

CONJUGACY CLASSES OF AUTOHOMEOMORPHISMS OF \mathbb{N}^*

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ABSTRACT. We present some problems related to the conjugacy classes of $\text{Aut}(\mathbb{N}^*)$.

INTRODUCTION

1. SOME DEFINITIONS AND NOTATION

As this note is about the autohomeomorphisms of \mathbb{N}^* we fix some notation regarding $\beta\mathbb{N}$. For a quick overview of $\beta\mathbb{N}$ we refer to Chapter D-18 of [12]; a more comprehensive introduction is [14] by the second author.

We let Aut denote the autohomeomorphism group of \mathbb{N}^* , rather than $\text{Aut}(\mathbb{N}^*)$, because \mathbb{N}^* will be the only space under discussion in this paper.

Let us first identify some easily described members of this group.

Trivial autohomeomorphisms. To begin: it is clear that an autohomeomorphism of $\beta\mathbb{N}$ leaves both \mathbb{N} and \mathbb{N}^* invariant and hence is determined by its restriction on \mathbb{N} , which is a permutation of \mathbb{N} . This provides us with the first source of autohomeomorphisms of \mathbb{N}^* : the permutation group $S_{\mathbb{N}}$ of \mathbb{N} .

For $\pi \in S_{\mathbb{N}}$ we let $\beta\pi$ denote its extension to $\beta\mathbb{N}$ and π^* the restriction of $\beta\pi$ to \mathbb{N}^* . Thus, permutations of \mathbb{N} determine autohomeomorphisms of \mathbb{N}^* . It is an elementary exercise to show that $\pi^* = \rho^*$ if and only if the set $\{n : \pi(n) \neq \rho(n)\}$ is finite. This identifies our first set of easily described members of Aut : the image $\{\pi^* : \pi \in S_{\mathbb{N}}\}$ under the homomorphism $\pi \mapsto \pi^*$.

Every permutation is built up from cyclic permutations and if two permutations σ and τ are conjugate, say $\sigma = \pi^{-1}\tau\pi$ then the permutation π provides a one-to-one correspondents between the sets of cycles of σ and τ . Note that there may also be infinite cycles; these look like the infinite cyclic group \mathbb{Z} with the map $n \mapsto n + 1$.

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This shows that conjugacy classes in $S_{\mathbb{N}}$ are determined by sequences of the form $\langle \kappa_n : n < \omega \rangle$, where κ_n is the number of n -cycles in the permutation if $n \geq 1$, and κ_0 is the number of infinite cycles. Of course $\kappa_n \leq \aleph_0$ for all n .

There are other autohomeomorphisms with an easy description. Every bijection $\varphi : A \rightarrow B$ between co-finite subsets of \mathbb{N} determines an autohomeomorphism of \mathbb{N}^* : the restriction φ^* of $\beta\varphi : \text{cl } A \rightarrow \text{cl } B$ is a homeomorphism from $A^* = \mathbb{N}^*$ to $B^* = \mathbb{N}^*$. As above, if $\psi : C \rightarrow D$ is another such bijection then $\varphi^* = \psi^*$ iff $\{n \in A \cap C : \varphi(n) = \psi(n)\}$ is co-finite in \mathbb{N} .

The autohomeomorphisms that we described thus far are called *trivial autohomeomorphisms*, they form a subgroup of Aut that we will denote Triv .

Shelah proved that it is consistent that all autohomeomorphisms of \mathbb{N}^* are trivial, see [17, IV §5].

2. MODERATELY EASY RESULTS

In this section we describe two situations where one can say quite a lot about conjugacy classes in Aut . These are at the opposite ends of the spectrum: one is the situation where all autohomeomorphisms are trivial and the other is where the Continuum Hypothesis holds and there is a wide (possibly the widest) collection of non-trivial autohomeomorphisms of \mathbb{N}^* .

2.1. Trivial autohomeomorphisms. As mentioned above it is consistent that all autohomeomorphisms are trivial hence we should look at conjugacy in this case.

We start by quoting a result by Van Douwen from [5]. To this end we associate an integer with every element of Triv . Let φ be a bijection between co-finite subsets of \mathbb{N} ; define

$$h(\varphi) = |\mathbb{N} \setminus \text{ran } \varphi| - |\mathbb{N} \setminus \text{dom } \varphi|$$

Now Theorem 6.1 from [5] states that h induces a homomorphism from Triv onto \mathbb{Z} . That is, if $\varphi^* = \psi^*$ then $h(\varphi) = h(\psi)$ and the induced map $\varphi^* \mapsto h(\varphi)$ is a homomorphism. We use h to denote this homomorphism.

If φ^* and ψ^* are conjugate in Triv then $h(\varphi^*) = h(\psi^*)$. Therefore we concentrate on conjugacy of autohomeomorphisms determined by members of $S_{\mathbb{N}}$.

Many conjugacy classes. Assume φ^* and ψ^* are conjugate in Triv , this means that there is a bijection $\tau : A \rightarrow B$ between co-finite sets such that $\varphi^*\tau^* = \tau^*\psi^*$ and this in turn means that the set $X = \{n : \varphi(\tau(n)) = \tau(\psi(n))\}$ is co-finite.

There are only finitely many cycles in φ and ψ whose domains (and their (pre)images under τ) meet the complement of X . The remaining cycles of φ are mapped by τ to cycles of ψ and vice versa. Therefore the sequences $\langle \kappa_n : n < \omega \rangle$ and $\langle \lambda_n : n < \omega \rangle$ of cycle numbers of φ and ψ respectively are almost equal.

This makes it easy to construct a family of \mathfrak{c} many permutations that represent members of Triv that are not conjugate. For every infinite subset x of \mathbb{N} take a partition $\{A_n : n \in x\}$ of \mathbb{N} such that $|A_n| = n$ for all n and create a permutation π_x of \mathbb{N}

by permuting each A_n cyclically —say $(a_1 a_2 \dots a_n)$, where $A_n = \{a_1, a_2, \dots, a_n\}$ listed in order.

If $x \neq y$ then π_x and π_y are not conjugate in $S_{\mathbb{N}}$, but they may of course be conjugate in Triv , say if $x = \{2, 5\} \cup \{n : n \geq 10\}$ and $y = \{3, 4\} \cup \{n : n \geq 10\}$. If the symmetric difference of x and y is infinite then π_x^* and π_y^* will not be conjugate. This implies that an almost disjoint family of cardinality \mathfrak{c} will provide us with \mathfrak{c} many conjugacy classes.

In Section 3 we raise some questions suggested by these considerations.

Infinite cycles. We should make a few remarks about infinite cycles in permutations of \mathbb{N} . Such a cycle is, as mentioned above, a copy of the set \mathbb{Z} of integers with the shift map $\sigma : n \mapsto n + 1$. If we work, for the moment, in $\beta\mathbb{Z}$ then we see that \mathbb{Z}^* is split into two clopen sets that are minimally σ^* -invariant. Indeed, it should be clear that $L = \{n \in \mathbb{Z} : n < 0\}^*$ and $R = \{n \in \mathbb{Z} : n \geq 0\}^*$ are both invariant under σ^* . It is only slightly more difficult to verify that if A is an infinite subset of L (or R) such that $\sigma^*[A^*] \subseteq A^*$ then $L \setminus A$ (or $R \setminus A$) is finite.

We see that when passing from $S_{\mathbb{N}}$ to Triv an infinite cycle ceases to be a unit: it splits into two independent autohomeomorphisms.

Any conjugation, even if non-trivial, will preserve the structure of these minimal invariant clopen subsets; in particular the cardinality of the family of these sets. This shows that for two permutations φ and ψ for which φ^* and ψ^* are conjugate in Aut the numbers of infinite cycles are the same.

Conclusion: if φ and ψ have cycle number sequences $\langle \kappa_n : n < \omega \rangle$ and $\langle \lambda_n : n < \omega \rangle$ respectively and if φ^* and ψ^* are conjugate in Triv then

- $\kappa_0 = \lambda_0$,
- for all $n \geq 1$ the equalities $\kappa_n = \aleph_0$ and $\lambda_n = \aleph_0$ are equivalent, and
- $\kappa_n = \lambda_n$ for all but finitely many $n \geq 1$.

The last condition becomes important only if there are infinitely many n for which κ_n and λ_n are finite.

Part of the analysis above was used in [4] to show that an autohomeomorphism of \mathbb{N}^* derived from a homeomorphism between \mathbb{N}^* and ω_1^* is non-trivial.

2.2. The Continuum Hypothesis. Many questions have a relatively easy answer under the assumption of CH. This is largely due to Parovichenko’s characterization of \mathbb{N}^* under that assumption.

The Continuum Hypothesis implies that Aut is a simple group. This was proven by Fuchino in [10] in a more general form: the automorphism group of a saturated Boolean algebra is simple, see [18, Theorem 5.12] for a more accessible proof. Since the Continuum Hypothesis implies that the Boolean algebra of clopen sets of \mathbb{N}^* is saturated the result follows. In [11] Fuchino proved that Aut is also simple in the \aleph_2 -Cohen model.

We shall show that CH implies that \mathbf{Aut} has $2^{\mathfrak{c}}$ many conjugacy classes. For this we need two known results about \mathbb{N}^* .

The first result is due to Hart and Vermeer.

Theorem 1 ([13], CH). *Every closed P -set in \mathbb{N}^* is the fixed-point set of an involution.* \square

In fact the proof is flexible enough to enable one to make any given P -set the fixed-point set of an autohomeomorphism of any prescribed finite order.

The next result is due to Dow, Gubbi and Szymański.

Theorem 2 ([7]). *There are $2^{\mathfrak{c}}$ many mutually non-homeomorphic (rigid) compact separable extremally disconnected spaces.* \square

We combine these two results using the well-known fact that under CH every such separable space can be embedded into \mathbb{N}^* as a P -set, see [14, Theorem 1.4.4].

This produces $2^{\mathfrak{c}}$ many mutually non-homeomorphic P -sets. Each of these is the fixed-point set of an involution. These involutions are never conjugated because conjugate autohomeomorphisms have homeomorphic fixed-point sets.

Although this set of involutions answers the question about the number of conjugacy classes of \mathbf{Aut} it is actually quite small. As noted above we can, almost for free, get autohomeomorphisms of any desired finite order. In addition, Theorem 1.4.4 from [14] states that every compact F -space of weight \mathfrak{c} can be embedded in \mathbb{N}^* as a nowhere dense P -set. This immediately gives us many more conjugacy classes.

It also suggests some questions that we shall mention in Section 3.

3. QUESTIONS

In this section we collect questions that are suggested by the results in Section 2 and by other results in the literature.

What happens to trivial autohomeomorphisms? We found \mathfrak{c} many conjugacy classes in $S_{\mathbb{N}}$ by exploiting the cycle structure of permutations. In the model where all members of \mathbf{Aut} are trivial these gave us the maximum possible number of conjugacy classes.

We have also seen that permutations that are not conjugate may induce the same autohomeomorphism of \mathbb{N}^* .

It also seems conceivable that quite distinct permutations may determine conjugate members of \mathbf{Aut} in case there are non-trivial autohomeomorphisms. The general question then is: what happens to conjugacy classes of trivial autohomeomorphisms when \mathbf{Aut} is not equal to \mathbf{Triv} ?

We give some specific versions of this question below, where we should emphasize that to the best of our knowledge these questions have not even been answered under the assumption of the Continuum Hypothesis when \mathbf{Aut} is much much richer than \mathbf{Triv} .

Question 1. What is the relationship between conjugacy classes of permutations in $S_{\mathbb{N}}$ and their conjugacy classes in \mathbf{Aut} ?

This question is quite general and we may specialize to the permutations we considered in Section 2.

Question 2. Let x and y be infinite subsets of \mathbb{N} such that $x \neq^* y$. Under what conditions will π_x^* and π_y^* become conjugate in \mathbf{Aut} ?

As toy problems one may consider $x = \{2^n : n \in \mathbb{N}\}$ and $y = \{3^n : n \in \mathbb{N}\}$, or $u = \{2^n : n \in \mathbb{N}\}$ and $v = \{4^n : n \in \mathbb{N}\}$.

As a variation we can use a function $f : \mathbb{N} \rightarrow \mathbb{N}$ to specify a permutation τ_f (up to conjugacy): partition \mathbb{N} into set A_n where $|A_n| = f(n)$ for all n and turn each A_n into an $f(n)$ -cycle. The difference with the π_x is that we allow repetitions of cardinalities.

Once one knows the effect of CH on these questions one can venture into models where there are autohomeomorphisms of varying degrees of (non-)triviality. A sample of such models can be found for example in [8, 9, 15, 16]

Infinite cycles. In the above questions we concentrated on finite cycles. We have seen that an infinite cycle ceases to be a building block when we move to \mathbf{Aut} . It gives us two autohomeomorphisms that are not induced by permutations:

- the forward shift $\sigma_{\mathbb{N}} : n \mapsto n + 1$ on \mathbb{N} , and
- the downward shift $\sigma_{\mathbb{N}}^{-1} : n \mapsto n - 1$ on \mathbb{N} .

The latter two are mapped to 1 and -1 respectively by the homomorphism h .

The two shifts are minimal in that \mathbb{N}^* (and the empty set) are the only clopen sets that are invariant. The two shifts are not conjugate in \mathbf{Triv} , but whether they can be conjugate is open, even under CH.

Question 3. Is it consistent that $\sigma_{\mathbb{N}}$ and $\sigma_{\mathbb{N}}^{-1}$ are conjugate?

An extensive study of this problem can be found in [3].

The shift map also has various universality properties, see [1, 2]; for example CH implies that the system $\langle \mathbb{N}^*, \sigma_{\mathbb{N}}^{-1} \rangle$ is a quotient of $\langle \mathbb{N}^*, \sigma_{\mathbb{N}} \rangle$.

Other ways of (dis)proving conjugacy. We exhibited, under CH, many conjugacy classes by exhibiting autohomeomorphisms with non-homeomorphic fixed-point sets. These fixed-point sets were all P -sets and that is no coincidence; the converse of Theorem 1 is a theorem of ZFC: every fixed-point set of an autohomeomorphism of \mathbb{N}^* is a P -set.

This, combined with the homeomorphism extension theorem for P -sets from [6], indicates that the fixed-points sets will play a key role in deciding conjugacy.

As we saw above a P -set can be the fixed-point set of autohomeomorphisms of all possible finite orders.

This suggests the following concrete problem, under CH:

Question 4. Assume h and g are two autohomeomorphisms with the same fixed-point set and the same finite order. Are h and g conjugate?

We end with a general question: what other invariants can we use to (dis)prove conjugacy of autohomeomorphisms of \mathbb{N}^* .

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