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DENSE TOPOLOGICAL GROUPS IN PAROVIČENKO SPACES

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Abstract. We show that the statement 'the Čech-Stone remainder of the discrete space ω contains a dense subspace which is (homeomorphic to) a topological group' is not a statement of ZFC. We also discuss the question of whether this result can be extended to Parovičenko spaces.

1. Introduction

All topological spaces under discussion are assumed to be Tychonoff. A space X is called a *Parovičenko space* if

(P1) X is a zero-dimensional compact space without isolated points with weight $\mathfrak{c}$,
(P2) every two disjoint open F_σ-subsets have disjoint closures, and
(P3) every nonempty G_δ in X has nonempty interior.

Moreover, X is called an F-space, [14], if each cozero-set in X is C^*-embedded in X. A normal space is an F-space if and only if X satisfies (P2), [21, 1.1.2(b)]. And, a space that satisfies (P3) is usually called an *almost P-space.*

Parovičenko [22] showed that under the Continuum Hypothesis (abbreviated: CH), every Parovičenko space is homeomorhic to ω^*, the Čech-Stone-remainder $\beta\omega \setminus \omega$ of the countable discrete space ω. In fact, CH is equivalent to the statement that every Parovičenko space is homeomorphic to ω^*, [9].

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The space ω^* is widely studied in set theory and set theoretic topology. In recent years, there was a lot of interest in the algebraic structure of $\beta\omega$ and ω^*. The space ω^* is not a topological group, for example because each compact topological group satisfies the countable chain condition while ω^* has cellularity $\mathfrak{c}$. But it *is* a compact right topological semigroup. For details, see e.g. [16]. The question of whether it is homogeneous was first answered in the negative under CH in [23], and later in ZFC in [13] (see also [20]).

If X is a compact subspace of ω^* then it is (homeomorphic to) a topological group if and only if it is finite. Such an X would be a closed subspace of ω^* satisfying the countable chain condition and hence it would be extremally disconnected (abbreviated: ED), [21, 1.2.2], and hence finite by [12]. There are many subspaces of ω^* that *are* topological groups, even uncountable ones. There exists a pairwise disjoint family consisting of nonempty open subsets of ω^* of cardinality $\mathfrak{c}$. Consequently, ω^* contains a discrete topological group of cardinality $\mathfrak{c}$.

These considerations prompt an interesting problem: *does there exist an infinite crowded subspace of ω^* which is (homeomorphic to) a topological group?* Here a space is called *crowded* if it contains no isolated points. We will show that the affirmative answer to this question follows from (unpublished) work of van Douwen. He proved that if X is a P-space of weight κ, then X can be embedded in the projective cover $E(2^\kappa)$ of 2^κ, and hence in $\beta(2^\kappa)$, see [10, §4]. This prompts the question of whether there exists a *dense* subspace of ω^* which is a topological group. Under CH, this question has an affirmative answer. In [6, 3.5] it was shown that (under CH), the subspace of ω^* consisting of all the P-points of ω^*, is homeomorphic to the space $X = 2^{\omega_1}$ with the G_δ-topology, which is a topological group. We will show that the statement 'ω^* *contains a dense topological group*' which follows from CH, is not a statement of ZFC. In fact, we will show that the existence of a dense topological group in ω^* implies the existence of a P-point in ω^*. Hence our conclusion follows from the famous Shelah P-point Independence Theorem, [25].

2. The main result

If X is a space, then X_δ denotes X with the topology generated by the G_δ-subsets of X. Our basic space of interest here is $G = (2^{\omega_1})_\delta$, where 2^{ω_1} is the Cantor cube of weight ω_1. This space was studied earlier, see e.g., [17] and [7]. It is clear that G is a P-space as well as a (Boolean) topological group.

We will first discuss the question how the subspaces of ω^* look like that are (homeomorphic to) topological groups. As we pointed out in §1, ω^* contains a discrete subspace of size $\mathfrak{c}$, which is obviously a topological group. So the question becomes interesting (and nontrivial) for the case of crowded subspaces.

Now consider the group G from above. It has weight $\mathfrak{c}$ $(= 2^\omega)$ and hence can be embedded in $2^\mathfrak{c}$. By van Douwen's result quoted in §1, G embeds in the projective cover (absolute) of $2^\mathfrak{c}$ which in turn embeds in $\beta\omega$ by [11] (for details, see Remark 2 in [10]). Hence, G embeds in ω^*.

We now turn to dense subspaces.

If x is a P-point of X, then we call x *nice* if the character of x in X equals the π-character of x in X.

The following lemma is well-known, its proof is included for the sake of completeness.

Lemma 2.1. *Let G be a topological group. Then for every neighborhood U of the neutral element e of G there is a G_δ-subgroup S of G that is contained in U.*

Proof. Let U_0 be a symmetric open neighborhood of e that is contained in U. For every $n < \omega$, let U_{n+1} be a symmetric open neighborhood of e such that $U_{n+1}^2 \subseteq U_n$. Then $S = \bigcap_{n<\omega} U_n$ is clearly as required. □

The following lemma may also be known, but we do not know a reference.

Lemma 2.2. *Any dense subspace of an almost P-space is an almost P-space.*

Proof. Let X be almost P, and let Y be dense in Y. If S is a G_δ-subset of Y containing the point $y \in Y$, then there is a G_δ-subset T of X such that $S = T \cap Y$. By assumption, S contains a nonempty open subset U. Hence S contains the nonempty open (in Y) subset $U \cap Y$. Observe that we used that Y is dense to ensure that $U \cap Y \neq \emptyset$. □

Theorem 2.3. *Let Y be a dense subspace of an almost P-space X. If Y is a topological group, then every point of Y is a nice P-point of X.*

Proof. For every n, let U_n be an open neighborhood in Y of the neutral element e of Y. By Lemma 2.1, there is a G_δ-subgroup S of Y such that $e \in S \subseteq \bigcap_{n<\omega} U_n$. Lemma 2.2 gives us that S has nonempty interior in Y. But this means that S is open being a subgroup of Y. Hence e is a P-point of Y which has the additional property that its character and its π-character in Y agree (since Y is a topological group). Since Y is dense in X, it consequently follows that e is a nice P-point in X. □

If τ is an infinite cardinal and X is a space, then A is a G_τ subset of X if A is the intersection of at most τ open subsets of X. If every nonempty G_τ subset of X has nonempty interior, then we say that X is an *almost P_τ-space.*

Suppose now that the almost P_τ-space X contains a dense in X topological group G. Then the neutral element e of G is a P_τ-point of X. The proof is practically the same as proof of Theorem 2.3 and so we leave it to the reader.

So being a P-space is a necessary condition for a topological group to have a compactification which is an almost P-space. It is natural to ask whether this is also sufficient. It is not, for example because no compactification of a countable infinite discrete group is an almost P-space. On the other hand, the Alexandroff one-point compactification of an uncountable discrete group *is* an almost P-space. Thus, we are left with the following question: is it true that every non-discrete topological group, which is a P-space, is homeomorphic to a dense subspace of some compact almost P-space? Let us show that the answer is in the negative. First observe that no noncompact Lindelöf space has a compactification which is an almost P-space. Hence no non-discrete Lindelöf topological group which is a P-space has a compactification which is an almost P-space. That such a group exists is well-known. An example is the free topological group of the one-point Lindelöffication of an uncountable discrete space. For details, see e.g. [1]. We will show in Theorem 3.1 below that the topological group G from above does have a compactification which is an almost P-space in ZFC. In §1 we observed that under CH, the space ω^* is such a compactification.

We conclude from Theorem 2.3 that if there is a dense subspace of ω^* which is a topological group, then there exist nice P-points in ω^*. But more can be concluded.

Corollary 2.4. *If a finite product of copies of ω^* has a dense subspace which is a topological group, then ω^* contains a nice P-point.*

Proof. An arbitrary finite product of almost P-spaces is again almost P. Hence, if a finite product X of copies of ω^* contains a dense subspace which is a topological group, then X contains a nice P-point which in turn implies that ω^* contains a nice P-point. □

It is convenient to introduce the following statement, where $\kappa \geq 1$ is a cardinal number:

$(*)_\kappa$ *The product of κ copies of ω^* contains a dense subspace which is a topological group.*

Under CH, any product of copies of ω^* contains a dense subspace which is a topological group since ω^* does. Hence CH implies $(*)_\kappa$ for all $\kappa \geq 1$. But for finite κ, $(*)_\kappa$ is not a theorem of ZFC by the famous Shelah P-point Independence Theorem, [25], and Corollary 2.4.

No infinite product of nontrivial spaces is almost P. Hence the argument that was used in the proof of Corollary 2.4, does not apply for infinite products of copies of ω^*. This prompts the following question.

Question 2.5. Is it consistent that $(*)_\kappa$ fails for some cardinal $\kappa \geq \omega$?

By the result in [6, 3.5] mentioned above we have that CH must fail in any model where the answer to this question is affirmative.

We finish this section by outlining an approach that could be useful for answering this. In the Bell-Kunen model from [4], $\mathfrak{c} = \aleph_{\omega_1} > \omega_1$ and every point in ω^* has π-character ω_1. We do not know whether in this model, ω^* contains a dense subspace which is a topological group. If such a subspace exists, it consists by Theorem 2.3 of P-points of character ω_1. It is known that such points in ω^* can exist, [19, VIII, Al0] (see also [15]).

Let $(**)$ denote the following statement:

$(**)$ *All points in ω^* have character $\mathfrak{c}$ but π-character less than $\mathfrak{c}$.*

Observe that it is known that ω^* contains points of character $\mathfrak{c}$, [18]. But there are models in which not all points in ω^* have character $\mathfrak{c}$, [19, VIII, Al0] (see also [15]). Clearly, $(**)$ fails under CH. This prompts the following question.

Question 2.6. Is it consistent that $(**)$ holds?

As usual, if κ is a cardinal number, then $\mathrm{cf}(\kappa)$ denotes its cofinality. In set theoretic language, our question boils down to asking whether it is consistent that $\mathfrak{d} \leq \mathfrak{r} \leq \mathrm{cf}(\mathfrak{c}) < \mathfrak{u} = \mathfrak{c}$ and every ultrafilter having π-character $< \mathfrak{c}$. This reformulation uses the result of Aubrey [3] stating that $\mathfrak{r} = \mathfrak{u}$ if $\mathfrak{r} < \mathfrak{d}$, as well as the fact that there always exists an ultrafilter of π-character $\mathrm{cf}(\mathfrak{c})$, [4]. (We are indebted to the referee for this observation.)

We think this problem is of independent interest. If a model such as in Question 2.6 exists, it would give by Theorem 2.3 another reason why it cannot be shown that ω^* contains a dense topological group in ZFC. And maybe this model will shed light upon Question 2.5, as the following result shows.

Theorem 2.7. *Under $(**)$, $(*)_\kappa$ fails for every κ such that $1 \leq \kappa < \mathrm{cf}(\mathfrak{c})$.*

Proof. Let $X = \prod_{i \in I} X_i$, where $1 \leq |I| < \mathrm{cf}(\mathfrak{c})$ and for every $i \in I$, $X_i \approx \omega^*$. Striving for a contradiction, assume that X contains a dense subspace G which is a topological group. Since G is dense in X, and in a topological group the character of each point equals its π-character, X contains a point x whose character and π-character agree. We will show that this is impossible. The character of x is obviously $\mathfrak{c}$. Moreover, by assumption, for each $i \in I$, the π-character κ_i of x_i is less than $\mathfrak{c}$. Since $1 \leq |I| < \mathrm{cf}(\mathfrak{c})$, this means that $\kappa = \sup_{i \in I} \kappa_i < \mathfrak{c}$. A moment's reflection shows that the π-character of x is at most κ. But since $\kappa < \mathfrak{c}$, this is a contradiction. $\square$

Since $\mathrm{cf}(\mathfrak{c}) > \omega$, we get

Corollary 2.8. *Under* $(**)$, $(*)_\omega$ *fails.*

As we noted above, in [4] it was shown that there exists a point in ω^* with π-character at least $\mathrm{cf}(\mathfrak{c})$. It consequently is a natural question of whether every Parovičenko space contains a point of π-character at least $\mathrm{cf}(\mathfrak{c})$. We were only able to answer this question for the important class of Parovičenko spaces consisting of all the Čech-Stone-remainders X^* of noncompact, zero-dimensional, locally compact spaces X of weight at most $\mathfrak{c}$ (see e.g. [24, Proposition 3.37] and [21, Theorem 1.2.5]).

Theorem 2.9. *Let X be a locally compact, noncompact (and zero-dimensional) space (of weight at most $\mathfrak{c}$). Then (the Parovičenko space) X^* contains a point with π-character at least* $\mathrm{cf}(\mathfrak{c})$.

Proof. By [8, Lemma 1.1(A)], there are an open subspace U of X^*, a copy H of ω^* in U, and an open retraction $r\colon U \to H$. From this, and the result from [4] just quoted, we obviously get what we want. □

Question 2.10. Does every Parovičenko space contain a point with π-character at least $\mathrm{cf}(\mathfrak{c})$?

3. The topological group G

If $\alpha < \omega_1$ and $f \in 2^\alpha$, then $[f]$ denotes the set $\{g \in 2^{\omega_1} : g\restriction\alpha = f\}$. We let S denote the set of all successor ordinals in ω_1.

Theorem 3.1. *The space G has a compactification bG which is an almost P-space. Moreover, bG is zero-dimensional and has weight $\mathfrak{c}$.*

Proof. First note that $X = 2^{\omega_1}$ is homeomorphic to $Y = K^{\omega_1}$, where $K = 2^\omega$. Hence X_δ and Y_δ are homeomorphic spaces. We prove the theorem for $G = Y_\delta$.

Let $\mathscr{B}$ denote the standard clopen base for Y_δ. That is, $\mathscr{B} = \{[f] : f \in K^\alpha, \alpha < \omega_1\}$. Consider the following subcollection of $\mathscr{B}$:

$$\mathscr{C} = \{[f] : f \in K^\alpha, \alpha \in S\}.$$

It is clear that $\mathscr{C}$ is a clopen base for Y_δ as well.

Now let $\mathscr{D}$ denote the Boolean algebra generated by $\mathscr{C}$. We claim that its Stone space is the compactification bG of G that we are looking for. Let A and B be countable subsets of S. For every $\alpha \in A$ and $\beta \in B$, pick countable $\mathscr{F}_\alpha \subseteq K^\alpha$ and $\mathscr{G}_\beta \subseteq K^\beta$ such that

$$(\dagger) \qquad \mathscr{P} = \{[f] : f \in \mathscr{F}_\alpha, \alpha \in A\} \cup \{G \setminus [g] : g \in \mathscr{G}_\beta, \beta \in B\}$$

has the finite intersection property. We will show that $\mathscr{P}$ has nonempty intersection. At the end of the proof we will verify that this suffices.

We start with some preliminary remarks. If $f, f' \in \mathscr{F}_\alpha$ are distinct for some $\alpha \in A$, then $[f] \cap [f'] = \emptyset$, which is absurd. Hence if $\mathscr{F}_\alpha \neq \emptyset$ then it consists of a single element, say f_α. Hence we are in fact dealing with a collection of the form:

$$(\ddagger) \qquad \mathscr{P} = \{[f_\alpha] : \alpha \in A\} \cup \{G \setminus [g] : g \in \mathscr{G}_\beta, \beta \in B\}.$$

Assume first that $A = \emptyset$. Pick a point $x \in K$ which is not in the range of any g, where $g \in \mathscr{G}_\beta, \beta \in B$. Then the function $g\colon \omega_1 \to K$ with constant values x is in $\bigcap_{g \in \mathscr{G}_\beta, \beta \in B} G \setminus [g]$. Next assume that $B = \emptyset$. In this case there is nothing to prove, since any function in K^{ω_1} that extends $\bigcup_{\alpha \in A} f_\alpha$ is in the intersection of the system $\mathscr{P}$.

Hence we may assume without loss of generality that both A and B are nonempty.

Let $\gamma = \sup A$, and pick $f_\gamma \in K^\gamma$ such that $[f_\gamma] = \bigcap_{\alpha \in A} [f_\alpha]$. We will show that some extension of f_γ belongs to $\bigcap \mathscr{P}$.

Pick $\beta \in B$ and an arbitrary $g \in \mathscr{G}_\beta$.

Claim 1. For every $\gamma_0 \in A$, f_{γ_0} is not an extension of g.

Indeed, if for some $\gamma_0 \in A$, f_{γ_0} extends g, then $[f_{\gamma_0}] \subseteq [g]$ and so $[f_{\gamma_0}] \cap (G \setminus [g]) = \emptyset$. But this contradicts the fact that any two members of $\mathscr{P}$ meet.

Assume first that $\beta \leq \gamma$. If $\beta = \gamma$, then γ is a successor since β is and hence $\gamma \in A$. Therefore, $f_\gamma \neq g$ by Claim 1. If $\beta < \gamma$ and if f_γ is an extension of g, then for some $\beta < \gamma_0 \leq \gamma$, where $\gamma_0 \in A$, f_{γ_0} is an extension of g. This is impossible by Claim 1. In either case, f_γ is not an extension of g so that $[f_\gamma] \subseteq G \setminus [g]$, which means that we can ignore β.

We are left with the case that $\gamma < \beta$. Pick $\delta_\beta < \omega_1$ such that $\delta_\beta + 1 = \beta$. Then $\gamma \leq \delta_\beta$, hence f_γ is undefined at the point δ_β, in contrast to g. Let x_β be any point in K which does not belong to the (countable) set $\{g(\delta_\beta) : g \in \mathscr{G}_\beta\}$.

Define $\varphi\colon \gamma \cup \{\delta_\beta : (\beta \in B) \,\&\, (\gamma < \beta)\} \to K$ as follows:

$$\varphi(\alpha) = \begin{cases} f_\gamma(\alpha) & (\alpha < \gamma), \\ x_\beta & (\beta \in B, \gamma < \beta, \alpha = \delta_\beta), \end{cases}$$

and let $\psi\colon \omega_1 \to K$ be any extension of φ. Then ψ is in the intersection of ($\ddagger$).

Now we check that what we proved suffices. Indeed, let $\mathscr{T}$ be a countable collection of open subsets of bG such that $T = \bigcap \mathscr{T}$ be nonempty, and pick $t \in T$. There is a countable subcollection $\mathscr{W}$ of $\mathscr{D}$ such that

$$t \in \bigcap_{W \in \mathscr{W}} \overline{W} \subseteq T.$$

Fix a member $W \in \mathscr{W}$. It has the form $\bigcup_{i\in F}\bigcap_{j\in G} W_{ij}$, where for every $i \in F$ and $j \in G$, W_{ij} is of the form $[f]$ for certain $f \in K^\alpha$, $\alpha \in S$, or $G \setminus [g]$ for certain $g \in K^\beta$, $\beta \in S$. Collect all such $[f]$'s that have the property that $t \in \overline{[f]}$ and all such $G \setminus [g]$'s that have the property that $t \in \overline{G \setminus [g]}$. Together they form a system $\mathscr{P}$ such as in (†), and for that $\mathscr{P}$ we have

$$t \in \bigcap_{P\in\mathscr{P}} \overline{P} \subseteq T.$$

We showed that $\bigcap \mathscr{P}$ is nonempty and hence clopen in G since G is a P-space. But this clearly implies that T has nonempty interior in bG. □

Theorem 3.2. *The space G has a compactification bG which is an F-space. Moreover, bG is zero-dimensional and has weight $\mathfrak{c}$.*

Proof. As we argued in §2, there is an embedding $i\colon G \to \omega^*$. Since every closed subspace of a compact F-space is an F-space, [21, Lemma 1.2.2], the closure of $i(G)$ is as required. □

Under CH, ω^* is a Parovičenko compactification of G (hence this compactification satisfies the conclusions of both Theorems 3.1 and 3.2). We do not know whether CH is essential in this result.

Question 3.3. Does there exist a Parovičenko compactification of G in ZFC? A Parovičenko space containing a dense subspace which is a topological group?

4. Possible generalizations

In the last decade there was a lot of focus on natural generalizations of the concept of a topological group. These include coset spaces, paratopological and semitopological groups, etc. For details, see e.g. [1]. In the light of our results, these generalizations leave many interesting problems open. We will not discuss them here, we will only concentrate on coset spaces.

Recall that a *coset space* is a space homeomorphic to a space of the form G/H, where G is a topological group and H is a closed subgroup of it. The class of coset spaces is much larger than the class of topological groups and includes for example all locally compact homogeneous spaces that are separable and metrizable. For details, see e.g. [2].

It is natural to ask whether the existence of a dense subspace of ω^* that is a coset space also implies the existence of a P-point. But it does not, as a simple counterexample demonstrates. Indeed, for every permutation $\pi\colon \omega \to \omega$, let $\beta\pi\colon \beta\omega \to \beta\omega$ denote its Čech extension. For a point $p \in \omega^*$ let

$$\tau(p) = \{\beta\pi(p) : \pi \in S_\omega\}$$

denote its *type*, [13]. It is not difficult to see that for every $p \in \omega^*$, $\tau(p)$ is a dense subspace of ω^*. It is also homogeneous. Hence it is a coset space, being zero-dimensional and homogeneous, [5].

Question 4.1. Suppose that κ is a cardinal number such that $\kappa \geq \mathfrak{c}$. Is it true in ZFC that G^κ can be topologically embedded in the product of κ copies of ω^* as a dense subspace?

Remark 4.2 (Added on March 3, 2020). After this paper was written, we noted that a result identical to our Theorem 2.3 was proved independently in the recent preprint *k-Markov and k-tactic for NONEMPTY in the Choquet Game* by S. Önal and S. Soyarslan.

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