Covering Tychonoff cubes by topological groups

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ABSTRACT

Let \( \tau \) be an uncountable cardinal. We prove that if \( \mathcal{A} \) is a cover of the Tychonoff cube \( I^\tau \) such that \( |\mathcal{A}| \leq \tau \), then some element \( A \in \mathcal{A} \) is not homeomorphic to a topological group.

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1. Introduction

If \( X \) is a space, then \( \{ \{ x \} : x \in X \} \) is a cover of \( X \) by \( |X| \)-many spaces that are homeomorphic to a topological group. One naturally wonders whether one can do better than this triviality.

Every finite-dimensional compact metrizable space without isolated points can be covered by a finite family consisting of topological copies of the space of irrational numbers. Hence each such space can be covered by a finite family of subspaces, each homeomorphic to a topological group. For the Hilbert cube \( Q = \mathbb{I}^\omega \) this is also true, but requires nontrivial results from infinite-dimensional topology. In fact, \( Q \) can be covered by two topological copies of the countable infinite product of lines. One of these copies is \( s = (0, 1)^\omega \). The other one comes from the following observation. There is a homeomorphism \( f \) of \( Q \) sending its pseudo-boundary into its pseudo-interior. Hence \( f^{-1}(s) \) is a topological copy of \( s \) containing the pseudo-boundary of \( Q \). For details, see van Mill [5, §6.5].
For non-metrizable Tychonoff cubes, the situation is dramatically different.

**Theorem 1.1.** Let $\tau$ be an uncountable cardinal. If $\mathcal{E}$ is a cover of the Tychonoff cube $I^\tau$ by subspaces that are each homeomorphic to a topological group (not necessarily the same one), then $|\mathcal{E}| \geq \tau^+$. 

So under the Generalized Continuum Hypothesis, it follows that for uncountable $\tau$, the Tychonoff cube $I^\tau$ cannot be covered by a family of fewer than $2^\tau$ subspaces, each homeomorphic to a topological group. We do not know whether this can be proved without additional set theoretical assumptions.

We use some ideas in our recent paper Arhangel’skii and van Mill [3].

2. Preliminaries

All topological spaces under discussion are assumed to be Tychonoff.

We use standard notation. If $B$ is a set and $\lambda$ is a cardinal number, then $[B]^{\lambda}$ denotes the collection $\{A \subseteq B : |A| \leq \lambda\}$.

Let $\lambda$ be an infinite cardinal. We say that a subset $B$ of a space $X$ is a $G_\lambda$-subset of $X$ provided there exists a family $\mathcal{U}$ of open subsets of $X$ such that $|\mathcal{U}| \leq \lambda$ and $B = \bigcap \mathcal{U}$. A subspace $D$ of $X$ is called $G_\lambda$-dense in $X$ provided that $D$ intersects every nonempty $G_\lambda$-subset of $X$.

Observe that since $\lambda$ is assumed to be infinite, each nonempty $G_\lambda$-subset of $X$ contains a nonempty closed $G_\lambda$-subset.

A $G_\omega$-dense set is usually called $G_\delta$-dense.

Let $\tau$ be an uncountable cardinal, and consider the Tychonoff cube $I^\tau$. If $A \subseteq \tau$ and $f : A \to I$, then

$$B(A, f) = \{x \in I^\tau : (\forall \alpha \in A)(x_\alpha = f(\alpha))\}.$$ 

Then $B(A, f)$ is a closed $G_\lambda$-subset of $I^\tau$, where $\lambda = |A| \cdot \omega$. Observe that if $\lambda \leq \tau$ and $B$ is a nonempty $G_\lambda$-subset of $I^\tau$, then there are a subset $A \in [\tau]^{\leq \lambda}$ and an $f \in I^A$ such that $B(A, f) \subseteq B$. Moreover, $B(A, f)$ is a closed $G_\lambda$-subset of $I^\tau$ and hence of $B$.

We will use the trivial fact that if $D$ is a $G_\lambda$-dense subset of a space $X$, and $S$ is a nonempty $G_\lambda$-subset of $X$, then $D \cap S$ is $G_\lambda$-dense in $S$.

3. The construction

Now let $\tau$ be an uncountable cardinal, and let $\mathcal{E}$ be a cover of $I^\tau$ such that $|\mathcal{E}| \leq \tau$.

**Lemma 3.1.** There are $E \in \mathcal{E}$, $\lambda$ and $A \in [\tau]^{\lambda}$, where $\omega \leq \lambda < \tau$, and $f \in I^A$ such that $E \cap B(A, f)$ is a $G_\lambda$-dense subset of $B(A, f)$.

**Proof.** Assume that this is not true. Enumerate $\mathcal{E}$ as $\{E_\alpha : \alpha < \tau\}$ (repetitions permitted). By transfinite induction on $\alpha < \tau$, we will construct a subset $A_\alpha$ of $\tau$ of size at most $|\alpha| \cdot \omega$, and an element $f_\alpha \in I^{A_\alpha}$ such that

1. if $\alpha' < \alpha < \tau$, then $A_\alpha' \subseteq A_\alpha$ and $f_\alpha | A_\alpha' = f_{\alpha'}$,
2. $B(A_\alpha, f_\alpha) \cap E_\alpha = \emptyset$.

By assumption, $E_0$ is not $G_\omega$-dense in $I^\tau$, hence there are a subset $A_0 \in [\tau]^{\omega}$ and a function $f_0 \in I^{A_0}$ such that $E_0 \cap B(A_0, f_0) = \emptyset$. Assume that we completed the construction for all $\alpha' < \alpha$, where $\alpha < \tau$. Put $A = \bigcup_{\alpha' < \alpha} A_{\alpha'}$ and $f = \bigcup_{\alpha' < \alpha} f_{\alpha'}$, respectively. Then $\omega \leq \lambda = |A| \leq |\alpha| \cdot \omega < \tau$. By assumption, $E_\alpha \cap B(A, f)$ is not $G_\lambda$-dense in $B(A, f)$. Hence there is a nonempty $G_\lambda$-subset $F$ of $B(A, f)$ such that
$E_\alpha \cap F = \emptyset$. Then, clearly, $F$ is a $G_\lambda$-subset of $\mathbb{I}^\tau$. Hence there are $A' \in [\tau]^{<\lambda}$ and $f' \in \mathbb{I}^{A'}$ such that $B(A', f') \subseteq F \subseteq B(A, f)$. Clearly, $A \subseteq A'$ and $f'|A = f$. Hence by putting $A_\alpha = A'$ and $f_\alpha = f'$, we satisfy our inductive requirements. This completes the transfinite construction.

Now put $A = \bigcup_{\alpha < \tau} A_\alpha$ and $f = \bigcup_{\alpha < \tau} f_\alpha$. There exists $x \in \mathbb{I}^\tau$ such that
\[ \forall \alpha \in A : x_\alpha = f(\alpha). \]

Then, by construction, $x \notin \bigcup \mathcal{E}'$, which contradicts the fact that $\mathcal{E}'$ covers $\mathbb{I}^\tau$. \qed

**Lemma 3.2.** Let $\lambda$ be an infinite cardinal, and assume that $X$ is a $G_\lambda$-dense subset of $Y$. Then for every subset $A$ of $X$ which is a $G_\lambda$-subset of $X$, there is a $G_\lambda$-subset $S$ of $Y$ such that $A \subseteq S \subseteq \overline{A}$ (here ‘closure’ means closure in $Y$), and $S \cap X = A$.

**Proof.** Let $\mathcal{W}$ be a family of at most $\lambda$ open subsets of $X$ such that $A = \bigcap \mathcal{W}$. For every $U \in \mathcal{W}$, pick an open subset $V(U)$ of $Y$ such that $V(U) \cap X = U$. Put $S = \bigcap_{U \in \mathcal{W}} V(U)$. Then, clearly, $S$ is a $G_\lambda$-subset of $Y$, $A \subseteq S$, and $S \cap X = A$. Assume that there exists an element $p \in S \setminus \overline{A}$. Then $S \cap (Y \setminus \overline{A})$ is a nonempty $G_\lambda$-subset of $Y$ which misses $X$, which contradicts the fact that $X$ is $G_\lambda$-dense in $Y$. \qed

The following lemma is well-known and its proof is left to the reader (or see [3, Lemma 3.1]).

**Lemma 3.3.** Let $G$ be a topological group. If $S$ is a $G_\delta$-subset of $G$ containing the neutral element $e$ of $G$, then there is a closed subgroup $N$ of $G$ such that

1. $N \subseteq S$,
2. $N$ is a $G_\delta$-subset of $G$.

After these preliminary observations, we will present the proof of Theorem 1.1.

By Lemma 3.1, there exist $E \in \mathcal{E}'$, an infinite cardinal $\lambda < \tau$ and $A \in [\tau]^\lambda$ such that $F = E \cap B(A, f)$ is $G_\lambda$-dense in $B(A, f)$. We claim that $E$ is not a topological group. Striving for a contradiction, let us assume that $E$ is a topological group. Since $E$ is homogeneous, we assume without loss of generality that the neutral element $e$ of $E$ belongs to $F$.

Since $B(A, f)$ is a $G_\lambda$-subset of $\mathbb{I}^\tau$, $F$ is a $G_\lambda$-subset of $E$. Hence by Lemma 3.3 there is a closed subgroup $N$ of $E$ such that $N$ is a $G_\lambda$-subset of $E$ which is contained in $F$. Hence $N$ is a $G_\lambda$-subset of $F$. By Lemma 3.2, there is a $G_\lambda$-subset $S$ of $B(A, f)$ such that $N \subseteq S \subseteq \overline{N}$ (here ‘closure’ is closure in $B(A, f)$ and hence in $\mathbb{I}^\tau$). Observe that $S$ is a $G_\lambda$-subset of $\mathbb{I}^\tau$.

Since $e \in S \subseteq B(A, f)$, we may consequently pick a subset $A_0 \in [\tau]^\lambda$ and a function $f_0 \in \mathbb{I}^{A_0}$ such that $e \in B(A_0, f_0) \subseteq S \subseteq \overline{N} \subseteq B(A, f)$. Observe that $A \subseteq A_0$ and $f_0|A = f$. Since $B(A_0, f_0)$ is a $G_\lambda$-subset of $\mathbb{I}^\tau$ and $F = E \cap B(A, f)$ is $G_\lambda$-dense in $B(A, f)$, it follows that $F_0 = E \cap B(A_0, f_0)$ is $G_\lambda$-dense in $B(A_0, f_0)$. Now pick a closed subgroup $N_0$ of $E$ such that $N_0$ is a $G_\lambda$-subset of $E$ which is contained in $F_0$.

Continuing in this way recursively, we construct:

1. subsets $A_0 \subseteq A_1 \subseteq \cdots$ of $\tau$, each of size $\lambda$,
2. functions $f_n : A_n \rightarrow \mathbb{I}$ for $n < \omega$ such that $f_n|A_{n-1} = f_{n-1}$,
3. closed subgroups $N_0 \supseteq N_1 \supseteq \cdots$ of $E$ which are $G_\lambda$-subsets of $E$ while moreover
\[ B(A_0, f_0) \supseteq N_0 \supseteq B(A_1, f_1) \supseteq N_1 \supseteq \cdots. \]

Put $M = \bigcap_{n < \omega} N_n$. Then $M$ is a closed subgroup of $E$ and is a $G_\lambda$-subset of $E$. Moreover, put $P = \bigcup_{n < \omega} A_n$ and $g = \bigcup_{n < \omega} f_n$, respectively. Then $M \subseteq B(P, g)$. Observe that
\[ E \cap B(P, g) = E \cap \bigcap_{n < \omega} B(A_n, f_n) = \bigcap_{n < \omega} N_n = M. \]

It is clear that \( P \) is of the size \( \lambda \) and that \( M \) is a \( G_\lambda \)-dense subset of \( B(P, g) \). Notice that \( B(P, g) \) is a non-metrizable Tychonoff cube since \( \lambda < \tau \). We see that \( M \) is a topological group which is \( G_\delta \)-dense in a non-metrizable Tychonoff cube. This will lead to a contradiction: in Claim 1 we will show that some Tychonoff cube is the Čech-Stone-compactification of \( M \), and then we will show that this is impossible.

**Claim 1.** The space \( M \) is pseudocompact, and some Tychonoff cube \( \Gamma^\kappa \) is the Čech-Stone compactification \( \beta M \) of \( M \).

To prove Claim 1, we need to use a factorization result of Arhangel’skii [1], the key feature of which is that it concerns continuous functions on dense subspaces of products of separable metrizable spaces [2, Corollary 1.7.8]. It implies that every continuous realvalued function on a dense subset of a Tychonoff cube depends on countably many coordinates. Thus if \( A \) is a \( G_\delta \)-dense subset of some Tychonoff cube \( \Gamma^\tau \), then for every continuous function \( f: A \to \mathbb{R} \) there is, by Corollary 1.7.8 in [2], a countable subset \( L \) of \( \tau \) and a continuous function \( g: \pi_L(A) \to \mathbb{R} \), where \( \pi_L: \Gamma^\tau \to \Gamma^L \) is the projection mapping, such that \( g(\pi_L(a)) = f(a) \) for all \( a \in A \). However, since \( A \) is \( G_\delta \)-dense in the Tychonoff cube \( \Gamma^\tau \), we have \( \pi_L(A) = \Gamma^L \), which evidently implies that \( f \) can be continuously extended over \( \Gamma^\tau \). We also see that every continuous function \( f: A \to \mathbb{R} \) is bounded since \( \Gamma^\tau \) is compact. Hence \( A \) is pseudocompact. We apply this argument to \( A = M \) and conclude that \( M \) is pseudocompact. Thus, Claim 1 holds.

However, since \( M \) is a pseudocompact topological group, \( \beta M \) is also a topological group by the Comfort-Ross theorem [4]. But \( \Gamma^\tau \) is not a topological group, for example because it has the Fixed-Point Property by Brouwer’s Theorem. (A similar argument was also used in [3]).

We repeat a problem posed in [3]: can every compact topological group be split into two homeomorphic and homogeneous parts?

Again, \( \tau \) be an uncountable cardinal. The referee remarked that our arguments actually prove a stronger result: If \( X \) is a compact space of countable weight such that \( X^\omega \) is not a topological group, and if \( \mathcal{E} \) is a cover of \( X^\tau \) by subspaces that are each homeomorphic to a topological group (not necessarily the same one), then \( |\mathcal{E}| \geq \tau^+ \). The proof is identical to the one above, except for the fact that we need an argument to conclude that \( X^\tau \) is not a topological group. But, as the referee noted, this is a direct consequence of Ridderbos [6, Theorem 2.3].

**References**


