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THE EXISTENCE OF A CONNECTED MEAGER IN ITSELF CDH SPACE IS INDEPENDENT OF ZFC

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ABSTRACT. We show that the existence of a countable dense homogeneous metric space which is connected and meager in itself is independent of ZFC.

1. INTRODUCTION

All spaces under discussion are separable and metrizable.

Recall that a space X is *countable dense homogeneous* (CDH) if, given any two countable dense subsets D and E of X, there is a homeomorphism $f\colon X \to X$ such that $f[D] = E$. This classical notion isolated by Bennett in [1] dates back to the works of Cantor, Brouwer, Fréchet, and others. Examples of CDH spaces are the Euclidean spaces, the Hilbert cube and the Cantor set. In fact, every strongly locally homogeneous Polish space is CDH, as was shown by Bessaga and Pełczyński [2].

A space X is called a λ-*set* if every countable subset of X is G_δ in X. Fitzpatrick and Zhou in [4] noted that every meager in itself CDH space is a λ-set. On the other hand, in [6] it is shown that there is a meager in itself CDH space of size κ if and only if there is a λ-set of the same size. Uncountable λ-sets exist in ZFC, though the existence of a λ-set of size $\mathfrak{c}$ is independent of ZFC ([11, Theorem 22]).

The natural question, whether meager in itself CDH spaces can be connected and, more generally, whether λ-sets can be connected arose recently. The above considerations show that the answer to both questions is consistently negative: If there are no λ-sets of size $\mathfrak{c}$, then every λ-set and, in particular, every meager in itself CDH space is of size less than $\mathfrak{c}$, hence is zero-dimensional, ergo not connected ([4]). In this note we shall see the consistency of a positive answer to the questions.

2. A CONNECTED λ-SET

As a warm up, we present the proof of the following:

Theorem 2.1. *The following are equivalent:*

(1) *There is a λ-set of size $\mathfrak{c}$, and*
(2) *there is a connected λ-set.*

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Proof. As every infinite connected space has size $\mathfrak{c}$ it suffices to show that (1)⇒(2). Let $Y \subseteq 2^\omega$ be a λ-set of size $\mathfrak{c}$ which is $\mathfrak{c}$-dense, i.e., has intersection of size $\mathfrak{c}$ with every non-empty open subset of 2^ω. To see that such a set exists let Y' be a λ-set of size $\mathfrak{c}$ contained in its completion Z. Let G be a G_δ subset of 2^ω for which there exists a continuous one-to-one surjection $f : G \to Z$. Note that $f^{-1}[Y']$ is a λ-set of size $\mathfrak{c}$ contained in 2^ω. Then let

$$P = 2^\omega \setminus \bigcup\{U : U \subseteq 2^\omega : |f^{-1}[Y'] \cap U| < \mathfrak{c}\}$$

and note that P is a perfect subset of 2^ω, hence is homeomorphic to 2^ω, and $Y = P \cap f^{-1}[Y']$ is a λ-set $\mathfrak{c}$-dense in P.

We shall construct the connected space X as a subspace of the one-point compactification $(2^\omega \times [0,1)) \cup \{\infty\}$ of $2^\omega \times [0,1)$.

Claim. There is a function $f : Y \to [0,1)$ such that $f \cap g \neq \emptyset$ for every Borel $g : U \to [0,1)$, where U is a non-empty open subset of 2^ω.

The proof of the claim is a straightforward diagonalization.

Having fixed a function f as above and identifying f with its graph, consider the space $X = f \cup \{\infty\}$.

To see that X *is a λ-set* it suffices to see that any countable subgraph of f is relatively G_δ, but this trivially follows from the fact that its domain is G_δ in Y. Now we shall check that X *is connected*. To see this, consider V, W disjoint open subsets of $2^\omega \times [0,1) \cup \{\infty\}$ both intersecting X. It suffices to show that $X \setminus (V \cup W) \neq \emptyset$. Since $\infty \in X$ we may consequently assume that $\infty \in W$. Now, let $(x, f(x)) \in X \cap V$. There are open $U \subseteq 2^\omega$ and $U' \subseteq [0,1)$ such that $(x, f(x)) \in U \times U' \subseteq V$. Define $g : U \to [0,1)$ by

$$g(x) = \max\{z \in [0,1) : (x,z) \notin W\}.$$

As $\infty \in W$ and $U \times U' \subseteq V$, the function is well defined, and is Borel, in fact of Baire class 1. Observe that for every $y \in U$ we have that $(y, g(y))$ belongs to the closure of W but not to W. By the claim there is a $y \in U$ such that $f(y) = g(y)$. Hence $(y, f(y)) \in X \setminus W$ and, moreover, $(y, f(y)) \in X \setminus V$ since V and W are disjoint and $(y, g(y))$ belongs to the closure of W. □

Note that the space constructed is homeomorphic to a subspace of the plane.

Similar arguments as in the proof of Theorem 2.1 can be found in Zindulka [13] and Mazurkiewicz and Szpilrajn [8] where λ-sets of positive dimension were constructed. The theorem can also be deduced from [7, §27, IX], as pointed out by the referee.

3. A connected meager in itself CDH space from CH

In this section we shall prove the main result of this note, namely we prove that assuming the Continuum Hypothesis there is a connected CDH meager in itself subspace of the Hilbert cube. Together with the above-mentioned observation that consistently every λ-set has size less than $\mathfrak{c}$ this shows that the existence of a countable dense homogeneous metric space which is connected and meager in itself is independent of ZFC.

We first review relevant material concerning the topology of the Hilbert cube.

3.1. **Topology of the Hilbert cube.** Let $Q = \prod_{n=1}^{\infty} [-1,1]_n$ denote the Hilbert cube with with product metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.$$

The *pseudo-boundary* of Q is

$$B(Q) = \{x \in Q : (\exists\, n \in \mathbb{N})(|x_n| = 1)\}$$

and its *pseudo-interior*

$$s = \prod_{n=1}^{\infty} (-1,1)_n$$

is the complement of $B(Q)$. Observe that $B(Q)$ and s are dense in Q, and that both are connected.

Recall that a closed set $A \subseteq Q$ is a *Z-set* if given a continuous function $f : Q \to Q$, and an $\varepsilon > 0$ if there is a continuous function $g : Q \to Q$ such that $g[Q] \cap A = \emptyset$ while $d(f,g) < \varepsilon$, i.e., if $d(f(x), g(x)) < \varepsilon$ for every $x \in Q$. A set $B \subseteq Q$ is a *σZ-set* if it is a countable union of Z-sets. We shall denote by $\mathscr{Z}(Q)$ the collection of all Z-sets of Q, and $\mathscr{Z}_\sigma(Q)$ denotes the family of all σZ-sets of Q. Many examples of Z-sets are given by the following simple lemma:

Lemma 3.1 ([9, Lemma 6.2.3 (ii)]). *A closed $A \subseteq Q$ such that there are infinitely many $n \in \mathbb{N}$ such that $\pi_n[A] \neq [-1,1]$ is a Z-set of Q.*

In particular, it follows that $B(Q) \in \mathscr{Z}_\sigma(Q)$, and that any compact subset K of s is a Z-set.

We denote by $\mathscr{H}(Q)$ the group of autohomeomorphisms of Q. Given $\varepsilon > 0$, call a homeomorphism $h \in \mathscr{H}(Q)$ *ε-small* if $d(h, \mathrm{id}) < \varepsilon$, i.e., if $d(x, h(x)) < \varepsilon$ for every $x \in Q$.

Theorem 3.2 ([9, Theorem 6.4.6]). *Let $f : E \to F$ be a homeomorphism between two Z-sets of Q such that $d(f, \mathrm{id}_E) < \varepsilon$. Then f extends to an ε-small homeomorphism $\tilde{f} \in \mathscr{H}(Q)$.*

Theorem 3.3 ([9, Theorem 6.4.8]). *Let A be a closed subset of a compact space X, and let $f : X \to Q$ be a continuous map such that $f \restriction A$ is an embedding and $f[X]$ is a Z-set of Q. Then for every $\varepsilon > 0$ there is an embedding $g : X \to Q$ such that $g \restriction A = f \restriction A$ and $f[X]$ is a Z-set.*

A set $A \in \mathscr{Z}_\sigma(Q)$ is

(1) a *capset* if there is a homeomorphism $f \colon Q \to Q$ so that $f[A] = B(Q)$,
(2) an *absorber* if for every $\varepsilon > 0$ and every pair of Z-sets K, L there is an ε-small $h \in \mathscr{H}(Q)$ such that $h \restriction K = \mathrm{id}$ and $h[L \setminus K] \subseteq A$,
(3) a *skeletoid* if A can be written as an increasing union of Z-sets A_n, $n \in \mathbb{N}$, so that for every $\varepsilon > 0$, $n \in \mathbb{N}$ and every Z-set K there are an $m \in \mathbb{N}$ and ε-small $h \in \mathscr{H}(Q)$ such that $h \restriction A_n = \mathrm{id}$ and $h[K] \subseteq A_m$.

Basic properties of these sets are given by the following result:

Theorem 3.4 ([9, Theorem 6.5.2]). *Let A and B be absorbers in Q.*

(1) *$h[A]$ is an absorber for any $h \in \mathscr{H}(Q)$,*
(2) *$A \cup C$ is an absorber for any $C \in \mathscr{Z}_\sigma(Q)$, and*
(3) *for every $\varepsilon > 0$ there is an ε-small $h \in \mathscr{H}(Q)$ such that $h[A] = B$.*

It turns out that a σZ-set is a capset if and only if it is an absorber if and only if it is a skeletoid (see [9, §6.5] for details; in particular, Theorems 6.5.1, 6.5.5 and 6.5.8). The pseudo-boundary $B(Q)$ is the standard capset of Q. There are, however, many more.

Theorem 3.5. *Let F be an infinite co-infinite subset of $\mathbb{N}$, and let $\{x_n : n \in F\} \subseteq [-1,1]$. Then the set*

$$\Sigma = \{y \in Q : \textit{for all but finitely many } n \in F,\ y_n = x_n\}$$

is a capset.

Proof. The proof is standard and is similar to [9, Proposition 6.5.4]. Write F as $\bigcup_{n=1}^{\infty} F_n$, where every F_n is finite and $F_n \subseteq F_{n+1}$. For every n, put

$$M_n = \{y \in Q : (\forall\, m \in F \setminus F_n)(y_m = x_m)\}.$$

Then $M_n \approx Q$ since F is co-infinite, and M_n projects onto a point in infinitely many coordinate directions, hence it is a Z-set by Lemma 3.1. Also $M_n \subseteq M_{n+1}$, and $\bigcup_{n=1}^{\infty} M_n = \Sigma$. Hence $\Sigma \in \mathscr{Z}_\sigma(Q)$. Put $E = \mathbb{N} \setminus F$, and for every n, put

$$Q_n = \{y \in M_n : (\forall\, m \in E \cup F_n)(|y_m| \leq 1 - {}^1\!/_n)\}.$$

Then $Q_n \approx Q$, $Q_n \subseteq Q_{n+1}$, and there is a natural retraction $r_n \colon Q \to Q_n$ defined by

$$r_n(y)_i = \begin{cases} -1+{}^1\!/_n & (y_i \leq -1+{}^1\!/_n, i \in E \cup F_n), \\ y_i & (-1+{}^1\!/_n \leq y_i \leq 1-{}^1\!/_n, i \in E \cup F_n), \\ 1-{}^1\!/_n & (-1+{}^1\!/_n \leq y_i, i \in E \cup F_n), \\ x_i & (i \in F \setminus F_n). \end{cases}$$

Observe that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, r_n moves no point more than ε. Also observe that Q_n is a Z-set in Q_{n+1} by Lemma 3.1.

We now aim at proving that the sequence $(Q_n)_n$ witnesses the fact that $\bigcup_{n \in \mathbb{N}} Q_n$ is a skeletoid. To this end, let K be a Z-set, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. There is an $N \in \mathbb{N}$ such that $n < N$, and such that the standard retraction $r_N \colon Q \to Q_N$ moves the points less than ${}^1\!/_2\varepsilon$. Since Q_n is a Z-set in Q_N, Theorem 3.2 implies that we can adjust r_N a little so that its restriction to $K \cup Q_n$ is an embedding and is the identity on Q_n. This embedding can be extended to an ε-small homeomorphism of Q by Theorem 3.3, and this homeomorphism is the one we are looking for.

Hence $\bigcup_{n=1}^{\infty} Q_n$ is an absorber as every skeletoid is an absorber, and hence $\bigcup_{n=1}^{\infty} M_n$ is an absorber as well, being a σZ-set by Theorem 3.4, and finally $\bigcup_{n=1}^{\infty} M_n$ is a capset as every absorber is a capset. □

Corollary 3.6. *Let A be a G_δ-subset of $[-1,1]$ such that $[-1,1] \setminus A \neq \emptyset$. Then $B(Q) \setminus A^\infty$ is a capset.*

Proof. Observe that $B(Q) \setminus A^\infty$ is σ-compact, hence belongs to $\mathscr{Z}_\sigma(Q)$. Pick an arbitrary $x \in [-1,1] \setminus A$, and split $\mathbb{N}$ into three pairwise disjoint infinite sets, say Y_0, Y_1 and Y_2. For every $n \in Y_1$, put $x_n = 1$, and for every $n \in Y_2$, put $x_n = x$. Then the set Σ defined in the previous theorem with the sequence x_n, $n \in Y_1 \cup Y_2$, is contained in $B(Q) \setminus A^\infty$. Hence $B(Q) \setminus A^\infty$ contains a capset by Theorem 3.5, and therefore is a capset itself by Theorem 3.4. □

Variations of the following lemma are well known.

Lemma 3.7. *Let M and N be capsets in Q. In addition, let D^0 be a countable dense subset of $Q \setminus M$ containing the dense subset E^0 such that $F^0 = D^0 \setminus E^0$ is dense as well. Moreover, let D^1 be a countable dense subset of $Q \setminus N$ containing the dense subset E^1 such that $F^1 = D^1 \setminus E^1$ is dense as well. Then there is a homeomorphism h of Q such that $h[M] = N$, $h[E^0] = E^1$ and $h[F^0] = F^1$.*

Proof. Write $M = \bigcup_{n=1}^{\infty} M_n$ and $N = \bigcup_{n=1}^{\infty} N_n$, where the sequences $(M_n)_n$ and $(N_n)_n$ witness the fact that M and N are skeletoids.

Let $i \in \{0, 1\}$. Write E^i as $\bigcup_{n=1}^{\infty} E^i_n$, where each E^i_n is finite and $E^i_n \subseteq E^i_{n+1}$. Write F^i similarly as $\bigcup_{n=1}^{\infty} F^i_n$.

It is clear that there is $n_1 > 0$ and an embedding $f_1 \colon M_1 \cup E^0_1 \cup F^0_1 \to N_{n_1} \cup E^1_{n_1} \cup F^1_{n_1}$ such that $f_1[M_1] \subseteq N_{n_1}$, $f_1[E^0_1] \subseteq E^1_{n_1}$, and $f_1[F^0_1] \subseteq F^1_{n_1}$. We can extend f_1 to a homeomorphism h_1 of Q (Theorem 3.2). Now let $\varepsilon > 0$. There clearly exists $n_2 > 1$ and an embedding $f_2 \colon N_{n_1} \cup E^1_{n_1} \cup F^1_{n_1} \to h_1[M_{n_2} \cup E^0_{n_2} \cup F^0_{n_2}]$ such that $f_2[N_{n_1}] \subseteq h_1[M_{n_2}]$, $f_2[E^1_{n_1}] \subseteq h_1[E^0_{n_2}]$ and $f_2[F^1_{n_1}] \subseteq h_1[F^0_{n_2}]$, while moreover $f_2 \restriction h_1[M_1 \cup E^0_1 \cup F^0_1] = \mathrm{id}$ and $d(f_2, \mathrm{id}) < \varepsilon$. We can extend f_2 to a homeomorphism g_2 of Q such that $d(g_2, \mathrm{id}) < \varepsilon$. Now put $h_2 = g_2^{-1}$. Then $(h_2 \circ h_1)[M_1] \subseteq N_{n_1} \subseteq (h_2 \circ h_1)[M_{n_2}]$, $(h_2 \circ h_1)[E^0_1] \subseteq E^1_{n_1} \subseteq (h_2 \circ h_1)[E^0_{n_2}]$, and $(h_2 \circ h_1)[F^0_1] \subseteq F^1_{n_1} \subseteq (h_2 \circ h_1)[F^0_{n_2}]$. And we can choose h_2 as close to the identity as we please. By the Inductive Convergence Criterion in [10, Theorem 1.6.2], we can consequently construct a sequence of homeomorphisms $(h_n)_n$ of Q the infinite left product of which is a homeomorphism h with the properties as stated in the lemma. □

3.2. **The construction.** Let us first note that both s and $B(Q)$ intersect every compact subset K of Q which disconnects Q.

It is a theorem of Hausdorff [5] that every Polish space can be written as the union of a strictly increasing sequence of G_δ-sets. In particular, we can write $[-1, 1]$ as $\bigcup_{\alpha<\omega_1} A_\alpha$, so that $A_0 = \emptyset$, each A_α is a G_δ-subset of $[-1, 1]$, $A_\alpha \subseteq A_\beta$ if $\alpha < \beta$, and $[-1, 1] \setminus A_\alpha \neq \emptyset$.

Enumerate all closed subsets of Q that separate Q by $\{K_\alpha : \alpha < \omega_1\}$, and enumerate all pairs of countable dense subsets of Q by $\{(E_\alpha, F_\alpha) : \alpha < \omega_1\}$ such that each pair is listed ω_1-many times.

We shall recursively construct a decreasing sequence $\{B_\alpha : \alpha < \omega_1\}$ of capsets and an increasing sequence $\{D_\alpha : \alpha < \omega_1\}$ of countable subsets of Q, together with an increasing sequence $\{H_\alpha : \alpha < \omega_1\}$ of countable subgroups of $\mathscr{H}(Q)$ so that (denoting $Q \setminus B_\alpha$ by s_α) for every $\alpha < \omega_1$:

(1) D_α is a countable dense subset of s_α, and $D_\alpha \cap K_\alpha \neq \emptyset$,
(2) there exists an ordinal $f(\alpha) < \omega_1$ such that $B(Q) \setminus A^\infty_{f(\alpha)} \subseteq B_\alpha$,
(3) D_α, s_α and B_α are invariant under H_α,
(4) if $E_\alpha \cup F_\alpha \subseteq D_\alpha$, and $D_\alpha \setminus (E_\alpha \cup F_\alpha)$ is dense, then there exists an element h of H_α such that $h[E_\alpha] = F_\alpha$,
(5) if $\gamma < \alpha$, $D_\alpha \setminus D_\gamma$ is a dense subset of Q contained in $s_\alpha \setminus s_\gamma$.

To start, put $s_0 = s$ and $B_0 = B(Q)$, and let D_0 be any countable dense subset of s_0 which meets K_0. Then consider the pair (E_0, F_0). Assume first that $E_0 \cup F_0 \subseteq D_0$, and that $D_0 \setminus (E_0 \cup F_0)$ is dense. Then there is by Lemma 3.7 a homeomorphism h of Q such that $h[E_0] = F_0$, $h[D_0] = D_0$ and $h[B(Q)] = B(Q)$. Observe that $h[s_0] = s_0$. Let H_0 denote the countable subgroup of $\mathscr{H}(Q)$ generated by $\{h\}$. If $E_0 \cup F_0 \not\subseteq D_0$ or if $D_0 \setminus (E_0 \cup F_0)$ is not dense, then we let H_0 denote the subgroup of $\mathscr{H}(Q)$ consisting only of the identity.

Now suppose that for some $\alpha < \omega_1$ we constructed for every $\beta < \alpha$ the sets D_β, s_β and B_β and a countable subgroup H_β of $\mathscr{H}(Q)$ satisfying the conditions (1)-(5) above.

Put $B = \bigcap_{\beta<\alpha} B_\beta$, $S = Q \setminus B$, and $H = \bigcup_{\beta<\alpha} H_\beta$, respectively. Observe that H is a countable subgroup of $\mathscr{H}(Q)$, and by (3), B and S are H-invariant.

By (2), we may pick an ordinal number $\xi < \omega_1$ such that $T = B(Q) \setminus A^\infty_\xi \subseteq B$. Since T is a capset by Corollary 3.6, it intersects K_α, say in the point x_0. Let X be a countable dense subset of T which contains x_0. Since the set $P = \{h(x) : x \in X, h \in H\}$ is countable and dense in Q, and each point of Q has countably many coordinates only, there exists $\xi < \eta < \omega_1$ such that $P \subseteq A^\infty_\eta$. Consider the capset $T' = B(Q) \setminus A^\infty_\eta$. It is contained in T and hence in B, and it misses P. Clearly, $S \cup P$ is H-invariant. Hence it misses the set

$$T'' = \bigcup_{h\in H} h[T'].$$

Observe that T'' is a capset since it contains the capset T' and $T' \subseteq T \subseteq B \subseteq B(Q)$ (hence T'' is a countable union of Z-sets), and $T' = B(Q) \setminus A^\infty_\eta \subseteq T'' \subseteq B$. By definition T'' is H-invariant.

Define $B_\alpha = T''$, $s_\alpha = Q \setminus B_\alpha$, $D_\alpha = D \cup P$, and $f(\alpha) = \eta$. Then all our inductive hypotheses are satisfied, except perhaps (4). However, we are basically back at the first step of the construction, so this can easily be taken care of. This completes the recursive construction.

Put $D = \bigcup_{\alpha<\omega_1} D_\alpha$. Then D is connected as D is dense in Q and by (1) intersects every closed set separating Q. To see that the set D is a λ-set note that every countable subset C of D is contained in one of the D_β, which in turn is a G_δ subset of D as $D_\beta = D \cap s_\beta$ by (5). Hence C itself is a G_δ-subset of D since $D_\beta \setminus C$ is countable, in particular, F_σ. To prove that D is CDH, let E and F be arbitrary countable dense subsets of D. Pick $\alpha < \omega_1$ such that $E \cup F \subseteq D_\alpha$. Then $D_{\alpha+1} \setminus D_\alpha$ is a countable dense subset of D which misses $E \cup F$. Let $\beta > \alpha+1$ be such that $(E, F) = (E_\beta, F_\beta)$. Then at stage β we took care of E and F.

4. Concluding remarks

There are several natural related questions. We suspect that a similar construction could be performed already in the plane:

Question 4.1. Is there, assuming CH, a connected meager in itself CDH space in the plane?

It is not clear to us to what extent the assumption of CH can be weakened. In light of Theorem 2.1 it is natural to ask:

Question 4.2. Is it consistent with ZFC that there is a connected λ-set yet there is no connected meager in itself CDH space?

Finally, let us note that every arc-connected CDH space is Baire, in particular, no arc-connected CDH space is meager in itself.

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