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Covering Tychonoff cubes by topological groups

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ABSTRACT

Let τ be an uncountable cardinal. We prove that if $\mathscr{A}$ is a cover of the Tychonoff cube $\mathbb{I}^\tau$ such that $|\mathscr{A}| \leq \tau$, then some element $A \in \mathscr{A}$ is not homeomorphic to a topological group.

1. Introduction

If X is a space, then $\big\{\{x\} : x \in X\big\}$ is a cover of X by $|X|$-many spaces that are homeomorphic to a topological group. One naturally wonders whether one can do better than this triviality.

Every finite-dimensional compact metrizable space without isolated points can be covered by a finite family consisting of topological copies of the space of irrational numbers. Hence each such space can be covered by a finite family of subspaces, each homeomorphic to a topological group. For the Hilbert cube $Q = \mathbb{I}^\omega$ this is also true, but requires nontrivial results from infinite-dimensional topology. In fact, Q can be covered by two topological copies of the countable infinite product of lines. One of these copies is $s = (0,1)^\omega$. The other one comes from the following observation. There is a homeomorphism f of Q sending its pseudo-boundary into its pseudo-interior. Hence $f^{-1}(s)$ is a topological copy of s containing the pseudo-boundary of Q. For details, see van Mill [5, §6.5].

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For non-metrizable Tychonoff cubes, the situation is dramatically different.

Theorem 1.1. *Let τ be an uncountable cardinal. If $\mathscr{E}$ is a cover of the Tychonoff cube $\mathbb{I}^\tau$ by subspaces that are each homeomorphic to a topological group (not necessarily the same one), then $|\mathscr{E}| \geq \tau^+$.*

So under the Generalized Continuum Hypothesis, it follows that for uncountable τ, the Tychonoff cube $\mathbb{I}^\tau$ cannot be covered by a family of fewer than 2^τ subspaces, each homeomorphic to a topological group. We do not know whether this can be proved without additional set theoretical assumptions.

We use some ideas in our recent paper Arhangel'skii and van Mill [3].

2. Preliminaries

All topological spaces under discussion are assumed to be Tychonoff.

We use standard notation. If B is a set and λ is a cardinal number, then $[B]^{\leq\lambda}$ denotes the collection $\{A \subseteq B : |A| \leq \lambda\}$.

Let λ be an infinite cardinal. We say that a subset B of a space X is a G_λ-*subset* of X provided there exists a family $\mathscr{U}$ of open subsets of X such that $|\mathscr{U}| \leq \lambda$ and $B = \bigcap \mathscr{U}$. A subspace D of X is called G_λ-*dense* in X provided that D intersects every nonempty G_λ-subset of X.

Observe that since λ is assumed to be infinite, each nonempty G_λ-subset of X contains a nonempty closed G_λ-subset.

A G_ω-dense set is usually called G_δ-dense.

Let τ be an uncountable cardinal, and consider the Tychonoff cube $\mathbb{I}^\tau$. If $A \subseteq \tau$ and $f\colon A \to \mathbb{I}$, then

$$B(A, f) = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A)(x_\alpha = f(\alpha))\}.$$

Then $B(A, f)$ is a closed G_λ-subset of $\mathbb{I}^\tau$, where $\lambda = |A| \cdot \omega$. Observe that if $\lambda \leq \tau$ and B is a nonempty G_λ-subset of $\mathbb{I}^\tau$, then there are a subset $A \in [\tau]^{\leq\lambda}$ and an $f \in \mathbb{I}^A$ such that $B(A, f) \subseteq B$. Moreover, $B(A, f)$ is a closed G_λ-subset of $\mathbb{I}^\tau$ and hence of B.

We will use the trivial fact that if D is a G_λ-dense subset of a space X, and S is a nonempty G_λ-subset of X, then $D \cap S$ is G_λ-dense in S.

3. The construction

Now let τ be an uncountable cardinal, and let $\mathscr{E}$ be a cover of $\mathbb{I}^\tau$ such that $|\mathscr{E}| \leq \tau$.

Lemma 3.1. *There are $E \in \mathscr{E}$, λ and $A \in [\tau]^\lambda$, where $\omega \leq \lambda < \tau$, and $f \in \mathbb{I}^A$ such that $E \cap B(A, f)$ is a G_λ-dense subset of $B(A, f)$.*

Proof. Assume that this is not true. Enumerate $\mathscr{E}$ as $\{E_\alpha : \alpha < \tau\}$ (repetitions permitted). By transfinite induction on $\alpha < \tau$, we will construct a subset A_α of τ of size at most $|\alpha| \cdot \omega$, and an element $f_\alpha \in \mathbb{I}^{A_\alpha}$ such that

(1) if $\alpha' < \alpha < \tau$, then $A_{\alpha'} \subseteq A_\alpha$ and $f_\alpha \restriction A_{\alpha'} = f_{\alpha'}$,
(2) $B(A_\alpha, f_\alpha) \cap E_\alpha = \emptyset$.

By assumption, E_0 is not G_ω-dense in $\mathbb{I}^\tau$, hence there are a subset $A_0 \in [\tau]^\omega$ and a function $f_0 \in \mathbb{I}^{A_0}$ such that $E_0 \cap B(A_0, f_0) = \emptyset$. Assume that we completed the construction for all $\alpha' < \alpha$, where $\alpha < \tau$. Put $A = \bigcup_{\alpha' < \alpha} A_{\alpha'}$ and $f = \bigcup_{\alpha' < \alpha} f_{\alpha'}$, respectively. Then $\omega \leq \lambda = |A| \leq |\alpha| \cdot \omega < \tau$. By assumption, $E_\alpha \cap B(A, f)$ is not G_λ-dense in $B(A, f)$. Hence there is a nonempty G_λ-subset F of $B(A, f)$ such that

$E_\alpha \cap F = \emptyset$. Then, clearly, F is a G_λ-subset of $\mathbb{I}^\tau$. Hence there are $A' \in [\tau]^{\leq\lambda}$ and $f' \in \mathbb{I}^{A'}$ such that $B(A', f') \subseteq F \subseteq B(A, f)$. Clearly, $A \subseteq A'$ and $f' \restriction A = f$. Hence by putting $A_\alpha = A'$ and $f_\alpha = f'$, we satisfy our inductive requirements. This completes the transfinite construction.

Now put $A = \bigcup_{\alpha<\tau} A_\alpha$ and $f = \bigcup_{\alpha<\tau} f_\alpha$. There exists $x \in \mathbb{I}^\tau$ such that

$$\forall\, \alpha \in A : x_\alpha = f(\alpha).$$

Then, by construction, $x \notin \bigcup \mathscr{E}$, which contradicts the fact that $\mathscr{E}$ covers $\mathbb{I}^\tau$. $\square$

Lemma 3.2. *Let λ be an infinite cardinal, and assume that X is a G_λ-dense subset of Y. Then for every subset A of X which is a G_λ-subset of X, there is a G_λ-subset S of Y such that $A \subseteq S \subseteq \overline{A}$ (here 'closure' means closure in Y), and $S \cap X = A$.*

Proof. Let $\mathscr{U}$ be a family of at most λ open subsets of X such that $A = \bigcap \mathscr{U}$. For every $U \in \mathscr{U}$, pick an open subset $V(U)$ of Y such that $V(U) \cap X = U$. Put $S = \bigcap_{U\in\mathscr{U}} V(U)$. Then, clearly, S is a G_λ-subset of Y, $A \subseteq S$, and $S \cap X = A$. Assume that there exists an element $p \in S \setminus \overline{A}$. Then $S \cap (Y \setminus \overline{A})$ is a nonempty G_λ-subset of Y which misses X, which contradicts the fact that X is G_λ-dense in Y. $\square$

The following lemma is well-known and its proof is left to the reader (or see [3, Lemma 3.1]).

Lemma 3.3. *Let G be a topological group. If S is a G_δ-subset of G containing the neutral element e of G, then there is a closed subgroup N of G such that*

(1) $N \subseteq S$,
(2) N is a G_δ-subset of G.

After these preliminary observations, we will present the proof of Theorem 1.1.

By Lemma 3.1, there exist $E \in \mathscr{E}$, an infinite cardinal $\lambda < \tau$ and $A \in [\tau]^\lambda$ such that $F = E \cap B(A, f)$ is G_λ-dense in $B(A, f)$. We claim that E is not a topological group. Striving for a contradiction, let us assume that E is a topological group. Since E is homogeneous, we assume without loss of generality that the neutral element e of E belongs to F.

Since $B(A, f)$ is a G_λ-subset of $\mathbb{I}^\tau$, F is a G_λ-subset of E. Hence by Lemma 3.3 there is a closed subgroup N of E such that N is a G_λ-subset of E which is contained in F. Hence N is a G_λ-subset of F. By Lemma 3.2, there is a G_λ-subset S of $B(A, f)$ such that $N \subseteq S \subseteq \overline{N}$ (here 'closure' is closure in $B(A, f)$ and hence in $\mathbb{I}^\tau$). Observe that S is a G_λ-subset of $\mathbb{I}^\tau$.

Since $e \in S \subseteq B(A, f)$, we may consequently pick a subset $A_0 \in [\tau]^\lambda$ and a function $f_0 \in \mathbb{I}^{A_0}$ such that $e \in B(A_0, f_0) \subseteq S \subseteq \overline{N} \subseteq B(A, f)$. Observe that $A \subseteq A_0$ and $f_0 \restriction A = f$. Since $B(A_0, f_0)$ is a G_λ-subset of $\mathbb{I}^\tau$and $F = E \cap B(A, f)$ is G_λ-dense in $B(A, f)$, it follows that $F_0 = E \cap B(A_0, f_0)$ is G_λ-dense in $B(A_0, f_0)$. Now pick a closed subgroup N_0 of E such that N_0 is a G_λ-subset of E which is contained in F_0.

Continuing in this way recursively, we construct:

(1) subsets $A_0 \subseteq A_1 \subseteq \cdots$ of τ, each of size λ,
(2) functions $f_n \colon A_n \to \mathbb{I}$ for $n < \omega$, such that $f_n \restriction A_{n-1} = f_{n-1}$,
(3) closed subgroups $N_0 \supseteq N_1 \supseteq \cdots$ of E which are G_λ-subsets of E while moreover

$$B(A_0, f_0) \supseteq \overline{N}_0 \supseteq B(A_1, f_1) \supseteq \overline{N}_1 \supseteq \cdots.$$

Put $M = \bigcap_{n<\omega} N_n$. Then M is a closed subgroup of E and is a G_λ-subset of E. Moreover, put $P = \bigcup_{n<\omega} A_n$ and $g = \bigcup_{n<\omega} f_n$, respectively. Then $M \subseteq B(P, g)$. Observe that

$$E \cap B(P,g) = E \cap \bigcap_{n<\omega} B(A_n, f_n) = \bigcap_{n<\omega} N_n = M.$$

It is clear that P is of the size λ and that M is a G_λ-dense subset of $B(P,g)$. Notice that $B(P,g)$ is a non-metrizable Tychonoff cube since $\lambda < \tau$. We see that M is a topological group which is G_δ-dense in a non-metrizable Tychonoff cube. This will lead to a contradiction: in Claim 1 we will show that some Tychonoff cube is the Čech-Stone-compactification of M, and then we will show that this is impossible.

Claim 1. The space M is pseudocompact, and some Tychonoff cube $\mathbb{I}^\kappa$ is the Čech-Stone compactification βM of M.

To prove Claim 1, we need to use a factorization result of Arhangel'skii [1], the key feature of which is that it concerns continuous functions on dense subspaces of products of separable metrizable spaces [2, Corollary 1.7.8]. It implies that every continuous realvalued function on a dense subset of a Tychonoff cube depends on countably many coordinates. Thus if A is a G_δ-dense subset of some Tychonoff cube $\mathbb{I}^\tau$, then for every continuous function $f\colon A \to \mathbb{R}$ there is, by Corollary 1.7.8 in [2], a countable subset L of τ and a continuous function $g\colon \pi_L(A) \to \mathbb{R}$, where $\pi_L\colon \mathbb{I}^\tau \to \mathbb{I}^L$ is the projection mapping, such that $g(\pi_L(a)) = f(a)$ for all $a \in A$. However, since A is G_δ-dense in the Tychonoff cube $\mathbb{I}^\tau$, we have $\pi_L(A) = \mathbb{I}^L$, which evidently implies that f can be continuously extended over $\mathbb{I}^\tau$. We also see that every continuous function $f\colon A \to \mathbb{R}$ is bounded since $\mathbb{I}^\tau$ is compact. Hence A is pseudocompact. We apply this argument to $A = M$ and conclude that M is pseudocompact. Thus, Claim 1 holds.

However, since M is a pseudocompact topological group, βM is also a topological group by the Comfort-Ross theorem [4]. But $\mathbb{I}^\tau$ is not a topological group, for example because it has the Fixed-Point Property by Brouwer's Theorem. (A similar argument was also used in [3]).

We repeat a problem posed in [3]: can every compact topological group be split into two homeomorphic and homogeneous parts?

Again, τ be an uncountable cardinal. The referee remarked that our arguments actually prove a stronger result: If X is a compact space of countable weight such that X^ω is not a topological group, and if $\mathscr{E}$ is a cover of X^τ by subspaces that are each homeomorphic to a topological group (not necessarily the same one), then $|\mathscr{E}| \geq \tau^+$. The proof is identical to the one above, except for the fact that we need an argument to conclude that X^τ is not a topological group. But, as the referee noted, this is a direct consequence of Ridderbos [6, Theorem 2.3].

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