let $\mathscr{V}$ be an open cover of Y such that $\mathrm{st}(\mathscr{V})$ refines $\mathscr{U}_Y$ and assume that α is chosen as in the premise of the proposition with $\pi_Y \circ \alpha$ is $\mathscr{V}$-close to π_Y. Then one can find an infinite sequence of continuous functions $s_i : X \longrightarrow (0, 1)$ $(i \in \mathbb{N})$ with $s_{i+1}(a) < s_i(a)$ and $\lim_{i\to\infty} s_i(a) = 0$ for every $a \in X$ with the following properties. For every set of the form $F = \alpha(\pi_Y^{-1}(\{y\}))$ where $y \in Y$ that is not a singleton in $X \times \{0\}$ we have that F lies in between the graphs of s_{i+2} and s_i for some $i \in \mathbb{N}$ if F intersects the closed region below the graph of s_2; F lies above the graph of s_3 if F intersects the closed region above the graph of s_2 and $\pi_X(F)$ is contained in an element of $\mathscr{U}_X$ if F intersects the closed region below the graph of s_1.

Put $t_0(a) = s_0(a) = 1$ for each $a \in X$. Consider the homeomorphism $\psi : X \times \mathbb{I} \longrightarrow X \times \mathbb{I}$ that maps every interval $\{a\} \times [t_{i+1}(a), t_i(a)]$ linearly onto $\{a\} \times [s_{i+1}(a), s_i(a)]$. Then $\Psi : M(f) \longrightarrow M(f)$ is the homeomorphism of $M(f)$ induced by ψ. We have that $\pi_Y = \pi_Y \circ \Psi$ and the sets of the form $\Psi^{-1}(\alpha(\pi_Y^{-1}(\{y\}))), y \in Y$ refine $\mathscr{U}_M$.

Recall that α is a proper near homeomorphism and that π_Y is proper. Then α can be approximated by a homeomorphism Φ so close to α that the sets of the form $\Psi^{-1}(\Phi(\pi_Y^{-1}(\{y\}))), y \in Y$ also refine $\mathscr{U}_M$. Clearly if Φ is sufficiently close to α, then $\pi_Y \circ \Phi^{-1} \circ \Psi$ is $\mathscr{U}_Y$-close to π_Y and the proposition follows from Bing's shrinking criterion. $\square$

A closed embedding $f : X \longrightarrow Y$ is said to be a Z-embedding if $f(X)$ is a Z-set in Y. Let us end this section with recalling the following **Z-embedding approximation properties** which play an important role in the proofs:

(i) Let Y be a complete ANR *strongly universal with respect to maps from complete spaces. Then every map $f : X \longrightarrow Y$ from a complete space X can arbitrarily*

closely be approximated by a Z-embedding $f' : X \longrightarrow Y$. Moreover, if f is a Z-embedding on a closed subset F of X, then f' can be chosen to coincide with f on F.

(ii) Let Y be a locally compact ANR *strongly universal with respect to maps from compact spaces. Then every proper map $f : X \longrightarrow Y$ from a locally compact space X can arbitrarily closely be approximated by a Z-embedding $f' : X \longrightarrow Y$. Moreover, if f is a Z-embedding on a closed subset F of X, then f' can be chosen to coincide with f on F.*

Although these properties are well known and often considered as folklore, we provide their elementary (that is, not based on characterization theorems) proof in Section 7 because an appropriate reference was not found. For the sake of simplicity some authors even regard the Z-embedding approximation properties as an alternative definition of strongly universal ANR's.

3. Topological characterization of the Hilbert cube

Everywhere in this section 'model space' means 'model space for Hilbert type compacta'. We will show that the existence of a model space implies the characterization of Hilbert type compacta.

The following proposition is very similar to Lemma 42.2 of Chapman [10] which is attributed to West [24] and Edwards in Note §42 on page 107 of [10].

Proposition 3.1. *Let X and Y be compact* AR*'s, $f : X \longrightarrow Y$ a cell-like map and A a closed subset of Y such that $Z = f^{-1}(A)$ is a Z-set in X and Z has a closed neighborhood H homeomorphic to a model space. Then the quotient map $\pi : X \longrightarrow X \cup_f A$ is a near homeomorphism and A is a Z-set in $X \cup_f A$.*

Proof. Note that, since H is an AR and Z is contained in the interior of H we get that Z is a Z-set in H as well. Fix a sufficiently small $\varepsilon > 0$ such that the ε-neighborhood Ω of Z is contained in H and let $\mathscr{U}$ be an open ε-cover of Y. Using that f is a fine homotopy equivalence (Haver [13]), we lift f restricted to Z to a Z-embedding $\phi : Z \longrightarrow H$ such that $\phi(Z) \subset \Omega$, $\phi(f^{-1}(a))$ is of diameter less than ε for every $a \in A$ and ϕ is homotopic to the identity map of Z by a homotopy controlled by $f^{-1}(\mathscr{U})$ and supported by Ω. Then, by Definition 1.2, ϕ extends to a homeomorphism $\Phi : X \longrightarrow X$ controlled by $f^{-1}(\mathscr{U})$ and supported by Ω. Hence Bing's shrinking criterion implies that π is a near homeomorphism. Then, since $Z = f^{-1}(A) = \pi^{-1}(A)$ is a Z-set in X, we get that A is a Z-set in $X \cup_f A$. Indeed, approximate the identity map of X by a map $f' : X \longrightarrow X \setminus Z$ and approximate π by a homeomorphism π'. Then the map $\pi \circ f' \circ (\pi')^{-1}$ witnesses that A is a Z-set in $X \cup_f A$. □

Corollary 3.2. *Let H be a model space, $f : H \longrightarrow Y$ a cell-like map and Y a compact* AR*. Then the projection from $H \times \mathbb{I}$ to $M(f)$ is a near homeomorphism. In particular, we get that $M(f) \approx H$.*

Proof. Follows from Proposition 3.1. □

Let $f : X \longrightarrow Y$ be a map of compacta and $\pi_Y : M(f) \longrightarrow Y$ the projection. We say that f is a **nice** map if the identity map of Y can arbitrarily closely be approximated by an embedding $g : Y \longrightarrow Y$ such that $\pi_Y^{-1}(g(Y))$ is a Z-set in $M(f)$.

Note that for every map $f : X \longrightarrow Y$ and a Hilbert type compactum C the induced map $f \times id : X \times C \longrightarrow Y \times C$ is a nice map. Indeed, observe that $M(f \times id) = M(f) \times C$ and $\pi_{Y \times C} = \pi_Y \times id$. Approximate the projection of $Y \times C$ to C by a Z-embedding $\phi : Y \times C \longrightarrow C$. Then the embedding $g : Y \times C \longrightarrow Y \times C$ defined by $g(y, c) = (y, \phi(y, c))$ for $(y, c) \in Y \times C$ witnesses that $f \times id$ is nice.

A similar argument also shows that the identity map on a Hilbert type compactum is nice.

This result is very similar to Theorem 43.1 of [10] which is attributed to Edwards in Note §43 on page 107.

Proposition 3.3. *Let H be a model space, let $f : H \longrightarrow Y$ be a nice cell-like map and Y a Hilbert type compactum. Then the projection $\pi_Y : M(f) \longrightarrow Y$ is a near homeomorphism.*

Proof. We aim to use Proposition 2.1. Since Y is a Hilbert type compactum and f is a nice map we conclude that π_Y restricted to $H \times \{0\}$ can be arbitrarily closely approximated by an embedding $g : H \times \{0\} \longrightarrow Y$ such that for $A = g(H \times \{0\})$ we have that $\pi_Y^{-1}(A)$ is a Z-set in $M(f)$. Let $\pi : M(f) \longrightarrow M(f) \cup_{\pi_Y} A$ and $\pi'_Y : M(f) \cup_{\pi_Y} A \longrightarrow Y$ be the natural projections. Observe that $M(f) \cup_{\pi_Y} A$ is an AR by Hu [14, Theorem VI.1.3 on page 181]. By Corollary 3.2 and Proposition 3.1 the projection π is a near homeomorphism and A is a Z-set in $M(f) \cup_{\pi_Y} A$. Hence one can choose a homeomorphism $h : M(f) \longrightarrow M(f) \cup_{\pi_Y} A$ so that the map $\pi \circ h^{-1} \circ \pi : M(f) \longrightarrow M(f) \cup_{\pi_Y} A$ is as close to π as we wish. Note that $\pi_Y = \pi'_Y \circ \pi$ and hence we may assume that $\pi_Y \circ h^{-1} \circ \pi = \pi'_Y \circ \pi \circ h^{-1} \circ \pi : M(f) \longrightarrow Y$ is as close to π_Y as we wish. Also note that π_Y is a fine homotopy equivalence. Then assuming that g is sufficiently close to π_Y on $H \times \{0\}$ and h is sufficiently close to π we can apply Definition 1.2 and Corollary 3.2 to replace h by its composition with a homeomorphism of $M(f)$ sending $H \times \{0\}$ to $h^{-1}(A)$ and still keeping $\pi_Y \circ h^{-1} \circ \pi$ and π_Y as close as we wish. Thus we now assume that h sends $H \times \{0\}$ onto A. Putting $\alpha = h^{-1} \circ \pi : M(f) \longrightarrow M(f)$ we note that α is a near homeomorphism and for every $y \in Y$ either $\alpha(\pi_Y^{-1}(y))$ is a singleton or $\alpha(\pi_Y^{-1}(y))$ does not meet $H \times \{0\}$, and the maps π_Y and $\pi_Y \circ \alpha$ are as close as we wish. Thus Proposition 2.1 applies with $X = H$. $\square$

Theorem 3.4. *Let H be a model space. Then*

(i) $H \approx H \times Q$ and

(ii) if $f : H \longrightarrow Y$ is a cell-like map and Y is a Hilbert type compactum, then f is a near homeomorphism.

Proof. Since the identity map $id : H \longrightarrow H$ is a nice map we get by Proposition 3.3 that the projection $M(id) = H \times \mathbb{I} \longrightarrow H$ is a near homeomorphism. As a consequence, the projection $H \times \mathbb{I} \longrightarrow H$ is shrinkable. Thus the projection $H \times \mathbb{I}^n \longrightarrow H$ is shrinkable for every n. But this clearly implies that the projection $H \times Q \longrightarrow H$ is shrinkable, hence a near homeomorphism by the Bing shrinking criterion and so $H \times Q \approx H$.

For part *(ii)* assume first that $f : H \longrightarrow Y$ is a nice map. Then f is a near homeomorphism because the projections $H \times \mathbb{I} \longrightarrow H$, $H \times \mathbb{I} \longrightarrow M(f)$ and $M(f) \longrightarrow Y$ are near homeomorphisms by Corollary 3.2 and Proposition 3.3.

Now consider the general case. Recall that $H \approx H \times Q$ and $f \times id : H \times Q \longrightarrow Y \times Q$ is a nice map, because Q is a Hilbert type compactum. Then $f \times id$ is a near homeomorphism and hence $Y \times Q \approx H$. Note that the projections $H \times Q \longrightarrow H$

and $Y \times Q \longrightarrow Y$ are nice maps and therefore they are also near homeomorphisms. All this implies that f is a near homeomorphism. $\square$

Theorem 3.5. *Any compact* AR *is a cell-like image of any model space.*

We will prove this theorem in the next section. Theorems 3.4 and 3.5 immediately imply the characterization theorem for Hilbert type compacta.

4. Cell-like resolution

In this section we prove Theorem 3.5. For its proof we need the following auxiliary propositions and constructions. Recall that a model space means a model space for Hilbert type compacta.

Proposition 4.1. *Let H be a model space, X a Z-set in H and $f : X \longrightarrow H$ any map. Then f extends to a cell-like map $H \longrightarrow H$.*

Proof. Let us first consider the case when $f(X)$ is a Z-set. By the Z-set unknotting theorem we may assume that X and $f(X)$ are disjoint. Take any Z-set $A \subset H$ such that A is an AR, $f(X) \subset A$ and $X \cap A = \emptyset$. Consider f as a map $f : X \longrightarrow A$. By extending f over a bigger Z-set we may assume that X is an AR. Let $M(f)$ be the mapping cylinder of f and $\pi_A : M(f) \longrightarrow A$ the projection. Embed $M(f)$ as a Z-set in H so that X and A are identified with the corresponding natural subsets of $M(f)$. Then the adjunction space $Y = H \cup_{\pi_A} A$ is an AR (Hu [14, Theorem VI.1.3 on page 181]), the projection $\pi : H \longrightarrow Y$ is cell-like and hence, by Proposition 3.1, π is a near homeomorphism and A is a Z-set in Y. Thus there is a homeomorphism $g : Y \longrightarrow H$ and by the Z-set unknotting theorem we may assume that g sends A to A by the identity map. Then π followed by g is the extension of f we are looking for.

Now consider the general case. Let the map $\phi : X \longrightarrow H \times \mathbb{I}$ be defined by $\phi(x) = (f(x), 0)$. By the previous case ϕ extends to a cell-like map $\Phi : H \longrightarrow H \times \mathbb{I}$. Then Φ followed by the projection of $H \times \mathbb{I}$ to H is the required cell-like extension of f. $\square$

Let $A \subset H$ be a compact AR and $r : H \longrightarrow A$ a retraction. We assume that the mapping cylinder $M(r)$ is obtained from $H \times \mathbb{I}$ by replacing $H \times \{1\}$ with A and denote by $\pi_A : M(r) \longrightarrow A$ the projection induced by r and by $\pi_I : M(r) \longrightarrow \mathbb{I}$ the projection to $\mathbb{I}$. We can rescale the interval $\mathbb{I}$ to another interval $[a, b]$ and consider $M(r)$ over $[a, b]$ assuming that the interval projection π_I sends $A \subset M(r)$ to the right end point b. We will also refer to $A \subset M(r)$ as the *right A-part* of $M(r)$ and $A \times \{0\} \subset H \times \{0\}$ as the *left A-part* of $M(r)$ and the projections π_A and π_I as the *A-projection* and the *interval projection* respectively.

By the *extended mapping cylinder* $E(r)$ of r we mean the union $M(r) \cup H \times [1/2, 1]$ in which we assume that $M(r)$ is the mapping cylinder over $[0, 1/2]$ and $A \times \{1/2\} \subset H \times [1/2, 1]$ is identified with $A \subset M(r)$ (the right A-part of $M(r)$). We will call $H \times [1/2, 1]$ the extension part of $E(r)$. We define the projection $\pi_I : E(r) \longrightarrow \mathbb{I}$ by the projections of $M(r)$ and $H \times [1/2, 1]$ to $[0, 1/2]$ and $[1/2, 1]$ respectively and we define the map $\pi_H^E : E(r) \longrightarrow H$ as the union of the projection $\pi_A : M(r) \longrightarrow A$ and the projection of $H \times [1/2, 1]$ to H. Clearly we can rescale $\mathbb{I}$ to an interval $[a, b]$ so that $M(r) \subset E(r)$ and the extension part of $E(r)$ will be sent by π_I to $[a, c]$ and $[c, b]$ respectively for some $a < c < b$. In that case we say that $E(r)$ is the extended cylinder over $[a, b]$ with $M(r) \subset E(r)$ being over $[a, c]$.

Proposition 4.2. *Let H be a model space and $r : H \longrightarrow A$ a retraction. Then there is a cell-like map from H to $E(r)$.*

Proof. By Proposition 4.1 there is a cell-like map $\phi : H \times [1/2, 1] \longrightarrow H \times [1/2, 1]$ so that $\phi(x, 1/2) = (r(x), 1/2)$ for $x \in H$. Consider $E(r)$ as the extended mapping cylinder over $\mathbb{I}$ with $M(r)$ being over $[0, 1/2]$ and identify $M(r)\setminus A$ with $H\times[0, 1/2)$. Extend ϕ to the map $\Phi : H \times \mathbb{I} \longrightarrow E(r)$ by $\Phi(x, t) = (x, t)$ for $x \in H$ and $0 \le t < 1/2$. Then Φ is the required cell-like map. $\square$

Let H be a model space. We say that a retraction $r : H \longrightarrow A$ is a *convenient* retraction if $E(r)$ is homeomorphic to H. Note that if H is a model space and $r : H \longrightarrow A$ is any retraction, then the induced retraction $r \times id : H \times Q \longrightarrow A \times Q$ is a convenient retraction because $E(r\times id) = E(r)\times Q$ is a Hilbert type compactum and because, by Proposition 4.2, $E(r \times id)$ admits a cell-like map from H and this map is a near homeomorphism by Theorem 3.4.

Let $r : H \longrightarrow A$ be a retraction. By the telescope $M(r, n)$ of n mapping cylinders of r we mean the union $M(r, n) = M_1(r)\cup\cdots\cup M_n(r)$ where $M_i(r)$ is the mapping cylinder of r over the interval $[t_{i-1}, t_i], t_i = i/n$ and the right A-part of M_i is identified with the left A-part of $M_{i+1}(r)$ by the identity map of A for $1 \le i \le n-1$. The projections of $M_i(r)$ to A and $[t_{i-1}, t_i]$ induce the corresponding projections $\pi_A : M(r, n) \longrightarrow A$ and $\pi_I : M(r, n) \longrightarrow \mathbb{I}$. Clearly $\mathbb{I}$ can be rescaled to any interval $[a, b]$.

In a similar way we define the infinite telescope $M(r, \infty) = \bigcup_{i=1}^{\infty} M_i(r)$ over an infinite partition $0 = t_0 < t_1 < t_2 < \dots, t_i \longrightarrow \infty$, of the ray $\mathbb{R}_+ = [0, \infty)$ with the mapping cylinder $M_i(r)$ being over the interval $[t_{i-1}, t_i]$. Again the A-projections and the interval projections of $M_i(r)$ define the projections $\pi_A : M(r, \infty) \longrightarrow A$ and $\pi_{\mathbb{R}} : M(r, \infty) \longrightarrow \mathbb{R}_+$ to which we will refer as the A-projection and the $\mathbb{R}$-projection respectively. Note that if $r : H \longrightarrow A$ is a convenient retraction, then $M(r, \infty)$ is homeomorphic to $H\times\mathbb{R}_+$. Indeed, assume that the cylinders of $M(r, \infty)$ are over the intervals $[i - 1, i]$, $i = 1, 2 \dots$. Then $\pi_{\mathbb{R}}^{-1}([0, 1/2])$ is homeomorphic to $H \times [0, 1/2]$ and $\pi_{\mathbb{R}}^{-1}([i - 1/2, i + 1/2])$ is homeomorphic to the extended cylinder $E(r)$ which is homeomorphic to $H \approx H \times [i - 1/2, i + 1/2]$ since r is convenient. Then using the Z-set unknotting theorem we can assemble all these pieces into a space homeomorphic to $H \times \mathbb{R}_+$.

The next two propositions are very similar to Theorems 41.2 and 41.3 of [10] which are attributed to Chapman and West in Note §41 on page 98 of [10].

Proposition 4.3. *Let H be a model space and $r : H \longrightarrow A$ be a convenient retraction. Then there is a homeomorphism $\phi : M(r) \longrightarrow M(r, 2) = M_1(r)\cup M_2(r)$ such that ϕ sends the right A-part and the left A-part of $M(r)$ to the right A-part of $M_2(r)$ and the left A-part of $M_1(r)$ respectively by the identity map of A. Moreover, ϕ can be chosen so that for the A-projections $\pi_A : M(r) \longrightarrow A$ and $\pi^*_A : M(r, 2) \longrightarrow A$ the composition $\pi^*_A \circ \phi$ is as close to π_A as we wish.*

Proof. Let $\pi_I : M(r) \longrightarrow \mathbb{I}$ and $\pi^*_I : M(r, 2) \longrightarrow \mathbb{I}$ be the interval projections. One can naturally identify $\pi_I^{-1}([0, 2/3])$ with $H \times [0, 2/3]$ and $(\pi^*_I)^{-1}([0, 2/3])$ with the extended mapping cylinder $E(r)$ of r over the interval $[0, 2/3]$ with mapping cylinder of $E(r)$ being over $[0, 1/2]$ and being identified with $M_1(r)$. Let the map $\pi^E_H : E(r) \longrightarrow H$ be as defined above. Since r is a convenient retraction and π^E_H is cell-like, we get that π^E_H is a near homeomorphism (Theorem 3.4). Since the projection $\pi_H : H \times [0, 2/3] \longrightarrow H$ is also a near homeomorphism one can find a homeomorphism $\psi : H \times [0,2/3] \longrightarrow E(r)$ so that $\pi_H^E \circ \psi$ is as close to π_H as we wish. Since $H \times \{2/3\}$ and the left A-part of $M_1(r)$ are Z-sets in $E(r)$, $A \times \{0\}$ and $H \times \{2/3\}$ are Z-sets in $H \times [0,2/3]$, and π_H^E and π_H similarly send these sets into H by the identity maps of A and H respectively, we can, in addition, by the Z-set unknotting theorem adjust ψ so that ψ sends $A \times \{0\}$ and $H \times \{2/3\}$ to the left A-part of $M_1(r)$ and $H \times \{2/3\}$ respectively by the identity maps. Extending ψ over $H \times \mathbb{I}$ by the identity map between $\pi_I^{-1}([2/3,1]) \subset M(r)$ and $(\pi_I^*)^{-1}([2/3,1]) \subset M_2(r)$ we get the required homeomorphism $\phi : M(r) \longrightarrow M(r,2)$. □

Proposition 4.4. *Let H be a model space and $r : H \longrightarrow A$ a convenient retraction. Then there is a cell-like map from* cone(H) *to* cone(A).

Proof. Denote by M^n the infinite telescope $M^n = \bigcup_{i=1}^{\infty} M_i^n$ of the mapping cylinders M_i^n of r over the intervals $[\frac{i-1}{2^n}, \frac{i}{2^n}]$ and let $\pi_A^n : M^n \longrightarrow A$ and $\pi_{\mathbb{R}}^n : M^n \longrightarrow \mathbb{R}_+$ be the A-projection and the $\mathbb{R}$-projection of M^n. We are going to construct a cell-like map $\phi : M^0 \longrightarrow A \times \mathbb{R}_+$. Let $\mathscr{E}_n, n = 1,2,\ldots$, be a sequence of open covers of A such that $\mathrm{st}(\mathscr{E}_{n+1})$ refines $\mathscr{E}_n$ and every set of $\mathscr{E}_n$ can be homotoped to a point inside a set of diam $< 1/2^n$. By Proposition 4.3 take a homeomorphism $\psi_{n+1}^n : M^n \longrightarrow M^{n+1}$ sending each mapping cylinder M_i^n of M^n to two consecutive mapping cylinders M_{2i-1}^{n+1} and M_{2i}^{n+1} of M^{n+1} as described in Proposition 4.3 and so that $\pi_A^{n+1} \circ \psi_{n+1}^n$ is $\mathscr{E}_{n+3}$-close to π_A^n. Denote $\psi_m^n = \psi_m^{m-1} \circ \cdots \circ \psi_{n+1}^n : M^n \longrightarrow M^m$ for $m > n$, $\psi_n^n = id$, $\phi_A^n = \pi_A^n \circ \psi_n^0$ and $\phi_{\mathbb{R}}^n = \pi_{\mathbb{R}}^n \circ \psi_n^0$. Note that if $x \in M^n$ belongs to M_i^n, then $(\pi_{\mathbb{R}}^m \circ \psi_m^n)(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n}]$ for every $m \geq n$. Thus we have that the sequences of maps $\{\phi_A^n\}$ and $\{\phi_{\mathbb{R}}^n\}$ converge and we denote their limits by $\phi_A : M^0 \longrightarrow A$ and $\phi_{\mathbb{R}} : M^0 \longrightarrow \mathbb{R}_+$ respectively. Consider $\phi = (\phi_A, \phi_{\mathbb{R}}) : M^0 \longrightarrow A \times \mathbb{R}_+$.

We will show that ϕ is cell-like. Take $x = (a,t) \in A \times \mathbb{R}_+$ and let $F = \phi^{-1}(x)$. Fix n and note that $\psi_n^0(F)$ is always contained in at most two consecutive mapping cylinders M_i^n and M_{i+1}^n of M^n. Also note that $\phi_A^n(F)$ refines $\mathscr{E}_{n+1}$. Then $\psi_n^0(F)$ can be homotoped to a point of the right A-part of M_{i+1}^n inside the set $(\pi_A^n)^{-1}(B) \cap (M_i^n \cup M_{i+1}^n)$ where $B \subset A$ is the closed $1/2^n$-neighborhood of $\phi_A^n(F)$ in A. Note $\mathrm{diam}\, B < 3/2^n$. Then $\mathrm{diam}\, \phi_A((\pi_A^n \circ \psi_n^0)^{-1}(B)) < 5/2^n$ and hence F can be homotoped to a point inside the set $\phi^{-1}(C \times [\frac{i-1}{2^n}, \frac{i+1}{2^n}])$ where C is the closed $5/2^n$-neighborhood of a in A. This implies that ϕ is cell-like.

It is also clear that ϕ is a proper map. Recall that M^0 is homeomorphic to $H \times \mathbb{R}_+$. Identify $H \times \mathbb{R}_+$ and $A \times \mathbb{R}_+$ with the complements of the vertices of cone(H) and cone(A) respectively. Then ϕ extends to a cell-like map from cone(H) to cone(A) by sending the vertex of cone(H) to the vertex of cone(A) and we are done. □

Proof of Theorem 3.5. Let A be a compact AR and H a model space. By Proposition 3.1, $H \approx H \times \mathbb{I} \approx$ cone(H). Recall that $A \times Q$ is a convenient retract of $H \approx H \times Q$. Then, by Proposition 4.4, cone$(A \times Q)$ is a cell-like image of cone(H). One can easily verify that if X is a Hilbert type compactum, then so is cone(X). Then, by Theorem 3.4, cone$(A \times Q) \approx$ cone(H). By the Z-set unknotting theorem we can identify the vertex of cone$(A \times Q)$ with the vertex of cone(H) and conclude from this that the set $A \times Q \times \{0\}$ has a closed neighborhood in $A \times Q \times \mathbb{I}$ homeomorphic to $H \approx H \times \mathbb{I}$. Then, by the symmetry, such a neighborhood also exists for $A \times Q \times \{1\}$ and hence, by Proposition 3.1, $A \times Q \approx A \times Q \times \mathbb{I}$ is homeomorphic to cone($A \times Q$). Thus $A \times Q \approx H$ and the projection of $A \times Q$ to A is the cell-like map we need. □

5. Topological characterization of Hilbert space

The results of the previous sections were intentionally presented in such a way that they apply with minor clarifications to the characterization of Hilbert type spaces. In this section we describe the adjustments in the proofs needed in the non-compact setting. Everywhere we replace Hilbert type compacta by Hilbert type spaces, a model space will mean a model space for the Hilbert type spaces and a cell-like map will mean a proper cell-like map. Almost all the proofs in Sections 3 and 4 will work in the non-compact setting with obvious trivial adjustments. So we will point out only those places that require clarifications.

Section 3.

Nice maps. The property that for a map $f : X \longrightarrow Y$ the induced map $f \times id : X \times Q \longrightarrow Y \times Q$ is nice remains true for non-compact spaces X and Y provided f is proper. We will show that using the properties that any proper map $g = (g_Y, g_Q) : X \longrightarrow Y \times Q$ can be arbitrarily closely approximated by

(*) a map $g' = (g'_Y, g'_Q) : X \longrightarrow Y \times Q$ such that $g'_Y = g_Y$ and $g'(X) \subset Y \times B(Q)$;

(**) an injective map $g' = (g'_Y, g'_Q) : X \longrightarrow Y \times Q$ such that $g'_Y = g_Y$ and $g'(X) \subset Y \times (Q \setminus B(Q))$.

Note that the maps g' in (*) and (**) are proper (and hence closed) since $g'_Y = g_Y$ and g is proper. Thus g' in (**) is a closed embedding.

Let us indicate how to prove (*) and (**). The case of compact X and Y is quite easy and, by a simple iterative procedure the compact case extends to the case of X and Y being locally compact. The general case reduces to the locally compact one as follows. Let $\mathscr{U}$ be an open cover of $Y \times Q$ determining the closeness of an approximation of g. Take any compactification Y^* of Y and extend g over a metric compactification X^* of X to a map $g^* : X^* \longrightarrow Y^* \times Q$ (for example embed X into Q, identify X with the graph Γ of g in $Q \times (Y^* \times Q)$ and set X^* to be the closure of Γ with g^* being the projection to $Y^* \times Q$). Extend $\mathscr{U}$ to an open cover of a set $U_Y \times Q$ where $Y \subset U_Y$ is open in Y^*. Thus we can replace X and Y by the locally compact sets $U_X = (g^*)^{-1}(U_Y \times Q)$ and U_Y respectively, g by g^* restricted to U_X and arrive at the locally compact setting.

Now we return to showing that $f \times id$ is nice. Let $\pi_Y : M(f) \longrightarrow Y$ be the projection. By (**) the identity map of $Y \times Q$ can be arbitrarily closely approximated by a closed embedding of $Y \times Q$ into $Y \times Q$ with the image A contained in $Y \times (Q \setminus B(Q))$. Then $(\pi_Y \times id)^{-1}(A)$ is contained in $M(f) \times (Q \setminus B(Q))$ and, hence by (*), $(\pi_Y \times id)^{-1}(A)$ is a Z-set in $M(f) \times Q$. Thus $f \times id$ is nice.

Theorem 3.4(i). The proof in the compact case can be extended to complete spaces using an infinite iterative procedure for constructing homeomorphisms of $H \times Q$ witnessing the shrinkability of the projection to H. Another way to prove Theorem 3.4(i) in the non-compact case is to show that the projections of the inverse limit of a sequence of complete spaces with bonding maps being proper near homeomorphisms are near homeomorphisms as well. It can be done by adjusting the proof of the similar result in the compact case [17, Theorem 6.7.4].

Theorem 3.5. The phrase compact AR should be replaced by the phrase Hilbert type space. The proof of this theorem is considered below (clarifications to Section 4).

Section 4. In Section 4 we need to consider only proper maps and retractions and everywhere assume that A is a Hilbert type space.

Proper retractions. A proper retraction to a Hilbert type space A always exists. Indeed, let $A \subset X$ be a closed subset of a complete space X. Note that $A \times \mathbb{I}$ is also a Hilbert type space with $A \times \{0\}$ being a Z-set in $A \times \mathbb{I}$. Then the identity map $A \longrightarrow A \times \{0\}$ extends to a Z-embedding $f : X \longrightarrow A \times \mathbb{I}$ and f followed by the projection of $A \times \mathbb{I}$ to A provides a proper retraction from X to A.

Convenient retractions. The property that a retraction $r : H \longrightarrow A$ from a model space H to $A \subset H$ induces a convenient retraction $r \times id : H \times Q \longrightarrow A \times Q$ remains true for non-compact spaces provided r is proper. Indeed, by Proposition 4.2, there is a proper cell-like map $f : H \longrightarrow E(r \times id)$. In order to show that $E(r \times id)$ is homeomorphic to H, it is enough to show, by Theorem 3.4, that $E(r \times id)$ is strongly universal. Take any map $g : X \longrightarrow E(r \times id)$ from a complete space X. Since f is a fine homotopy equivalence and H is a Hilbert type space we can lift g to a closed embedding $h : X \longrightarrow H$ so that $f \circ h$ is arbitrarily close to g. Note that $E(r \times id) = E(r) \times Q$. Then by (**) there is an arbitrarily close approximation of f by a closed embedding $f' : H \longrightarrow E(r \times id)$ and the closed embedding $f' \circ h : X \longrightarrow E(r \times id)$ witnesses the strong universality of $E(r \times id)$.

Proposition 4.4. We assume that the metrics on A and M^0 are complete, and the horizontal shift of M^0 one unit to the right is an isometric embedding. The properness of ϕ can be achieved as follows. Note that ϕ^n_A is proper on each cylinder M^0_i of M^0. Then we can additionally assume that ϕ^{n+1}_A is so close to ϕ^n_A that every fiber of $\phi^{n+1}_A|M^0_i$ is contained in the $1/2^{n+1}$-neighborhood of the corresponding fiber of $\phi^n_A|M^0_i$ for every i. Take a compact set K in $A \times \mathbb{R}_+$ and let a compact set $K_A \subset A$ and a natural number k be so that $K \subset K_A \times [0, k]$. Then for every n we have that $\phi^{-1}(K)$ is contained in the $1/2^n$-neighborhood of the compact set $(\phi^n_A, \phi^n_{\mathbb{R}})^{-1}(K_A \times [0, k+1])$. This implies $\phi^{-1}(K)$ is complete and totally bounded and, therefore, compact. Hence ϕ is proper.

Proof of Theorem 3.5. By cone(X) of a space X we mean the metric cone whose topology is defined by declaring $X \times [0, 1)$ to be an open subspace of cone(X) and the complements of $X \times [0, t]$ for $0 \le t < 1$ to form a basis of the vertex. We need to assume that X is a Hilbert type space and to verify that $X \times \mathbb{I} \approx$ cone(X) if $X \times \{1\}$ has a closed neighborhood H in $X \times \mathbb{I}$ so that H is homeomorphic to a model space. Let $\pi : X \times \mathbb{I} \longrightarrow$ cone(X) be the projection. Fix $\varepsilon > 0$, take a point $x \in U_\varepsilon = X \times (1 - \varepsilon, 1]$ and a neighborhood $U_x \subset U_\varepsilon$ of x. Applying an obvious homeomorphism of $X \times \mathbb{I}$ we can replace H by a homeomorphic set and assume that $U_\varepsilon \subset H$. Approximate the constant map sending $X \times \{1\}$ to x by a Z-embedding $\phi : X \times \{1\} \longrightarrow U_\varepsilon$. Note that $X \approx X \times \{1\}$ is an AR since it is a Hilbert type space. Then ϕ and the identity map of $X \times \{1\}$ are homotopic in U_ε and, by the Z-set unknotting theorem, ϕ extends to a homeomorphism $\Phi : X \times \mathbb{I} \longrightarrow X \times \mathbb{I}$ supported by U_ε. Now take another obvious homeomorphism $\Psi : X \times \mathbb{I} \longrightarrow X \times \mathbb{I}$ supported by $U_{2\varepsilon}$ such that Ψ sends U_ε into $\Phi^{-1}(U_x)$. Then, by Bing's shrinking criterion, $\Phi \circ \Psi$ witnesses that π is a near homeomorphism. □

6. Topological characterization of Hilbert cube and Hilbert space manifolds

In this section we apply the characterization of the Hilbert cube and Hilbert space to show that Hilbert cube and Hilbert space manifolds are topologically characterized as being strongly universal locally compact ANR's and strongly universal complete ANR's respectively where the strong universality of a locally compact space means the strong universality with respect to the maps from compact spaces.

Let X be a strongly universal locally compact ANR. If X is compact, then set $Y = \mathrm{cone}(X)$. If X is not compact, then denote by $X^* = X \cup \{*\}$ the one-point compactification of X and denote by Y the reduced cone over X^* which is obtained from $\mathrm{cone}(X^*)$ by collapsing the interval connecting the point $*$ with the vertex to a singleton. Note that in both cases Y is a compact AR and $X \times [0,1)$ embeds into Y as an open subset. That Y is an AR in the compact case follows from [17, Theorem 5.4.2], and in the non-compact case this can be proved in an almost identical way. Since Q and $Y \times Q$ are both Hilbert type compacta, then the characterization of the Hilbert cube implies $Y \times Q \approx Q$. Then $X \times [0,1) \times Q$ is a Hilbert cube manifold and hence $X \times \mathbb{I} \times Q \approx X \times Q$ is a Hilbert cube manifold as well.

Now assume that X is a strongly universal complete ANR. Define a space Y as the union of $X \times \mathbb{I}$ and the Hilbert space where $X \times \{1\}$ is identified with a homeomorphic Z-set in the Hilbert space. One can easily verify that Y is a strongly universal complete AR and hence, by the characterization of Hilbert space, both Y and $Y \times Q$ are homeomorphic to the Hilbert space. Observe that $X \times [0,1)$ is an open subset of Y and hence $X \times [0,1) \times Q$ is an open subset of $Y \times Q$. Thus we have that $X \times [0,1) \times Q$ is a Hilbert space manifold and hence $X \times \mathbb{I} \times Q \approx X \times Q$ is a Hilbert space manifold as well.

Note that the Z-set unknotting theorem for Hilbert cube implies the corresponding Z-set unknotting theorem for Hilbert cube manifolds in the form of (ii) of Definition 1.2 with the additional requirement that the homotopy is proper. Also note that the Z-set unknotting theorem for Hilbert space implies the corresponding Z-set unknotting theorem for Hilbert space manifolds exactly in the form of (ii) of Definition 1.2. Such unknotting theorems can easily be derived from [5] and [4].

Then, after obvious adjustments, the proofs of Propositions 3.1 and 3.3 apply for proper cell-like maps with model spaces H being replaced by Hilbert cube (space) manifolds M satisfying $M \approx M \times \mathbb{I}$ and Hilbert type compacta (spaces) replaced by strongly universal locally compact ANR's (strongly universal complete ANR's). Now let X be a strongly universal locally compact ANR (strongly universal complete ANR). Recall that $X \times Q$ is a Hilbert cube (space) manifold. Then, by Corollary 3.2, the mapping cylinder of the projection from $X \times Q$ to X is homeomorphic to $X \times Q$. Note that the projection from $X \times Q$ to X is a nice proper cell-like map. Then Proposition 3.3 implies that X is homeomorphic to $X \times Q$ and hence X is a Hilbert cube (space) manifold.

Let us finally note that Theorem 3.4 also applies to show that a proper cell-like map of Hilbert cube (space) manifolds is a near homeomorphism.

7. Appendix

In this section we provide an elementary proof of the Z-embedding approximation properties (i) and (ii) presented in Section 2.

Proof of (i). We recall some basic properties of (a complete strongly universal ANR) Y:

(A1) every Z-set A in Y is a strong Z-set, i.e., the identity map of Y can arbitrarily closely be approximated by a map with the closure of the image not intersecting A ([11], Lemma 2.3.15);
(A2) every compact subset of Y is a Z-set ([11], Lemma 2.3.16);
(A3) a closed subset A of Y is a Z-set if and only if there is countable dense subset $\mathscr{G}$ of the function space $C(Q,Y)$ such that the images of Q under the maps in $\mathscr{G}$ do not intersect A ([11], Proposition 2.2.3).

First assume that $F \neq \emptyset$, identify F with $f(F)$ and consider F as a closed subset of both X and Y. Fix a countable dense collection $\mathscr{G} = \{g_0, g_1, g_2, \dots\}$ of maps in $C(Q,Y)$ whose images do not intersect F. Take an open cover $\mathscr{U}$ of Y and fix complete metrics d_X and d_Y on X and Y respectively such that d_X and d_Y are bounded by 2 and the unit balls of Y with respect to d_Y refine $\mathscr{U}$. We will construct by induction maps $f_n : X \longrightarrow Y$ that do not move the points of F, a decreasing sequence of open neighborhoods U_n of F in X and compatible metrics d_Y^n on Y such that the following holds:

(B1) for every $x \in U_n$ there is a point x_F in F such that $d_X(x, x_F) \leq 2/2^{n-1}$ and $d_Y(f_{n-1}(x), x_F) \leq 2/2^{n-1}$ for every $n \geq 1$;
(B2) f_n restricted to $X_n = X \setminus U_n$ is a closed embedding such that $Y_n^X = f_n(X_n)$ does not intersect both F and $Y_n^Q = g_0(Q) \cup \dots \cup g_n(Q)$;
(B3) $d_Y^i(f_{n-1}(x), f_n(x)) \leq 1/2^n$ for every $i \leq n-1$, $n \geq 1$ and $x \in X$;
(B4) $$d_Y^n(y_1,y_2) = \begin{cases} d_X(f_n^{-1}(y_1) \cap X_n, f_n^{-1}(y_2) \cap X_n), & \text{if } y_1, y_2 \in Y_n^X, \\ d_Y(y_1,y_2), & \text{if } y_1,y_2 \in F \cup Y_n^Q, \\ 2, & \text{if } y_1 \in Y_n^X \text{and } y_2 \in F \cup Y_n^Q. \end{cases}$$

Set $U_0 = X$, $f_0 = f$, $d_Y^0 = d_Y$ and proceed from n to $n+1$ as follows. Define $U_{n+1} \subset U_n$ to be so close to F that (B1) holds. Recall that F and Y_{n+1}^Q are Z-sets in Y. Then, applying (A1) and a controlled version of the Borsuk Homotopy Extension Theorem, approximate f_n by a map f_{n+1} so that f_{n+1} does not move the points of F and (B2) holds. We may also assume that f_{n+1} is so close to f_n that $(B3)$ holds as well. Define the metric d_Y^{n+1} on $Y_{n+1}^X \cup F \cup Y_{n+1}^Q$ to satisfy (B4) and, by Hausdorff's metric extension theorem, extend this partial metric to a metric d_Y^{n+1} on Y. The construction is completed.

Recall that $d_Y^0 = d_Y$ is a complete metric and, hence, by (B3), $f' = \lim f_n$ is well defined. Fix $\epsilon > 0$ and let $x_1, x_2 \in X_n$ be distinct points with $d_X(x_1,x_2) \geq \epsilon$. Take $i > n$ such that $1/2^i \leq \epsilon/8$. Then, by (B4), $d_Y^i(f_i(x_1), f_i(x_2)) = d_X(x_1,x_2) \geq \epsilon$ and hence, by (B3), $d_Y^i(f'(x_1), f'(x_2)) \geq \epsilon/2$. This implies that f' restricted to X_n is a closed embedding. (The contrapositive of the previous argument shows $(f'|X_n)^{-1}$ is uniformly continuous with respect to the metrics d_Y^i and d_X.)

By (B3) and (B4), $d_Y^n(f'(x), F \cup Y_n^Q) \geq 1$ for every $x \in X_n$. Thus we get that $f'(X_n) \cap F = \emptyset$ and $f'(X_n) \cap Y_n^Q = \emptyset$. Hence f' is injective and $f'(X) \cap Y_n^Q = \emptyset$ for every n.

Now, in order to show that f' is a closed embedding, we only need to verify that if for a sequence x_n in X such that $x_n \in U_{n+1}$ the sequence $f'(x_n)$ converges in Y, then x_n converge in X as well. Indeed, take a point $x_F^n \in F$ witnessing the property (B1) for $x = x_n$. By (B3), $d_Y(f'(x_n), f_n(x_n)) = d_Y^0(f'(x_n), f_n(x_n)) \leq 2/2^n$. Hence by (B1), $d_Y(f'(x_n), x_F^n) \leq 4/2^n$ and $\lim x_n = \lim x_F^n = \lim f'(x_n)$.

Recall that $f'(X) \cap Y_n^Q = \emptyset$ for every n and hence, by (A3), f' is a Z-embedding. By (B3), f' and $f = f_0$ are 1-close with respect to $d_Y = d_Y^0$. Thus f' is a Z-embedding $\mathscr{U}$-close to f and coinciding with f on F.

The above construction also works for $F = \emptyset$ by just letting $U_n = \emptyset$ for every $n \geq 1$ and removing the condition (B1).

Proof of (ii). The proof of (i) also applies to prove (ii) as follows. Assume that Y is locally compact. Observe that (A1) trivially holds in this case because in this context, the map which pushes Y off a Z-set can be taken to be proper and hence having closed image.

Replace (A2) by the property that every map from a compactum to Y can arbitrarily closely be approximated by a Z-embedding ([17, Theorem 7.3.5]).

Assume in (A3) that the maps in $\mathcal{G}$ are Z-embeddings ([17], Proposition 7.3.2). Proposition 2.2.3 in [11] shows that if A lies in the complement of the images of a dense subset of $C(Q, Y)$, then id_Y can be approximated by maps whose images miss A. Since Y is locally compact, sufficiently close approximations of id_Y are necessarily proper. So id_Y can be approximated by proper maps whose images miss A. Thus, A satisfies the definition of a Z-set in the locally compact context. Now choose d_Y to be a proper metric; then all the maps $f_n : X \to Y$ and $f' : X \to Y$ generated in the proof of (i) will be proper.

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