PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 9, Pages 71–74 (March 23, 2022)
https://doi.org/10.1090/bproc/106

UNIVERSAL AUTOHOMEOMORPHISMS OF $\mathbb{N}^*$

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(Communicated by Vera Fischer)

To the memory of Cor Baayen, who taught us many things

Abstract. We study the existence of universal autohomeomorphisms of $\mathbb{N}^*$. We prove that the Continuum Hypothesis (CH) implies there is such an autohomeomorphism and show that there are none in any model where all autohomeomorphisms of $\mathbb{N}^*$ are trivial.

Introduction

This paper is concerned with universal autohomeomorphisms on $\mathbb{N}^*$, the Čech-Stone remainder of $\mathbb{N}$.

In very general terms we say that an autohomeomorphism h on a space X is *universal* for a class of pairs (Y, g), where Y is a space and g is an autohomeomorphism of Y, if for every such pair there is an embedding $e : Y \to X$ such that $h \circ e = e \circ g$, that is, h extends the copy of g on $e[Y]$.

In [1, Section 3.4] one finds a general way of finding universal autohomeomorphisms. If X is homeomorphic to X^ω then the shift mapping $\sigma : X^\mathbb{Z} \to X^\mathbb{Z}$ defines a universal autohomeomorphism for the class of all pairs (Y, g), where Y is a subspace of X. One embeds Y into $X^\mathbb{Z}$ by mapping each $y \in Y$ to the sequence $\langle g^n(y) : n \in \mathbb{Z}\rangle$.

Thus, the Hilbert cube carries an autohomeomorphism that is universal for all autohomeomorphisms of separable metrizable spaces and the Cantor set carries one for all autohomeomorphisms of zero-dimensional separable metrizable spaces. Likewise the Tychonoff cube $[0, 1]^\kappa$ carries an autohomeomorphism that is universal for all autohomeomorphisms of completely regular spaces of weight at most κ, and the Cantor cube 2^κ has a universal autohomeomorphism for all zero-dimensional such spaces.

Our goal is to have an autohomeomorphism h on $\mathbb{N}^*$ that is universal for all autohomeomorphisms of all *closed* subspaces of $\mathbb{N}^*$. The first result of this paper is that there is no trivial universal autohomeomorphism of $\mathbb{N}^*$, and hence no universal autohomeomorphism at all in any model where all autohomeomorphisms of $\mathbb{N}^*$ are trivial. On the other hand, the Continuum Hypothesis implies that there is a universal autohomeomorphism of $\mathbb{N}^*$. The proof of this will have to be different from the results mentioned above because $\mathbb{N}^*$ is definitely not homeomorphic to its power $(\mathbb{N}^*)^\omega$; it will use group actions and a homeomorphism extension theorem.

Received by the editors July 27, 2021, and, in revised form, September 6, 2021.
2020 *Mathematics Subject Classification.* Primary 54D40; Secondary 03E50 54A35.
Key words and phrases. Autohomeomorphism, $\mathbb{N}^*$, universality.

We should mention the dual notion of universality where one requires the existence of a surjection $s : X \to Y$ such that $g \circ s = s \circ h$. For the space $\mathbb{N}^*$ this was investigated thoroughly in [2] for general group actions.

1. Some preliminaries

Our notation is standard. For background information on $\mathbb{N}^*$ we refer to [5].

We denote by Aut the autohomeomorphism group of $\mathbb{N}^*$. We call a member h of Aut *trivial* if there are cofinite subsets A and B of $\mathbb{N}$ and a bijection $b : A \to B$ such that h is the restriction of βb to $\mathbb{N}^*$.

In both sections we shall use the G_δ-topology on a given space (X, τ); this is the topology τ_δ on X generated by the family of all G_δ-subsets in the given space. It is well-known that $w(X, \tau_\delta) \leqslant w(X, \tau)^{\aleph_0}$; we shall need this estimate in Section 3.

2. What if all autohomeomorphisms are trivial?

To begin we observe that fixed-point sets of trivial autohomeomorphism of $\mathbb{N}^*$ are clopen. Therefore, to show that no trivial autohomeomorphism is universal it would suffice to construct a compact space that can be embedded into $\mathbb{N}^*$ and that has an autohomeomorphism whose fixed-point set is not clopen.

The example. We let L be the ordinal $\omega_1 + 1$ endowed with its G_δ-topology. Thus all points other than ω_1 are isolated and the neighbourhoods of ω_1 are exactly the co-countable sets that contain it. Then L is a P-space of weight $\aleph_1$ and hence, by the methods in [4, Section 2], its Čech-Stone compactification βL can be embedded into $\mathbb{N}^*$.

We define $f : L \to L$ such that ω_1 is the only fixed point of βf. We put

$$f(\omega_1) = \omega_1,$$
$$f(2 \cdot \alpha) = 2 \cdot \alpha + 1, \text{ and}$$
$$f(2 \cdot \alpha + 1) = 2 \cdot \alpha.$$

This defines a continuous involution on L.

If $p \in \beta L \setminus L$ then $p \in \operatorname{cl} \alpha$ for some $\alpha < \omega_1$ and then either $E = \{2 \cdot \beta : \beta < \alpha\}$ or $O = \{2 \cdot \beta + 1 : \beta < \alpha\}$ belongs to the ultrafilter p. But $f[E] \cap E = \emptyset = f[O] \cap O$, hence $\beta f(p) \neq p$.

Since ω_1 is not an isolated point of βL, no matter how this space is embedded into $\mathbb{N}^*$ there is no trivial autohomeomorphism of $\mathbb{N}^*$ that would extend βf.

3. The Continuum Hypothesis

Under the Continuum Hypothesis the space $\mathbb{N}^*$ is generally very well-behaved and one would expect it to have a universal autohomeomorphism as well. We shall prove that this is indeed the case. We need some well-known facts about closed subspaces of $\mathbb{N}^*$.

First we have Theorem 1.4.4 from [5] which characterizes the closed subspaces of $\mathbb{N}^*$ under CH: they are the compact zero-dimensional F-spaces of weight $\mathfrak{c}$, and, in addition: every closed subset of $\mathbb{N}^*$ can be re-embedded as a nowhere dense closed P-set.

Second we have the homeomorphism extension theorem from [3]: CH implies that every homeomorphism between nowhere dense closed P-sets of $\mathbb{N}^*$ can be extended to an autohomeomorphism of $\mathbb{N}^*$.

Step 1. We consider the natural action of Aut on $\mathbb{N}^*$; the map $\sigma : \mathsf{Aut} \times \mathbb{N}^* \to \mathbb{N}^*$ given by $\sigma(f,p) = f(p)$. This action is continuous when Aut carries the compact-open topology τ and hence also when Aut carries the G_δ-modification τ_δ of τ. For the rest of the construction we consider the topology τ_δ.

Using this action we define an autohomeomorphism $h : \mathsf{Aut} \times \mathbb{N}^* \to \mathsf{Aut} \times \mathbb{N}^*$ by $h(f,p) = (f, f(p))$. The map h is continuous because its two coordinates are and it is a homeomorphism because its inverse $(f,p) \mapsto (f, f^{-1}(p))$ is continuous as well.

Now if X is a closed subset of $\mathbb{N}^*$ and $g : X \to X$ is an autohomeomorphism then we can re-embed X as a nowhere dense closed P-set and we can then find an $f \in \mathsf{Aut}$ such that $f \restriction X = g$. We transfer this embedded copy of X to $\{f\} \times \mathbb{N}^*$ in $\mathsf{Aut} \times \mathbb{N}^*$; for this copy of X we then have $h \restriction X = g$. It follows that h satisfies the universality condition.

Step 2. We embed $\mathsf{Aut} \times \mathbb{N}^*$ into $\mathbb{N}^*$ in such a way that there is an autohomeomorphism H of $\mathbb{N}^*$ such that $H \restriction (\mathsf{Aut} \times \mathbb{N}^*) = h$. Then H is the desired universal autohomeomorphism of $\mathbb{N}^*$.

To this end we list a few properties of this product.

Weight. The weight of the product is equal to $\mathfrak{c}$, as both factors have weight $\mathfrak{c}$. For $\mathbb{N}^*$ this is clear and for Aut this follows because the topology τ has weight $\mathfrak{c}$ and one obtains a base for τ_δ by taking the intersections of all countable subfamilies of a base for τ.

Zero-dimensional and F. The product is a zero-dimensional F-space as the product of the P-space Aut and the compact zero-dimensional F-space $\mathbb{N}^*$, see [6, Theorem 6.1].

Strongly zero-dimensional. The product $\mathsf{Aut} \times \mathbb{N}^*$ is not compact, but we shall construct a compactification of it that is also a zero-dimensional F-space of weight $\mathfrak{c}$.

For this we need to prove that $\mathsf{Aut} \times \mathbb{N}^*$ is actually strongly zero-dimensional. We prove more: the product is ultraparacompact, meaning that every open cover has a *pairwise disjoint* open refinement.

Let $\mathcal{U}$ be an open cover of the product consisting of basic clopen rectangles.

For each $f \in \mathsf{Aut}$ there is a finite subfamily $\mathcal{U}_f$ of $\mathcal{U}$ that covers $\{f\} \times \mathbb{N}^*$, say $\mathcal{U}_f = \{C_i \times D_i : i < k_f\}$. Let $C_f = \bigcap_{i<k} C_i$ and $D_{f,i} = D_i \setminus \bigcup_{j<i} D_j$ for $i < k_f$. Then $\mathcal{C}_f = \{C_f \times D_{f,i} : i < k_f\}$ is a disjoint family of clopen rectangles that covers $\{f\} \times \mathbb{N}^*$ and refines $\mathcal{U}$.

Because Aut has weight $\mathfrak{c}$, and we assume CH, there is a sequence $\langle f_\alpha : \alpha \in \omega_1 \rangle$ in Aut such that $\{C_{f_\alpha} : \alpha \in \omega_1\}$ covers Aut. Next we let $V_\alpha = C_{f_\alpha} \setminus \bigcup_{\beta<\alpha} C_{f_\beta}$ for all α. Because Aut is a P-space the family $\{V_\alpha : \alpha \in \omega_1\}$ is a disjoint open cover of Aut.

The family $\{V_\alpha \times D_{f_\alpha,i} : i < k_{f_\alpha}, \alpha \in \omega_1\}$ then is a disjoint open refinement of $\mathcal{U}$.

A compactification. To complete Step 2 we construct a compactification of $\mathsf{Aut} \times \mathbb{N}^*$ that is a zero-dimensional F-space of weight $\mathfrak{c}$ and that has an autohomeomorphism that extends h. The Čech-Stone compactification would be the obvious canditate, were it not for the fact that its weight is equal to $2^\mathfrak{c}$. More precisely, using some continuous onto function from (Aut, τ) onto $[0,1]$ one obtains a clopen partition of $(\mathsf{Aut}, \tau_\delta)$ of cardinality $\mathfrak{c}$. This shows that $\beta(\mathsf{Aut} \times \mathbb{N}^*)$ admits a continuous surjection onto the space $\beta\mathfrak{c}$ (where $\mathfrak{c}$ carries the discrete topology).

To create the desired compactification we build, either by transfinite recursion or by an application of the Löwenheim-Skolem theorem, a subalgebra $\mathbb{B}$ of the algebra of clopen subsets of $\mathsf{Aut} \times \mathbb{N}^*$ that is closed under h and h^{-1}, of cardinality $\mathfrak{c}$, and that has the property that for every pair of countable subsets A and B of $\mathbb{B}$ such that $a \cap b = \emptyset$ whenever $a \in A$ and $b \in B$ there is a $c \in \mathbb{B}$ such that $a \subseteq c$ and $c \cap b = \emptyset$ for all $a \in A$ and $b \in B$. The latter condition can be fulfilled because $\mathsf{Aut} \times \mathbb{N}^*$ is an F-space — $\bigcup A$ and $\bigcup B$ have disjoint closures — and strongly zero-dimensional — the closures can be separated using a clopen set.

The Stone space $\mathrm{St}(\mathbb{B})$ of $\mathbb{B}$ is then a compactification of $\mathsf{Aut} \times \mathbb{N}^*$ that is a compact zero-dimensional F-space of weight $\mathfrak{c}$, with an autohomeomorphism $\bar{h}$ that extends h. We embed $\mathrm{St}(\mathbb{B})$ into $\mathbb{N}^*$ as a nowhere dense P-set and extend $\bar{h}$ to an autohomeomorphism H of $\mathbb{N}^*$.

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