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GROUPWISE EMBEDDED SUBSPACES OF TYCHONOFF CUBES

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ABSTRACT. We discuss the notion of groupwise embeddability in Tychonoff cubes of uncountable weight and pose some problems.

1. INTRODUCTION

All spaces under discussion are Tychonoff. A map stands for a continuous function.

In [2], the authors of the present note proved that if τ is any uncountable cardinal, then the Tychonoff cube $\mathbb{I}^\tau$ cannot be covered by a family consisting of τ-many subspaces, each homeomorphic to a topological group (not necessarily, the same one). Here $\mathbb{I}$ denotes the closed interval $[-1, 1]$. Hence the minimal number of subspaces that are homeomorphic to a topological group needed to cover $\mathbb{I}^\tau$ for uncountable τ is at least τ^+. In contrast, for $\tau = \omega$, the case of the Hilbert cube $Q = \mathbb{I}^\omega$, this number is 2. It is an open problem whether τ^+ in the uncountable case can be generalized to 2^τ, we will come back to this later.

Let τ be uncountable. If $\mathscr{A}$ is a cover of $\mathbb{I}^\tau$ such that for each member $A \in \mathscr{A}$ there is a subspace B of $\mathbb{I}^\tau$ such that $A \subseteq B$ and B is homeomorphic to some topological group, then by the above, $|\mathscr{A}| \geq \tau^+$. This motivates to ask for which classes of subspaces of $\mathbb{I}^\tau$ this holds, and prompts the following definition. A subspace Y of a space X is called *groupwise embedded* in X if there exists a subspace $Z \subseteq X$ such that $Y \subseteq Z$ and Z is homeomorphic to a topological group. Clearly, $\mathbb{I}^\tau$ is not groupwise embedded in $\mathbb{I}^\tau$ since it has the fixed-point property and hence is not a topological group. So for a subspace to be groupwise embedded says something about the way it is embedded in the ambient space.

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For several natural and important subclasses of *compact* subspaces of $\mathbb{I}^\tau$, powerful homeomorphism extension results obtained by Mednikov [8] and Chigogidze [4] (inspired by results of Anderson [1]), give interesting results. It can be shown, for example, that every compact subspace of $\mathbb{I}^\tau$ of weight smaller than τ and every Eberlein compact subspace of $\mathbb{I}^\tau$, is groupwise embedded. We obtain more results by not relying on these homeomorphism extension results but instead by directly applying Anderson's classical results for the Hilbert cube.

2. Preliminaries

2.1. **The Hilbert cube.** For all undefined notions, see [9]. The results in this subsection are well-known to experts in infinite-dimensional topology and follow by applications of standard methods.

Let $B(Q) = \{x \in Q : (\exists\, n < \omega)(|x_n| = 1)\}$ denote the pseudo-boundary of Q. In addition, for every $n \geq 1$, let $\Sigma_n = [-1+2^{-n}, 1-2^{-n}]^\omega$, i.e. $\Sigma_n = \{x \in Q : (\forall\, i < \omega)(|x_i| \leq 1-2^{-n})\}$. Put $\Sigma = \bigcup_{n\in\mathbb{N}} \Sigma_n$.

Let $\{E_n : n < \omega\}$ be a partition of ω, where each E_n is infinite. For each $n < \omega$, let $P(n) \subseteq [-1,1]^{E_n}$ be a Z-set homeomorphic to Q, and put

$$B = \{x \in Q : (\exists\, n < \omega)(x_n \in P(n))\}.$$

Obviously, B is a countable union of Z-sets in Q.

Lemma 2.1. *There is a homeomorphism h of Q such that $h(\Sigma) \subseteq B$.*

Proof. By the Homeomorphism Extension Theorem for Z-sets, [9, 6.4.6], there exists for each $n < \omega$, a homeomorphism $h_n\colon [-1,1]^{E_n} \to [-1,1]^{E_n}$ such that

$$h_n([-1+2^{-n}, 1-2^{-n}]^{E_n}) = P(n).$$

Clearly, $h = \prod_{n<\omega} h_n$, is as required. □

Corollary 2.2. *For each $n < \omega$, let Q_n be a Hilbert cube. Take an arbitrary $p_n \in Q_n$ for every n, and put $B = \{x \in \prod_{n<\omega} Q_n : (\exists\, n)(x_n = p_n)\}$. Then there is a homeomorphism of pairs $(Q, B(Q)) \approx (\prod_{n<\omega} Q_n, B)$.*

Proof. Let $\{F_n : n < \omega\}$ be a partition of ω, where each F_n is infinite. For each n, let $a(n) = \min F_n$. In the Hilbert cube $M_n = \prod_{i\in F_n} Q_i$, put $B_n = \{p_{a(n)}\} \times \prod_{i\in F_n\setminus\{a(n)\}} Q_i$. Then B_n is a Z-set copy of Q in M_n. By Lemma 2.1, there is a homeomorphism $h\colon Q \to \prod_{n<\omega} Q_n$ such that $h(\Sigma) \subseteq B$. Since B is a countable union of Z-sets in $\prod_{n<\omega} Q_n$, it follows that B is an absorber by [9, 6.5.2(2)], from which by an application of [9, 6.5.8], the result is obvious. □

2.2. **Tychonoff cubes of uncountable weight.** Assume that τ is an uncountable cardinal. The homeomorphism extension result for $\mathbb{I}^\tau$ that we mentioned in §1 is stated in terms of so-called Z_τ-sets. They were characterized in Chigogidze [4, 5.3], as follows: a closed subset A of $\mathbb{I}^\tau$ is a Z_τ-set if and only if it does not contain any G_κ-subset of $\mathbb{I}^\tau$, for any $\kappa < \tau$. Here a set is a G_κ-*set* if it is an intersection of a family consisting of κ-many open sets. Any homeomorphism between Z_τ-subsets of $\mathbb{I}^\tau$ can by [4, 4.10] be extended to a homeomorphism of $\mathbb{I}^\tau$. See also [8] for a more general result.

Proposition 2.3. *Let τ be uncountable, and let K be a compact subspace of $\mathbb{I}^\tau$ which does not contain a topological copy of $\mathbb{I}^\tau$. Then K is groupwise embedded in $\mathbb{I}^\tau$.*

Proof. If K would contain a G_κ-subset of $(-1,1)^\tau$ for certain $\kappa < \tau$ it would contain a closed copy of $\mathbb{I}^\tau$ which is not the case. Hence K is a Z_τ-subset of $\mathbb{I}^\tau$. The same reasoning applies to all topological copies of K in $\mathbb{I}^\tau$. Since there is a topological copy of K in $(-1,1)^\tau$, the homeomorphism extension theorem for Z_τ-subsets of $\mathbb{I}^\tau$ implies that K is contained in a subspace of $\mathbb{I}^\tau$ that is homeomorphic to $\mathbb{R}^\tau$. Hence K is groupwise embedded in $\mathbb{I}^\tau$. □

There are many classes of compact subspaces each element of which does not contain a closed topological copy of $\mathbb{I}^\tau$. They all consist of groupwise embedded spaces. Examples are the class of all Eberlein compact subspaces of $\mathbb{I}^\tau$, the class of all hereditarily normal compact subspaces of $\mathbb{I}^\tau$, and the class of all compact subspaces of weight less that τ, and, more generally, the class of all compact subspaces of tightness less than τ.

Šapirovskiĭ [10] (see also [7, 3.18]) proved that a compact space X can be mapped onto $\mathbb{I}^\tau$ if and only if there is a closed subspace F of X with $\pi_\chi(p,F) \geq \tau$ for each $p \in F$. Hence closed subspaces of $\mathbb{I}^\tau$ that do not satisfy Šapirovskiĭ's criterion, are Z_τ-sets and hence are groupwise embedded. This is clear since if the compact subspace X of $\mathbb{I}^\tau$ does not satisfy Šapirovskiĭ's criterion, then it does not contain a closed copy of $\mathbb{I}^\tau$ since such a copy would be a retract of X.

In the next section, we will generalize Proposition 2.3 somewhat. Observe that the groupwise embedded subspaces of $\mathbb{I}^\tau$ that we get from it are subspaces of compact subspaces that are groupwise embedded. Hence it is not clear for example whether every *countable* subspace of $\mathbb{I}^\tau$ is groupwise embedded. Indeed, the closure of any countable dense subset of $\mathbb{I}^\mathfrak{c}$ is not groupwise embedded so the compact case does not help here. Nonetheless, they are groupwise embedded, as we will show in the next section.

3. Main results

We again consider the case of Tychonoff cubes $\mathbb{I}^\tau$ for uncountable τ.

If $B \subseteq \tau$, then $\pi_B\colon \mathbb{I}^\tau \to \mathbb{I}^B$ denotes the projection. Moreover, if $S, T \subseteq \tau$ and $S \subseteq T$, then $\pi^T_S\colon \mathbb{I}^T \to \mathbb{I}^S$ denotes the projection.

Theorem 3.1. *Let X be a subspace of $\mathbb{I}^\tau$, where $\tau > \omega$, such that X cannot be mapped onto a G_δ-dense subset of $\mathbb{I}^\tau$. Then X is groupwise embedded in $\mathbb{I}^\tau$.*

Proof. Let $\mathscr{B}$ be a maximal family consisting of countably infinite subsets B of τ such that $\pi_B(X) \neq \mathbb{I}^B$.

Assume first that $E = \tau \setminus \bigcup \mathscr{B}$ has size τ. By assumption, $\pi_E(X)$ is not G_δ-dense in $\mathbb{I}^E$. There consequently is a countably infinite subset F of E such that $\pi_F(X) \neq \mathbb{I}^F$. But this means that $\mathscr{B}$ is not maximal.

Hence $|\tau \setminus \bigcup \mathscr{B}| < \tau$, and so $|\mathscr{B}| = \tau$. If $B \in \mathscr{B}$ and $\alpha \in \tau \setminus \bigcup \mathscr{B}$, then $\pi_{B\cup\{\alpha\}}(X) \neq \mathbb{I}^{B\cup\{\alpha\}}$. This means that we can add the points of $\tau \setminus \bigcup \mathscr{B}$ one by one to distinct members of $\mathscr{B}$ so that may assume without loss of generality that $\bigcup \mathscr{B} = \tau$.

Write $\mathscr{B}$ as $\{B_{\alpha,n} : (\alpha < \tau) \,\&\, (n < \omega)\}$, where $B_{\alpha,n} \neq B_{\alpha',n'}$ if $(\alpha, n) \neq (\alpha', n')$. For every $\alpha < \tau$, put $E_\alpha = \bigcup_{n<\omega} B_{\alpha,n}$ and let Q_α be the Hilbert cube $\mathbb{I}^{E_\alpha}$. We think of Q_α as the product $\prod_{n<\omega} \mathbb{I}^{B_{\alpha,n}}$. Observe that for every $n < \omega$, $\pi^{E_\alpha}_{B_{\alpha,n}}(\pi_{E_\alpha}(X))$ is a proper subset of $\mathbb{I}^{B_{\alpha,n}}$. By Corollary 2.2, there is a homeomorphism $f_\alpha\colon Q_\alpha \to Q_\alpha$ such that $f_\alpha(\pi_{E_\alpha}(X)) \subseteq (-1,1)^{E_\alpha}$. The product $f = \prod_{\alpha<\tau} f_\alpha\colon \mathbb{I}^\tau \to \mathbb{I}^\tau$ is a homeomorphism and $f(X) \subseteq \prod_{\alpha<\tau}(-1,1)_\alpha$. But $\prod_{\alpha<\tau}(-1,1)_\alpha$ is homeomorphic to the topological group $\mathbb{R}^\tau$, hence X is groupwise embedded in $\mathbb{I}^\tau$. □

There are several unrelated classes of subspaces of $\mathbb{I}^\tau$ that are groupwise embbedded by this theorem.

Corollary 3.2. *Let τ be an uncountable cardinal. The following classes of subspaces of $\mathbb{I}^\tau$ are groupwise embedded in $\mathbb{I}^\tau$.*

(1) *All subspaces of size less than $\mathfrak{c}$ (hence all countable subspaces).*
(2) *All subspaces that can be written as $\bigcup_{n<\omega} X_n$, where each X_n is a compact space that cannot be mapped onto $\mathbb{I}^\tau$.*
(3) *All subspaces that can be written as $\bigcup_{n<\omega} X_n$, where each X_n is a compact space of tightness less than τ.*
(4) *All P-spaces of Lindelöf degree less than $\mathfrak{c}$ (hence all Lindelöf P-spaces).*

Proof. The proof of (1) is clear, since every G_δ-dense subspace of $\mathbb{I}^\tau$ has size at least $\mathfrak{c}$.

For (2), let $X = \bigcup_{n<\omega} X_n$ be as stipulated. Assume that there is a continuous map $f\colon X \to S$, where $S \subseteq \mathbb{I}^\tau$ is G_δ-dense. Clearly, S is both pseudocompact and Lindelöf, hence compact. This means that f maps X onto $\mathbb{I}^\tau$. By the Baire Category Theorem, there exists n such that $f(X_n)$ has nonempty interior in $\mathbb{I}^\tau$. But then $f(X_n)$ containes a copy of $\mathbb{I}^\tau$ on which it can be retracted. This is a contradiction, since we assumed that X_n cannot be mapped onto $\mathbb{I}^\tau$.

Statement (3) is a consequence of (2), since the tightness does not increase under continuous maps of compacta, and since the tightness of $\mathbb{I}^\tau$ equals τ.

For (4), simply observe that $\mathbb{I}^\tau$ can be split into $\mathfrak{c}$-many nonempty G_δ-sets. □

Theorem 3.1 is in some sense sharp since for $\tau > \omega$, no G_δ-dense subspace of $\mathbb{I}^\tau$ is groupwise embedded. Indeed, let $X \subseteq \mathbb{I}^\tau$ be G_δ-dense. Then any subspace Y of $\mathbb{I}^\tau$ containing X is G_δ-dense as well. But in [2, Page 4, lines 3-5] it was shown by employing the celebrated Comfort-Ross theorem from [5] that no topological group in $\mathbb{I}^\tau$ is G_δ-dense.

4. Remarks and questions

Let us begin by stating the problem that has motivated this note.

Question 4.1. Let $\tau > \omega$, and let $\mathscr{A}$ be a cover of $\mathbb{I}^\tau$ such that $|\mathscr{A}| < 2^\tau$. Does there exist $A \in \mathscr{A}$ such that A is not homeomorphic to a topological group?

There are proper subspaces of Q that are not groupwise embedded. Indeed, take any $x \in Q$, and consider $X = Q \setminus \{x\}$. By Fathi and Visetti [6], X is not a topological group. Hence the only candidate for a subspace of Q that contains X and is a topological group, is Q itself. But Q has the fixed-point property. The following question seems open.

Question 4.2. Let X be a proper closed subspace of Q. Is X groupwise embedded?

An inspection of our proofs reveals that all the enveloping topological groups in $\mathbb{I}^\tau$ that we found are topological copies of $\mathbb{R}^\tau$. This prompts the following problem:

Question 4.3. Let $\tau \geq \omega$. Is there a subspace X of $\mathbb{I}^\tau$ that is a topological group for which there does not exist a subspace Y of $\mathbb{I}^\tau$ which is homeomorphic to $\mathbb{R}^\tau$ and contains X?

The problem seems open even in the case that $\tau = \omega$.

Question 4.4. Let $\tau \geq \omega$ and assume that X and Y are groupwise embedded (compact) subspaces of $\mathbb{I}^\tau$. Is $X \cup Y$ groupwise embedded?

A positive answer to this question for uncountable τ would follow from a positive answer to the following problem.

Question 4.5. Let $\tau > \omega$ and assume that $X \subseteq \mathbb{I}^\tau$ is compact. Is X groupwise embedded in $\mathbb{I}^\tau$ if and only if X is a Z_τ-set in $\mathbb{I}^\tau$?

For $\tau = \omega$, this is not true. Wong [11] constructed a Cantor set in the Hilbert cube which is not a Z-set.

Let us call a subspace X of a space Y *weakly groupwise embedded* in Y if it is contained in the union of a countable family of subspaces of Y each of which is homeomorphic to a topological group. Clearly, this notion is countably additive. Moreover by [2], each cover of $\mathbb{I}^\tau$ for uncountable τ by weakly groupwise embedded subspaces has size at least τ^+.

Question 4.6. Is every (compact) weakly groupwise embedded subspace of $\mathbb{I}^\tau$ for uncountable τ groupwise embedded in $\mathbb{I}^\tau$?

If the answer to Question 4.4 is in the negative, then so is the answer to this question.

Let us take this opportunity to correct an inaccuracy in our paper [3] that was brought to our attention by Benjamin Vejnar. We are grateful to him for informing us about this. The statement on Page 6, line 3, that the complement of A would be compact if A would be locally compact, is not true. All that can be concluded is that the complement of A in its closure is compact. But the proof of the theorem can be completed easily by the ideas that are in the paper. Assume that $\mathbb{I}^\tau \setminus A$ is not dense. Then A contains a basic closed subcube of $\mathbb{I}^\tau$. Since A is a topological group, one can use the technique as in the proof of Theorem 3.2 to construct a compact subcube of $\mathbb{I}^\tau$ which is contained in A and which has uncountable weight and is a topological group. Then one reaches the same contradiction as in the paper. Besides, a stronger result was proven in our subsequent paper [2].

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