Acta Math. Hungar., **157** (**2**) (2019), 364–370
https://doi.org/10.1007/s10474-018-0881-0
First published online October 22, 2018

HOMOGENEOUS CONTINUA THAT ARE NOT SEPARATED BY ARCS

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(Received June 5, 2018; revised August 13, 2018; accepted August 22, 2018)

Abstract. We prove that if X is a strongly locally homogeneous and locally compact separable metric space and G is a region in X with $\dim G = 2$, then G is not separated by any arc in G.

1. Introduction

By a *space* we mean a separable metric space. Kallipoliti and Papasoglu [4] proved that any locally connected, simply connected, homogeneous metric continuum can not be separated by arcs, and asked if this is true without the assumption of simply connectedness. A partial answer to this question was provided in [8] for homogeneous metric continua of dimension two having a non-trivial second integral Čech cohomology group. In the present paper we prove the following partial answer to Kallipoliti and Papasoglu's question.

Theorem 1.1. *Let X be a locally compact strongly locally homogeneous space and G be a region in X with* $\dim G = n \geq 2$. *Then G is not separated by any arc $J \subset G$.*

Recall that a space is strongly locally homogeneous if every point $x \in X$ has a local basis of open sets U such that for every $y, z \in U$ there is a homeomorphism h on X with $h(y) = z$ and h is identity on $X \setminus U$. Obviously, every

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† The first author is pleased to thank the Nipissing University for generous hospitality and support.

‡ The second author was partially supported by NSERC Grant 261914-13.

Key words and phrases: connected space, homogeneous continuum, locally compact separable metric space, locally connected space.

Mathematics Subject Classification: primary 54F15, secondary 54F45.

open subset of a strongly locally homogeneous space is also strongly locally homogeneous. Since strongly locally homogeneous connected spaces are homogeneous, any region G satisfying the hypotheses of Theorem 1.1 should be homogeneous. We claim that it is locally connected as well. Indeed, since any strongly homogeneous Polish space is countable dense homogeneous [1] and a locally compact countable dense homogeneous connected space is locally connected [3], we have that any region G from Theorem 1.1 is locally connected. (There is also a simple direct proof of this fact.) According to [6], no region of homogeneous locally compact space of dimension $n \geq 1$ can be separated by a closed set of dimension $\leq n-2$. So, Theorem 1.1 is interesting only for regions G of dimension two.

2. Some preliminary results

LEMMA 2.1. *Let A be a closed nowhere dense subset of X such that $\dim X \setminus A = 0$ and $d(A, X \setminus A) = 0$. Then there is a retraction $r\colon X \to A$ such that $r(X \setminus A)$ is countable.*

PROOF. The technique is similar to that in [5]. In brief, one constructs a cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ by disjoint nonempty clopen subsets of $X \setminus A$ such that

(1) $\operatorname{diam} V_n < d(V_n, A)$ for each n,
(2) there is a sequence $\{a_n : n \in \mathbb{N}\}$ in A such that

$$\lim_{n\to\infty} d(a_n, V_n) = 0.$$

Then define $r\colon X \to A$ as follows: $r(a) = a$ for every a and $r(V_n) = \{a_n\}$ for every n. It is easy to check that r is as required. □

If J is an arc and $p, q \in J$, then (p, q) and $[p, q]$ denote, respectively, the open and closed subintervals in J with endpoints p, q.

PROPOSITION 2.2. *Let $J = [a, b]$ be an arc in a space (X, d) which is everywhere 2-dimensional. Then b has arbitrarily small open neighborhoods U such that $\operatorname{bd}(U)$ is at most 1-dimensional and intersects J in exactly one point.*

PROOF. Fix $\varepsilon > 0$ and let U be an open neighborhood of b in X such that $\operatorname{diam} \overline{U} < \varepsilon$ and $\dim \operatorname{bd} U \leq 1$. We may assume without loss of generality that $J \setminus U \neq \emptyset$ and $J \cap U$ is uncountable. Put $Y = J \cup \overline{U}$. Moreover, put $A = J \cup \operatorname{bd} U$, $B = (J \setminus U) \cup \operatorname{bd} U$ and $C = (J \cap \overline{U}) \cup \operatorname{bd} U$, respectively.

Let D be a zero-dimensional dense subset of U such that $\dim U \setminus D = 1$. Since $\dim J = 1$, we may clearly assume that $D \cap J = \emptyset$.

Because C is a closed nowhere dense subset of $C \cup D$ and $d(C, D) = 0$, there is a retraction $r_1\colon C \cup D \to C$ such that $r_1(D)$ is countable (Lemma 2.1).

Let $r\colon A \cup D \to A$ be defined by $r(x) = r_1(x)$ if $x \in C \cup D$ and $r(x) = x$ if $x \notin C \cup D$. Obviously r is a retraction such that $r(D)$ is countable. Pick an arbitrary $s \in U \cap J$ such that $s \neq b$, $[s, b] \subset U$ and $s \notin r(D)$. Choose also two points $s_1, s_2 \in J \cap U$ different from s and b such that $s \in (s_1, s_2)$, and let $V_1 = A \setminus [s_1, b]$ and $V_2 = (s_2, b]$. Obviously V_1 and V_2 are open subsets of A containing B and $\{b\}$, respectively. Moreover, $\overline{V}_1 = A \setminus (s_1, b]$ and $\overline{V}_2 = [s_2, b]$.

Claim 1. *$\{s\}$ is a partition in A between $\overline{V}_1$ and $\overline{V}_2$.*

Indeed, put $P = [s, b]$ and $Q = [a, s] \cup \operatorname{bd} U$. Then P and Q are closed subsets of A such that $P \cup Q = A$, $\overline{V}_2 \subset P$, $\overline{V}_1 \subset Q$ and $P \cap Q = \{s\}$.

Claim 2. *$\{s\}$ is a partition in $A \cup D$ between $r^{-1}(\overline{V}_1)$ and $r^{-1}(\overline{V}_2)$.*

Since $r^{-1}(s) = \{s\}$, this is a direct consequence of Claim 1.

By [7, Lemma 3.1.4], there is a partition S between $\{b\}$ and B in Y such that $S \cap (A \cup D) \subset \{s\}$. If $s \notin S$, then $S \cup \{s\}$ is also a partition between $\{b\}$ and B in Y, hence we may assume without loss of generality that $s \in S$. But then $S \cap J = \{s\}$. Write $Y \setminus S$ as $E \cup F$, where E and F are disjoint relatively open subsets of Y such that $b \in E$ and $B \subset F$.

Claim 3. *$E \subset U$.*

Indeed, since $E \cap B = E \cap \big((J \setminus U) \cup \operatorname{bd} U\big) = \emptyset$, this is clear.

Since E is open in U and U is open in X we have that E is open in X. Moreover, $\operatorname{diam} E < \varepsilon$. Also, $E \cup S$ is closed in Y and hence in X. As a consequence $\operatorname{bd} E \subset S$. Since $S \subset U \setminus D$, we have $\dim S \leq 1$, as required. □

It will be convenient to use additive notation for the topological group $\mathbb{S}^1$.

The following result can be proved by tools from algebraic topology. For the convenience of the reader, we include a simple direct proof.

Proposition 2.3. *Let X be a space and let A be a closed subspace of it. Moreover, let $\gamma\colon A \to \mathbb{S}^1$ be continuous. Suppose that there are closed subsets P_1, P_2 of X satisfying the following conditions:*

- *$P_1 \cup P_2 = X$ and if $C = P_1 \cap P_2$ then $C \cap A$ is a singleton, say c;*
- *$\gamma | P_i \cap A$ is extendable over P_i for each $i = 1, 2$, but γ is not extendable over X.*

Then there is a continuous function $\beta\colon C \to \mathbb{S}^1$ such that $\beta(c) = 0$ and β is not nullhomotopic.

Proof. Let $\alpha_i\colon P_i \to \mathbb{S}^1$ for $i = 1, 2$ be a continuous extension of $\gamma | P_i \cap A$. Define $\beta\colon C \to \mathbb{S}^1$ by $\beta(x) = \alpha_1(x) - \alpha_2(x)$ $(x \in C)$. Then, clearly, $\beta(c) = 0$. We claim that β is as required, and argue by contradiction. Assume that β is nullhomotopic. Let $H\colon C \times \mathbb{I} \to \mathbb{S}^1$ be a homotopy such that $H_0 \equiv 0$ and $H_1 = \beta$. Define $S\colon C \times \mathbb{I} \to \mathbb{S}^1$ by $S(x, t) = H(x, t) - H(c, t)$. Then $S_0 \equiv 0$,

$S_1 = \beta$ and $S(c,t) = 0$ for every t. Define a homotopy $T\colon (C \cup (P_2 \cap A)) \times \mathbb{I} \to \mathbb{S}^1$ by

$$T(x,t) = \begin{cases} S(x,t) & (x \in C,\ t \in \mathbb{I}), \\ 0 & (x \in P_2 \cap A,\ t \in \mathbb{I}). \end{cases}$$

Then $T_0 \equiv 0$ and hence can be extended to the constant function with value 0 on P_2. By the Borsuk Homotopy Extension Theorem [7, 1.4.2], the function T_1 can be extended to a continuous function $\delta\colon P_2 \to \mathbb{S}^1$. Now define $\varepsilon\colon X \to \mathbb{S}^1$ as follows:

$$\varepsilon|P_1 = \alpha_1, \quad \varepsilon|P_2 = \delta + \alpha_2.$$

If $x \in C$, then $\varepsilon|P_1(x) = \alpha_1(x)$ and

$$\varepsilon|P_2(x) = \delta(x) + \alpha_2(x) = S_1(x) + \alpha_2(x) = \beta(x) + \alpha_2(x) = \alpha_1(x).$$

Hence ε is well defined and continuous. Also observe that if $x \in P_2 \cap A$, then

$$\varepsilon(x) = 0 + \alpha_2(x) = \alpha_2(x).$$

Hence ε extends γ, which is a contradiction. □

3. Proof of Theorem 1.1

Throughout, let X be a locally compact and strongly locally homogeneous space, and G be a region in X of dimension 2. Suppose G is separated by an arc $J = [a,b] \subset G$. Recall that G is homogeneous and locally connected (see §1). Write $G \setminus J$ as $G_1 \cup G_2$, where G_1 and G_2 are disjoint nonempty open subsets of G. Everywhere below $\overline{K}$ denotes the closure of K in G for any set $K \subset G$.

We say that a space Y has no local cut points if no connected open subset $U \subset Y$ has a cut point.

LEMMA 3.1. *G has no local cutpoints.*

PROOF. By Kruspki [6, Theorem 2.1] it follows that every nonempty open connected subset U of G is a Cantor manifold of dimension 2. Hence U cannot be separated by a zero-dimensional closed set. □

A space X is *crowded* if it has no isolated points.

LEMMA 3.2. *The set $S = \overline{G}_1 \cap \overline{G}_2$ is a 1-dimensional closed and crowded subspace of J which separates G.*

PROOF. Assume first that $J \setminus (\overline{G}_1 \cup \overline{G}_2) \neq \emptyset$. Then G is somewhere at most 1-dimensional. Hence G is at most 1-dimensional at every point by homogeneity. But this contradicts G being 2-dimensional.

Hence $J \subset \overline{G}_1 \cup \overline{G}_2$ and so $G = \overline{G}_1 \cup \overline{G}_2$. If S is empty, then G is covered by the disjoint nonempty closed sets $\overline{G}_1$ and $\overline{G}_2$ which contradicts the connectivity of G.

Now assume that x is an isolated point of S. Let U be an open connected neighborhood of x in G such that $U \cap S = \{x\}$. Then x is a cutpoint of U. But this contradicts Lemma 3.1.

We conclude that S separates G and consequently has to be 1-dimensional by Krupski [6]. $\square$

Let s be the maximum of S (as a subset of $[a,b]$). Then $J_s = [a,s]$ also separates G and $G \setminus J_s$ is the union of the disjoint open sets G_1' and G_2', where $G_i' = \overline{G}_i \setminus J_s$. Moreover, $s \in \overline{G_1'} \cap \overline{G_2'}$. Hence, we can assume without loss of generality that $b \in \overline{G}_1 \cap \overline{G}_2$.

LEMMA 3.3. *There is an open neighborhood $U \subset G$ of b having compact closure and a compact set $F \subset G$ such that for every open neighborhood V of b with $\overline{V} \subset U$ there exist a compact set $M_U \subset \overline{U}$ and a continuous function $f\colon \operatorname{bd}_F(U \cap F) \to \mathbb{S}^1$ such that*

(1) *$b \in U \cap F$;*

(2) *M_U is everywhere 2-dimensional and $M_U \cap V \neq \varnothing$;*

(3) *$\dim \operatorname{bd} U \leq 1$ and $J \cap \operatorname{bd} U$ is a point;*

(4) *f is not extendable over $\operatorname{bd}_F(U \cap F) \cup M_U$, but it is extendable over $\operatorname{bd}_F(U \cap F) \cup P$ for every proper closed set P of M_U.*

PROOF. Choose a compact neighborhood O_b of b in G. Since every neighborhood of b is of dimension 2, there is a compact subset $Y \subset O_b$, a closed set $A \subset Y$ and a continuous function $g\colon A \to \mathbb{S}^1$ not extendable over Y. Let F be a minimal closed subset of Y containing A such that g is not extendable over F. Then for every open subset W of $F \setminus A$ with $\overline{W} \cap A = \varnothing$ there is a function $f_W\colon F \setminus W \to \mathbb{S}^1$ extending g such that f_W can not be extended to a continuous function $\bar{f}_W\colon F \to \mathbb{S}^1$. This means that $f_W | \operatorname{bd}_F W$ is not extendable over $\overline{W}$. Consequently, $F \setminus A$ is everywhere two-dimensional. We can assume by homogeneity that $b \in F \setminus A$. Indeed, by Effros' theorem [2], we take O_b so small that for every point $x \in O_b$ there is a homeomorphism h on G with $h(b) = x$. Then, consider the set $h(J)$ instead of J.

By Proposition 2.2, there are an open neighborhood U of b whose closure in G is a compact and a point $c \in (a,b)$ such that $\operatorname{bd} U \cap J = \{c\}$, $\dim \operatorname{bd} U \leq 1$ and $\overline{U} \cap A = \varnothing$. Suppose V is an open neighborhood of b such that $\overline{V} \subset U$, and consider a continuous function $f_V\colon F \setminus V \to \mathbb{S}^1$ extending g which is not extendable over F. Let $f = f_V | \operatorname{bd}_F(U \cap F)$. Clearly, f cannot be extended to a continuous function $\bar{f}\colon \overline{U \cap F} \to \mathbb{S}^1$, but f can be extended to a continuous function from $(\overline{U \cap F}) \setminus V$ into $\mathbb{S}^1$. Let M_U be a minimal closed subset of $\overline{U \cap F}$ with the property that f cannot be extended to a continuous function $\widetilde{f}\colon \operatorname{bd}_F(U\cap F)\cup M_U\to\mathbb{S}^1$. The minimality of M_U implies that f is extendable over $\operatorname{bd}_F(U\cap F)\cup P$ for any closed set $P\subsetneqq M_U$. Because f is extendable over $(\overline{U\cap F})\setminus V$, $M_U\cap V\neq\varnothing$. It is clear that M_U is a continuum.

Assume that O is a nonempty open subset of M_U such that $\dim O\leq 1$. Taking a smaller open subset of O, we may assume that $\dim\overline{O}\leq 1$. There are two possibilities, either $O\subset \operatorname{bd}_F(U\cap F)$ or $O\setminus\operatorname{bd}_F(U\cap F)\neq\varnothing$. If $O\subset\operatorname{bd}_F(U\cap F)$, $M_U\setminus O$ is a proper closed subset of M_U having the same properties as M_U, which contradicts minimality. If $O'=O\setminus\operatorname{bd}_F(U\cap F)$ $\neq\varnothing$, then $P=M_U\setminus O'$ is a proper closed subset of M_U. So, there is an extension $f_1\colon \operatorname{bd}_F(U\cap F)\cup P\to\mathbb{S}^1$ of f. Since $\dim\overline{O'}\leq 1$, we can extend f_1 over $\operatorname{bd}_F(U\cap F)\cup M_U$, a contradiction. Therefore, M_U is everywhere 2-dimensional. □

Now, we can complete the proof of Theorem 1.1. Choose open neighborhoods U and V of b, closed sets $F\subset G$ and $M_U\subset\overline{U\cap F}$ and a continuous function $f\colon \operatorname{bd}_F(U\cap F)\to\mathbb{S}^1$ satisfying the conditions (1)–(4) from Lemma 3.3. Let also $J\cap\operatorname{bd}U=\{c\}$ and $C=[c,b]$. We can also assume that V satisfies the additional property that for every two points $p,q\in V$ there is a homeomorphism φ of G supported on V with $\varphi(p)=q$. We may consequently assume without loss of generality that $b\in M_U$. Indeed, if $b\notin M_U$ we take a point $x\in M_U\cap V$ and a homeomorphism φ of G supported on V such that $\varphi(x)=b$. Then the set $\varphi(M_U)$ satisfies all condition from Lemma 3.3 and contains b. Since M_U is everywhere 2-dimensional, $\dim(M_U\cap V)=2$. Hence, $M_U\cap V$ meets at least one of the sets G_i, $i=1,2$.

Assume first that $M_U\cap V\cap G_1\neq\varnothing$ but $M_U\cap V\cap G_2=\varnothing$.

Then $M_U\cap W$ meets G_1 for every neighborhood W of b with $W\subset V$. Indeed, because $\dim M_U\cap W=2$ and $M_U\cap W\cap G_2=\varnothing$ it follows that $M_U\cap G_1\cap W\neq\varnothing$. Consequently there is a neighborhood W of b in G such that

(5) $\overline{W}\subset V$, $(M_U\cap V)\cap(G_1\setminus\overline{W})\neq\varnothing$ and $M_U\cap G_1\cap W\neq\varnothing$;

(6) For every $x,y\in W$ there is a homeomorphism h of G supported on W with $h(x)=y$.

Finally, choose points $x\in M_U\cap G_1\cap W$ and $y\in W\cap G_2$ and a homeomorphism $h\colon G\to G$ supported on W with $h(x)=y$. Since $h(z)=z$ for all points $z\in(M_U\cap V)\cap(G_1\setminus\overline{W})$, the set $\widetilde{K}=h(M_U)$ meets both G_1 and G_2. Moreover, the function f is not extendable over $\operatorname{bd}_F(U\cap F)\cup\widetilde{K}$ (otherwise f would be extendable over $\operatorname{bd}_F(U\cap F)\cup M_U$). On the other hand, since each of the sets $Q_i=h^{-1}(\widetilde{K}\cap\overline{G}_i)$, $i=1,2$, is a proper closed subset of M_U, f is extendable over each of the sets $\operatorname{bd}_F(U\cap F)\cup(\widetilde{K}\cap\overline{G}_i)$. Let $\gamma:\operatorname{bd}U$ $\to\mathbb{S}^1$ be an extension of f (recall that $\dim\operatorname{bd}U\leq 1$ and $\operatorname{bd}_F(U\cap F)$ is a closed subset of $\operatorname{bd}U$, so such γ exists). Because f is not extendable over

$\mathrm{bd}_F(U \cap F) \cup \widetilde{K}$, γ is not extendable over the set $K = \mathrm{bd}\, U \cup \widetilde{K} \cup C$. Denote $P_i = C \cup (K \cap \overline{G}_i)$, $i = 1, 2$. Obviously, $P_1 \cup P_2 = K$ and $P_1 \cap P_2 = C$. Then for each i we have $P_i \cap \mathrm{bd}\, U = \{c\} \cup (\mathrm{bd}\, U \cap \overline{G}_i)$. So, the function $\gamma|(P_i \cap \mathrm{bd}\, U)$ is extendable over the set P_i because $\dim C \cup \mathrm{bd}\, U = 1$. Hence, we can apply Proposition 2.3 (with $A = \mathrm{bd}\, U$) to conclude that there is a continuous function $\beta \colon C \to \mathbb{S}^1$ such that β is not nullhomotopic, a contradiction.

Assume next that $M_U \cap V$ meets both G_1 and G_2. We can now proceed as above (considering M_U instead of $\widetilde{K}$) to obtain the desired contradiction.

Acknowledgement. The authors express their gratitude to the referee for helpful comments.

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