Topology and its Applications 311 (2022) 107968

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Universal based-free involutions

Jan van Mill [a,*], James E. West [b]

[a] KdV Institute for Mathematics, University of Amsterdam, Science Park 105-107, P.O. Box 94248, 1090 GE Amsterdam, the Netherlands
[b] Department of Mathematics, Cornell University, Ithaca, NY 14053-4201, USA

ARTICLE INFO

Article history:
Received 28 September 2020
Received in revised form 5 January 2021
Available online 22 December 2021

MSC:
57N20
57S99

Keywords:
Universal based-free involution
Hilbert cube
Hilbert space
Z-set

ABSTRACT

By using a recent characterization result, we show that there are universal objects for various classes of based-free involutions.

1. Introduction

In this note, *space* means *separable metric space*. Let X and Y be spaces with involutions σ and τ, respectively. We say that a map $f : (X, \sigma) \to (Y, \tau)$ is *equivariant* provided that for every $x \in X$, $\tau(f(x)) = f(\sigma(x))$.

Let σ_Q, σ_{ℓ^2} and $\sigma_{\mathbb{R}^\infty}$ be the standard involutions of Q, ℓ^2 and $\mathbb{R}^\infty$, respectively, given by the formula $x \mapsto -x$. Here Q is the Hilbert cube $\prod_{n=1}^{\infty}[-1, 1]_n$, ℓ^2 is separable Hilbert space and $\mathbb{R}^\infty$ is the countable infinite product of real lines.

An involution on a space is called *based-free* if it has a unique fixed-point. Anderson asked in 1966 whether all based-free involutions on Q are conjugate. This formidable and classical problem which motivated the results in the present note as well as in its predecessor [12], is still unsolved. For more information on it, see Wong [14] and Berstein and West [4].

* Corresponding author.
E-mail addresses: j.vanMill@uva.nl (J. van Mill), west@math.cornell.edu (J.E. West).
URL: http://staff.fnwi.uva.nl/j.vanmill/ (J. van Mill).

https://doi.org/10.1016/j.topol.2021.107968
0166-8641/

In an earlier version of this note, we proved that (Q, σ_Q) is universal for the class of all compact spaces with a based-free involution. That is, for every compact space X with based-free involution τ, there exists an equivariant embedding $i : (X, \tau) \to (Q, \sigma_Q)$. We are indebted to the referee for pointing out that this result was proved earlier by Antonyan in [3, Corollary 2]. He also provided us with a simple and direct proof of this which we include here with his permission (see Theorem 3.3).

The compactness of X is essential in Antonyan's result, since, as we will show here, there does not exist an equivariant embedding $(\ell^2, \sigma_{\ell^2}) \to (Q, \sigma_Q)$. We will also show that $(\ell^2, \sigma_{\ell^2})$ is universal for *all* spaces with based-free involutions. And, if the space under consideration is Polish, i.e., complete, then the embedding can be chosen to be a closed embedding. The same results are not true for $(\mathbb{R}^\infty, \sigma_{\mathbb{R}^\infty})$: there does not exist an equivariant embedding $(\ell^2, \sigma_{\ell^2}) \to (\mathbb{R}^\infty, \sigma_{\mathbb{R}^\infty})$.

For background on infinite-dimensional topology, see [5], [6] and [10,11].

2. Preliminaries

If $f : X \to X$ is a function, then $\mathrm{Fix}(f)$ denotes its fixed-point set. We let $\iota : [-1, 1] \to [-1, 1]$ denote the standard reflection.

By s we denote $\prod_{n=1}^\infty (-1, 1)_n$. Moreover, σ_s denotes the standard involution on s. Clearly, the pairs $(\mathbb{R}^\infty, \sigma_{\mathbb{R}^\infty})$ and (s, σ_s) are topologically conjugate.

Let $\underline{0}$ be the point in s all coordinates of which are 0.

It is a consequence of a result in Engelking [8] that for every space X and homeomorphism f of X there exists a compactification bX of X such that f can be extended to a homeomorphism $\bar{f} : bX \to bX$. Observe that if f is an involution, then so is $\bar{f}$. This consequence of Engelking's theorem can also be proved rather easily by using Wallman compactifications. For background, see [11, §A.9]. We use these compactifications to prove the following result that will be important later on.

Theorem 2.1. *Let X be a space with involution τ. Then*

(1) *X has a compactification bX such that τ can be extended to an involution $\bar{\tau} : bX \to bX$.*
(2) *If (X, τ) is based-free, then the following statements are equivalent:*
   (a) *There is a compactification bX of X such that τ can be extended to a based-free involution $\bar{\tau} : bX \to bX$.*
   (b) *The unique fixed-point $\star$ of τ has arbitrarily small neighborhoods U in X such that $X \setminus U$ can be covered by a finite collection $\mathscr{F}$ of closed subsets of X such that for every $F \in \mathscr{F}$, $\tau(F) \cap F = \emptyset$.*

Proof. As we already noted, (1) follows from [8]. Alternatively, one can use the method of proof of (2).

For (2), first observe that (a) $\Rightarrow$ (b) is trivial. For if $\star$ is the unique fixed-point of $\bar{\tau}$, then it is the unique fixed-point of τ. If U is a small open neighborhood of $\star$ in bX, then for every $p \in bX \setminus U$ has the property that $\tau(p) \neq p$. Hence the existence of the family $\mathscr{F}$ follows from a straightforward compactness argument. For (b) $\Rightarrow$ (a), let $\{U_n : n \in \mathbb{N}\}$ be a neighborhood base for $\star$ in X such that for every n, there exists a finite family $\mathscr{F}_n$ of closed subsets of X such that $X \setminus U_n \subseteq \bigcup \mathscr{F}_n$ and for every $F \in \mathscr{F}_n$ we have $\tau(F) \cap F = \emptyset$. The countable collection $\{\overline{U}_n : n \in \mathbb{N}\} \cup \bigcup_{n=1}^\infty \mathscr{F}_n$ of closed subsets of X can by [11, A.9.1] be extended to a Wallman base $\mathscr{G}_0$. The collection $\mathscr{G}_0 \cup \tau(\mathscr{G}_0)$ (here $\tau(\mathscr{G}_0)$ denotes $\{\tau(G) : G \in \mathscr{G}_0\}$) can by the same result be extended to a Wallman base $\mathscr{G}_1$. Continue in this way recursively, and obtain Wallman bases $\mathscr{G}_0 \subseteq \mathscr{G}_1 \subseteq \cdots \subseteq \mathscr{G}_n \subseteq \cdots$ such that $\mathscr{G} = \bigcup_{n=0}^\infty \mathscr{G}_n$ is a Wallman base (trivial from the definition of a Wallman base, [11, p. 494]) which is invariant under τ. Now let bX be the corresponding Wallman compactification $\omega(X, \mathscr{G})$. Since $\mathscr{G}$ is τ-invariant, it is easy to see that τ can be extended to a homeomorphism $\bar{\tau} : bX \to bX$. (See e.g., [11, p. 189 lines 1-3, and Exercise A.9.4].) Now let p be any point of bX different from $\star$. There exists n such that $p \notin \overline{U}_n$, where the closure is taken in bX. Consider the collection $\mathscr{F}_n$. The point p has to be in the closure in bX of an element F of $\mathscr{F}_n$ since $\mathscr{F}_n$ is finite. But the disjoint sets F and $\tau(F)$ belong to the Wallman base $\mathscr{G}$ and hence F and $\tau(F)$ have disjoint closures in bX, [11, A.9.4]. Since $\bar{\tau}(p)$ is in the closure of $\tau(F)$ in bX, this shows that p is not a fixed-point of $\bar{\tau}$. In other words, $\bar{\tau}$ is based-free. □

Corollary 2.2. *For every space X with based-free involution σ, there exists a Polish space Y with based-free involution τ such that (X, σ) can be equivariantly embedded in (Y, τ). Moreover, if the unique fixed-point $\star$ of σ satisfies the condition in Theorem 2.1(b), then Y can be chosen to be compact.*

Proof. Let $\star$ be the unique fixed-point of σ, and let bX and $\bar{\sigma}$ be such as in Theorem 2.1(1). Put $F = \operatorname{Fix}(\bar{\sigma}) \setminus \{\star\}$, and $Y = bX \setminus F$. Then Y and $\tau = \bar{\sigma}\restriction Y$ are clearly as required. The second statement is obvious from Theorem 2.1(2). □

3. The compact case

We will now present the argument of the referee for the universality of (Q, σ_Q).

Lemma 3.1. *Let X a compact space with involution τ. Then for every closed τ-invariant subset A of X and $x \in X \setminus A$, there is an equivariant map $\varphi : (X, \tau) \to ([-1, 1], \iota)$ such that $\varphi(A) = 0$ and $\varphi(x) \neq 0$ (and hence $\varphi(\tau(x)) \neq 0$).*

Proof. Pick an arbitrary $t \in (0, 1]$, and define $f : \{x, \tau(x)\} \cup A \to [-1, 1]$ by $f(x) = t$, $f(\tau(x)) = -t$ and $f(a) = 0$ for every $a \in A$. By Tietze's theorem, we can extend f to a continuous function $\bar{f} : X \to [-1, 1]$. Define $\varphi : X \to [-1, 1]$ by $\varphi(z) = \frac{\bar{f}(z) - \bar{f}(\tau(z))}{2}$. Then φ is equivariant, $\varphi(x) = t > 0$, and since A is τ-invariant, $\varphi(A) = 0$. □

Corollary 3.2. *Let X a compact space with based-free involution τ. Then for all distinct points x and y in X, there is an equivariant map $\varphi : (X, \tau) \to ([-1, 1], \iota)$ such that $\varphi(x) \neq \varphi(y)$.*

Proof. Let $\star$ denote the unique fixed-point of τ, and let $x, y \in X$ be distinct. Then $x = \star$ or $x = \tau(y)$ or $x \notin \{\star, y, \tau(y)\}$. In all cases we are done by Lemma 3.1. □

So if X is a compact space with a based-free involution τ, then the equivariant maps $(X, \tau) \to ([-1, 1], \iota)$ separate the points of X. A standard reduction argument shows that countably many of such maps are sufficient to separate the points of X. The diagonal product of these functions yields the desired equivariant embedding $(X, \tau) \to (Q, \sigma_Q)$. This completes the proof of the following result.

Theorem 3.3 *(Antonyan [3]). Every compact space with a based-free involution admits an equivariant embedding in (Q, σ_Q).*

We now use the method in the proof of Lemma 3.1 to prove a generalization for later use. The pseudo-boundary, $B(Q)$, of Q is $\bigcup_{i=1}^{\infty} (\{-1, 1\}_i \times \Pi_{j \neq i} [-1, 1]_j)$.

Proposition 3.4. *Let X be a compact space with involution τ. Let S be a nonempty invariant G_δ-subset of X such that $S \cap \operatorname{Fix}(\tau) = \emptyset$. Then there is an equivariant map $f : (X, \tau) \to (Q, \sigma_Q)$ having the following properties:*

(1) $f^{-1}(\{\underline{0}\}) = \operatorname{Fix}(\tau)$,
(2) $f^{-1}(B(Q)) = X \setminus (S \cup \operatorname{Fix}(\tau))$,

(3) *$f\restriction S : S \to f(S)$ is a homeomorphism.*

Proof. Clearly, $Y = X \setminus (S \cup \mathrm{Fix}(\tau))$ is σ-compact and τ-invariant. Since τ is fixed-point free on Y, we can write Y as $\bigcup_{n=1}^{\infty} C_n$, where each C_n is compact and $C_n \cap \tau(C_n) = \emptyset$.

For every $n \in \mathbb{N}$ and $x \in X \setminus (C_n \cup \tau(C_n) \cup \mathrm{Fix}(\tau))$, let U_x^n be an open neighborhood of x in X such that $\overline{U_x^n} \cap \overline{\tau(U_x^n)} = \emptyset$,

$$\overline{U_x^n} \cup \overline{\tau(U_x^n)} \subseteq X \setminus (C_n \cup \tau(C_n) \cup \mathrm{Fix}(\tau))$$

and $\mathrm{diam}(U_x^n) < \frac{1}{n}$. For every n, there is a countable subset X_n of $X \setminus (C_n \cup \tau(C_n) \cup \mathrm{Fix}(\tau))$ such that

$$\bigcup\{U_x^n : x \in X_n\} = X \setminus (C_n \cup \tau(C_n) \cup \mathrm{Fix}(\tau)).$$

Let $\{A_n : n \geq 0\}$ be a partition of $\mathbb{N}$ into infinite sets. Moreover, let $\pi_0 : A_0 \to \mathbb{N}$ be a surjection, and for $n \geq 1$, let $\pi_n : A_n \to \{n\} \times X_n$ be a surjection.

For every $n \in A_0$, let $f_n : X \to [-1, 1]$ be a continuous map having the following properties: $f_n^{-1}(\{1\}) = C_{\pi_0(n)}$, $f_n^{-1}(\{-1\}) = \tau(C_{\pi_0(n)})$ and $f_n(\mathrm{Fix}(\tau)) = 0$. As in the proof of Lemma 3.1, define $\psi_n : X \to [-1, 1]$ by $\psi_n(z) = \frac{f_n(z)-f_n(\tau(z))}{2}$. Then ψ_n is equivariant, $\psi_n^{-1}(\{1\}) = C_{\pi_0(n)}$, and $\psi_n^{-1}(\{-1\}) = \tau(C_{\pi_0(n)})$. To prove that $\psi_n^{-1}(\{1\}) = C_{\pi_0(n)}$, first note that $\psi_n(C_{\pi_0(n)}) = 1$. If $p \in X$ is such that $\psi_n(p) = 1$, then $f_n(p) - f_n(\tau(p)) = 2$ and so $f_n(p) = 1$. It follows similarly that $\psi_n^{-1}(\{-1\}) = \tau(C_{\pi_0(n)})$.

Pick an arbitrary $t \in (0, 1]$. For $n \in \mathbb{N}$ and $x \in X_n$, let $f_x^n : X \to [-t, t]$ be a continuous map such that $(f_x^n)^{-1}(\{t\}) = \overline{U_x^n}$, $(f_x^n)^{-1}(\{-t\}) = \overline{\tau(U_x^n)}$ and $f_x^n(\mathrm{Fix}(\tau)) = 0$. Define $\varphi_x^n : X \to [-t, t]$ by $\varphi_x^n(z) = \frac{f_x^n(z)-f_x^n(\tau(z))}{2}$. Then φ_x^n is equivariant, and, as above, $(\varphi_x^n)^{-1}(\{t\}) = \overline{U_x^n}$ and $(\varphi_x^n)^{-1}(\{-t\}) = \overline{\tau(U_x^n)}$.

Define $f : (X, \tau) \to (Q, \sigma_Q)$ in the standard way by

$$f(z)_m = \begin{cases} \psi_m(z) & (m \in A_0), \\ \varphi_x^n(z) & (m \in A_n \text{ with } n \geq 1, \text{ and } \pi_n(m) = (n, x)). \end{cases}$$

Then f is equivariant and we claim that it satisfies (1) through (3).

For (1), first observe that $f(\mathrm{Fix}(\tau)) = \underline{0}$. Next, assume that $z \in X \setminus \mathrm{Fix}(\tau)$. First assume that $z \in C_n$ for certain $n \in \mathbb{N}$. Pick $m \in A_0$ such that $\pi_0(m) = n$. By construction, $f(z)_m = \psi_m(z) = 1$ since $z \in C_{\pi_0(m)}$. Next assume that $z \in S$. There exists $x \in X_1$ such that $z \in U_x^1$. But then for $m \in A_1$ with $\pi_1(m) = (1, x)$ we have $f(z)_m = \varphi_x^1(z) = t > 0$. Hence, indeed, $f^{-1}(\{\underline{0}\}) = \mathrm{Fix}(\tau)$.

For (2), pick an arbitrary z in some C_n. There is $m \in A_0$ such that $\pi_0(m) = n$. Then $f(z)_m = \psi_m(z) = 1$, hence $f(z) \in B(Q)$. Conversely, take an arbitrary $z \in X \setminus \bigcup_{n=1}^{\infty} C_n$. If $m \in \mathbb{N} \setminus A_0$, then $f(z)_m = \varphi_x^n(z)$ for some n and x and hence by construction, $|f(z)_m| \leq t < 1$. Let $m \in A_0$. Now $\psi_m^{-1}(\{1\}) = C_{\pi_0(m)}$, and $\psi_m^{-1}(\{-1\}) = \tau(C_{\pi_0(m)})$ and since $z \notin C_{\pi_0(m)} \cup \tau(C_{\pi_0(m)})$, we get $f(z)_m \in (-1, 1)$. Hence $f(z) \in s$. We conclude that, indeed, $f^{-1}(B(Q)) = X \setminus (S \cup \mathrm{Fix}(\tau))$.

For (3), we will first show that $g = f\restriction S : S \to f(S)$ is one-to-one. To this end, pick arbitrary distinct $z_0, z_1 \in S$. Pick n so large that $\frac{2}{n} < d(z_0, z_1)$. There exists $x \in X_n$ such that $z_0 \in U_x^n$. Since $\mathrm{diam}(U_x^n) < \frac{1}{n}$, $z_1 \notin \overline{U_x^n}$. If $z_1 \in \overline{\tau(U_x^n)}$, then $t = \varphi_x^n(z_0) \neq \varphi_x^n(z_1) = -t$. Moreover, if $z_1 \notin \overline{\tau(U_x^n)}$, then $z_1 \notin \overline{U_x^n} \cup \overline{\tau(U_x^n)}$, and hence by the above, $t = \varphi_x^n(z_0) \neq \varphi_x^n(z_1) < t$. Hence if $m \in A_n$ is such that $\pi_n(m) = (n, x)$, then $f(z_0)_m \neq f(z_1)_m$.

To prove that g is a homeomorphism, it suffices to prove that for every $x \in S$ we have that the set $f^{-1}(\{f(x)\})$ equals $\{x\}$. Assume that $z \in X$ and $x \in S$ are distinct while $f(z) = f(x)$. By we just proved $z \notin S$, and there are two cases. Assume first that $z \in \mathrm{Fix}(\tau)$. But then $x \in \mathrm{Fix}(\tau)$ by (1) which contradicts our assumptions. Assume next that $z \notin \mathrm{Fix}(\tau)$. By (2) it consequently follows that $f(z) \in B(Q)$ and hence $f(x) \in B(Q)$. But this is impossible since $f(S) \subseteq s$, again by (2). $\square$

4. Universality of (s, σ_s)

In this section we will characterize for which spaces and based-free involutions, (s, σ_s) is universal.

Theorem 4.1. *Let X be a space with based-free involution τ. The following statements are equivalent:*

(1) *The unique fixed-point $\star$ of τ has arbitrarily small neighborhoods U in X such that $X \setminus U$ can be covered by a finite collection of closed subsets $\mathscr{F}$ of X such that for every $F \in \mathscr{F}$, $\tau(F) \cap F = \emptyset$.*
(2) *(X, τ) can be equivariantly embedded in (s, σ_s).*

Moreover, if X is Polish, then the embedding in (2) can be chosen to be a closed embedding.

Proof. Since (Q, σ_Q) is based-free, (2) $\Rightarrow$ (1) follows by a trivial compactness argument. For (1) $\Rightarrow$ (2), assume that $\star$ is the unique fixed-point of τ. We may assume by Corollary 2.2, that X is Polish (to prove all there remains simultaneously). Let $(bX, \bar{\tau})$ be the compactification of X with based-free involution $\bar{\tau}$ that we get from Theorem 2.1. By Proposition 3.4, there is an equivariant map $f : bX \to Q$ such that $f^{-1}(\{\underline{0}\}) = \mathrm{Fix}(\bar{\tau}) = \{\star\}$ and $f^{-1}(B(Q)) = bX \setminus ((X \setminus \{\star\}) \cup \{\star\}) = bX \setminus X$. Hence $f{\restriction}X$ is a closed equivariant embedding of X into (s, σ_s). $\square$

It is a natural question whether the condition (1) in this result is always satisfied. But it is not, in [12, p. 7331] it was shown that any equivariant compactification of $(\ell^2, \sigma_{\ell^2})$ has fixed-points in its remainder. Hence, as was noted in [12], $(\ell^2, \sigma_{\ell^2})$ can not be equivariantly embedded in (s, σ_s). In particular, it follows that the involutions on ℓ^2 and s are not conjugate, despite the fact that by Anderson [1], ℓ^2 and s *are* homeomorphic. In other words, as the referee of [12] noted, there does not exist an odd homeomorphism $\ell^2 \to s$, that is, a homeomorphism $\alpha : \ell^2 \to s$ such that for every $x \in \ell^2$, $\alpha(-x) = -\alpha(x)$. Recently, the unpublished report of Wong [13] from 1972 came to our attention.[1] It turns out that the question of whether there is an odd homeomorphism $\ell^2 \to s$ is due to him [13, Question 4]. It was answered in [12] by the authors of the present note, unaware of the fact that it was asked as early as in 1972! Our result also answers Question 1 in [13] in the negative. There are also some universality results in [13], but they are of a different nature than ours.

Just for fun, we will now show by a direct argument that essentially can be found in [12], that $(\ell^2, \sigma_{\ell^2})$ does not satisfy condition (1) in Theorem 4.1. Indeed, consider the open ball $U = \{x \in \ell^2 : \|x\| < \frac{1}{2}\}$. Assume that $\mathscr{F}$ is a finite collection of closed subsets of ℓ^2 covering $\ell^2 \setminus U$ such that for every $F \in \mathscr{F}$, $F \cap \sigma_{\ell^2}(F) = \emptyset$. Then the sphere $S = \{x \in \ell^2 : \|x\| = 1\}$ is covered by $\mathscr{F}$. Pick m so large that $|\mathscr{F}| - 2 < m$. The set

$$S_m = \{(x_1, \ldots, x_{m+1}, 0, 0, 0, \cdots) : \sum_{i=1}^{m+1} x_i^2 = 1\}$$

is a copy of the m-sphere in S and $\mathscr{G} = \{F \cap S_m : F \in \mathscr{F}\}$ is a closed cover of S_m, no pair of points contains an antipodal pair. But this contradicts the Lusternik-Schnirelmann Theorem, [7, Chapter 16, Corollary 6.2(3)].

5. A 'model' for $(\ell^2, \sigma_{\ell^2})$

As was shown in the previous section, (s, σ_s) and $(\ell^2, \sigma_{\ell^2})$ are very different. It looks simpler to study (s, σ_s) than $(\ell^2, \sigma_{\ell^2})$ since it is so nicely placed in its equivariant compactification (Q, σ_Q). In this section,

[1] This report can be downloaded from the website of the Centre of Mathematics and Computer Science in Amsterdam: https://ir.cwi.nl/pub/7452.

we use the characterization theorem 2.7 from [12] to show that $(\ell^2, \sigma_{\ell^2})$ can be placed as nicely in Q as (s, σ_s). As we saw in the previous section, the involution on Q for this must be different from σ_Q. But a slightly adapted one does work, as we will show here. Our 'model' for $(\ell^2, \sigma_{\ell^2})$ will be used in the next section to show that $(\ell^2, \sigma_{\ell^2})$ is universal for all spaces with a based-free involution.

Put $\hat{Q} = Q \times \mathbb{I}$ and let the involution $\sigma_{\hat{Q}}$ on $\hat{Q}$ be defined by $\sigma_{\hat{Q}} = \sigma_Q \times 1_{\mathbb{I}}$. Here $1_{\mathbb{I}}$ denotes the identity on $\mathbb{I} = [0, 1]$. Let

$$M = ((s \setminus \{\underline{0}\}) \times \mathbb{I}) \cup \{(\underline{0}, 0)\}.$$

Lemma 5.1. *$M \approx \ell^2$.*

Proof. Its complement is the capset $(B(Q) \cup \{\underline{0}\}) \times \mathbb{I}) \setminus \{(\underline{0}, 0)\}$ from which the result easily follows. For details, see [10, §6.5 and Exercises 1 and 2 therein]. $\square$

It is trivial that $\sigma_{\hat{Q}}(M) = M$. Let τ denote the restriction of $\sigma_{\hat{Q}}$ to M.

Theorem 5.2. *$(\ell^2, \sigma_{\ell^2})$ and (M, τ) are topologically conjugate.*

We will use the characterization theorem 2.7 in [12]: a based-free involution of a space E homeomorphic to ℓ^2 is topologically conjugate to $(\ell^2, \sigma_{\ell^2})$ if and only if the fixed point $\star$ has a basis $V_1 \supseteq \overline{V}_2 \supseteq V_2 \supseteq \overline{V}_3 \supseteq \cdots V_n \supseteq \overline{V}_{n+1} \supseteq \cdots$ of invariant (open) neighborhoods such that for infinitely many n, V_n is contractible and for infinitely many n, $E \setminus \overline{V}_n$ is contractible.

Proof. Let U be a 'small' open contractible σ_Q-invariant neighborhood of $\underline{0}$ in Q. Then $V = U \times [0, t)$ is for suitable $t > 0$ a 'small' contractible open $\sigma_{\hat{Q}}$-invariant neighborhood of $(\underline{0}, 0)$ in $\hat{Q}$. Clearly, $W = V \cap M$ is contractible since $V \setminus M$ is a σ-Z-set in V and hence homotopically negligible, [10, 7.2.9]. Moreover, W is τ-invariant. Now consider $\tilde{V} = \overline{U} \times [0, t]$ and assume that $t < 1$. A moment's reflection shows that the closure of W in M equals $\tilde{V} \cap M$. The set $M \setminus \tilde{V}$ can be deformed onto $(s \setminus \{\underline{0}\}) \times \{1\}$, which is contractible (use for example that $\{\underline{0}\}$ is a Z-set in s). Hence $M \setminus \tilde{V}$ is contractible and we are done. $\square$

So we conclude that $(\ell^2, \sigma_{\ell^2})$ can be as conveniently placed in $(\hat{Q}, \sigma_{\hat{Q}})$ as (s, σ_s) in (Q, σ_Q). The pair $(\hat{Q}, \sigma_{\hat{Q}})$ is an equivariant compactification of $(\ell^2, \sigma_{\ell^2})$. This implies among other things that the involution σ_{ℓ^2} seen as a homeomorphism on s (via any homeomorphism between ℓ^2 and s) is conjugate to a homeomorphism that extends to a homeomorphism on Q. Not all homeomorphisms on s share this property, see [9].

6. Universality of $(\ell^2, \sigma_{\ell^2})$

We will prove here the announced universality properties of $(\ell^2, \sigma_{\ell^2})$ by using the 'model' (M, τ) described in §5.

Theorem 6.1. *Let X be a space with based-free involution τ. Then (X, τ) can be equivariantly embedded in $(\ell^2, \sigma_{\ell^2})$. Moreover, if X is Polish, then the embedding can be chosen to be a closed embedding.*

By Corollary 2.2 it suffices to prove that for Polish X we can find a closed equivariant embedding.

So let X be a Polish space with based-free involution σ and denote the unique fixed-point of σ by $\star$. By Theorem 2.1(1), there is a compactification bX of X such that σ can be extended to an involution $\bar{\sigma} : bX \to bX$. By abuse of notation, we will denote $\bar{\sigma}$ also by σ. Observe that $\mathrm{Fix}(\sigma) \cap X = \{\star\}$.

Let ϱ be any admissible metric on bX bounded by $\frac{1}{2}$. Then $d(x,y) = \varrho(x,y) + \varrho(\sigma(x),\sigma(y))$ is admissible as well, and σ is an isometry with respect to d. Moreover, d is bounded by 1. This is the metric on bX that we will use in the sequel.

Put $S = X \setminus \{\star\}$. By Proposition 3.4, there is an equivariant map $f : (bX,\sigma) \to (Q,\sigma_Q)$ such that $f^{-1}(\{\underline{0}\}) = \mathrm{Fix}(\sigma)$, $f^{-1}(B(Q)) = bX \setminus (S \cup \mathrm{Fix}(\sigma))$, and $f{\restriction}S : S \to f(S)$ is a homeomorphism. Observe that

$$f(bX) \cap s = f(S) \cup \{\underline{0}\} = f(X),$$

and $g = f{\restriction}X$ is one-to-one and continuous. But it is not necessarily a homeomorphism. For if $\mathrm{Fix}(\sigma)$ contains more than one point, then there is a sequence $(x_n)_n$ in X which converges to an element $p \in \mathrm{Fix}(\sigma) \setminus \{\star\}$ (simply use that X is dense in bX). Hence the sequence $(x_n)_n$ is closed and discrete in X, but $(f(x_n))_n$ converges to $\underline{0}$ in $f(X)$. Hence g is in that case not a homeomorphism.

Observe that $f(X)$ is closed in s and σ_s-invariant. Define $\varphi : X \to M$ by $\varphi(x) = (f(x), d(x,\star))$. Clearly, φ is continuous and one-to-one. We will check that φ is the required embedding.

Assume that $\varphi(x_n) \to (p,q)$ in M for some sequence $(x_n)_n$ in X. We will show that for some $x \in X$ we have $\varphi(x) = (p,q)$ and $x_n \to x$. This will show simultaneously that $\varphi(X)$ is closed in M and $\varphi : X \to \varphi(X)$ is a homeomorphism.

Indeed, since $f(x_n) \to p$, and $f(X)$ is closed in s, there exists $x \in X$ such that $p = f(x)$. There are two cases. If $x \in S = X \setminus \{\star\}$, then $x_n \to x$ since $f{\restriction}S : S \to f(S)$ is a homeomorphism. And then, $d(x_n,\star) \to d(x,\star)$ and so $(p,q) = \lim_{n\to\infty}(f(x_n), d(x_n,\star) = (f(x), d(x,\star)) = \varphi(x)$. Assume next that $x = \star$ and so $p = \underline{0}$. If $x_n \to \star$, then $f(x_n) \to \underline{0}$, $d(x_n,\star) \to 0$ and so $(p,q) = (\underline{0},0) = \varphi(x)$. So it remains to check what happens if $x_n \not\to \star$. Hence we may assume without loss of generality that for some $\varepsilon > 0$ we have that $d(x_n,\star) \geq \varepsilon$ for every n. Now $\varphi(x_n) = (f(x_n), d(x_n,\star)) \to (\underline{0},q)$ and $q \geq \varepsilon$. Hence $(p,q) \notin M$, which is a contradiction.

It remains to show that φ is equivariant. But this is easy. Indeed, if $x \in X$ is arbitrary, then since σ is an isometry with respect to d,

$$\varphi(\sigma(x)) = (f(\sigma(x)), d(\sigma(x),\star)) = (\sigma_Q(f(x)), d(x,\star)) = \tau(\varphi(x)),$$

as required.

Remark 6.2. In an earlier version of this note, we proved Theorem 6.1 by a different method. It was based on the negligibility results in Anderson, Henderson and West [2]. The proof presented here, which uses the argument of the referee that we mentioned earlier, is, in our opinion easier, more direct and transparent. For that reason we decided to present the present proof instead of the old one. The results in §§4 and 5 of the present note are new compared to the results in the earlier version.

References

[1] R.D. Anderson, Hilbert space is homeomorphic to the countable infinite product of lines, Bull. Am. Math. Soc. 72 (1966) 515–519.
[2] R.D. Anderson, D.W. Henderson, J.E. West, Negligible subsets of infinite-dimensional manifolds, Compos. Math. 21 (1969) 143–150.
[3] S.A. Antonyan, On based-free compact Lie group actions on the Hilbert cube, Mat. Zametki 65 (2) (1999) 163–174 (in Russian); English transl.: Math. Notes 65 (1–2) (1999) 135–143.
[4] I. Berstein, J.E. West, Based free compact Lie group actions on Hilbert cubes, in: Proc. Sympos. Pure Math., vol. XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 373–391.
[5] C. Bessaga, A. Pełczyński, Selected Topics in Infinite-Dimensional Topology, Monografie Matematyczne, vol. 58, PWN—Polish Scientific Publishers, Warsaw, 1975.
[6] T.A. Chapman, Lectures on Hilbert cube manifolds, in: Expository Lectures from the CBMS Regional Conference Held at Guilford College, October 11–15, 1975, in: Regional Conference Series in Mathematics, vol. 28, American Mathematical Society, Providence, R. I., 1976.

[7] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.
[8] R. Engelking, Sur la compactification des espaces métriques, Fundam. Math. 48 (1960) 321–324.
[9] J. van Mill, A homeomorphism on s not conjugate to an extendable homeomorphism, Proc. Am. Math. Soc. 105 (1989) 250–253.
[10] J. van Mill, Infinite-Dimensional Topology: Prerequisites and Introduction, North-Holland Publishing Co., Amsterdam, 1989.
[11] J. van Mill, The Infinite-Dimensional Topology of Function Spaces, North-Holland Publishing Co., Amsterdam, 2001.
[12] J. van Mill, J.E. West, Involutions of ℓ^2 and s with unique fixed points, Trans. Am. Math. Soc. 373 (2020) 7327–7346.
[13] R.Y.T. Wong, Involutions on the Hilbert spheres and related properties in (I−D) spaces, in: Rapport Mathematisch Centrum (Amsterdam) ZN, vol. 50, 1972, pp. 1–22.
[14] R.Y.T. Wong, Periodic actions on the Hilbert cube, Fundam. Math. 83 (1974) 203–210.