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SOME REALCOMPACT SPACES

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Abstract. We present examples of realcompact spaces with closed subsets that are C^*-embedded but not C-embedded, including one where the closed set is a copy of $\mathbb{N}$.

Introduction

The purpose of this note is to provide some examples of realcompact (but not compact) spaces that have closed subspaces that are C^*-embedded but not C-embedded, and, in particular, an example where the closed subspace is a copy of the discrete space $\mathbb{N}$ of natural numbers—what we henceforth call *a closed copy of* $\mathbb{N}$.

The reason for our interest is that we are not aware of any such examples. For instance, the examples in [5] of C^*- but not C-embedded subsets are not all closed and, when they are closed, the pseudocompactness of the ambient space makes C-embedding impossible.

The only explicit question of this nature that we could find is in [7, Question 1], which asks whether C^*-embedded subsets (not necessarily closed) of first-countable spaces are C-embedded. In that case, there is an independence result: There is a counterexample if $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$, and, in the model obtained by adding supercompact many random reals, the implication holds; see [1].

The more specific question of having a closed copy of $\mathbb{N}$, that is, C^*-embedded but not C-embedded, arises from an analysis of their position in powers of the real line; see section 2 for an explanation.

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It is clear that our examples should be non-normal Tychonoff spaces. After some preliminaries, we briefly discuss two classical examples, the Tychonoff and Dieudonné planks, and introduce a further plank $\mathbb{J}$.

The latter is pseudocompact, but we modify it in two steps. The first step yields a plank that is neither pseudocompact nor realcompact, and the second step gives us our first example.

Our second example is constructed in section 5 and it contains a closed copy of $\mathbb{N}$ that is C^*- but not C-embedded.

1. Preliminaries

We follow [4] and [5] as regards general topology and rings of continuous functions. As is common, $C(X)$ and $C^*(X)$ denote the rings of real-valued continuous and bounded continuous functions, respectively.

A subset A of a space X is *C-embedded* if every continuous function $f : A \to \mathbb{R}$ admits a continuous extension $\bar{f} : X \to \mathbb{R}$. It is *$C^*$-embedded* if every bounded continuous function $f : A \to \mathbb{R}$ admits a bounded continuous extension $\bar{f} : X \to \mathbb{R}$.

We define a space X to be *realcompact* if it can be embedded into a power of the real line as a closed subset. The most useful characterization for this paper is that every zero-set ultrafilter with the countable intersection property has a non-empty intersection; see [4, Theorem 3.11.11].

1.1. Planks.

As noted above, our examples will be non-normal Tychonoff spaces—non-normal because we need a closed subset that is not C-embedded and Tychonoff because that is part of the definition of realcompactness.

There are various examples of such spaces, such as the Tychonoff plank $\mathbb{T}$ ([9] or [8, Example 87]), and the Dieudonné plank $\mathbb{D}$ ([2] or [8, Example 89]). Both start with the product set $X = (\omega_1 + 1) \times (\omega_0 + 1)$ and take the subset $P = X \setminus \{\langle \omega_1, \omega_0 \rangle\}$ as the underlying set of the space. In each case, P has the subspace topology where X has a product topology induced by topologies on the factors. For $\mathbb{T}$, one takes the order topologies on both ordinals. For $\mathbb{D}$, one enlarges the order topology of $\omega_1 + 1$ by making all points of ω_1 isolated.

We shall consider a third variation in section 3 below.

2. Context

We begin with the following proposition, which may be well-known but bears repeating here because it shows that if one has a non-C-embedded copy of $\mathbb{N}$ in a realcompact space, then that copy contains many infinite subsets that *are* C-embedded.

Proposition 2.1. *Let X be realcompact and A a subset whose closure is not compact, then A contains a countably infinite subset that is closed, discrete, and C-embedded in X.*

Proof. Take a point x_0 in $\beta X \setminus X$ that is in the closure of A. Apply [4, Theorem 3.11.10] to find a continuous function $f : \beta X \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ if $x \in X$. Because x_0 is in the closure of A, we can find a sequence $\langle a_n : n \in \mathbb{N}\rangle$ in A such that $\langle f(a_n) : n \in \mathbb{N}\rangle$ is strictly decreasing with limit 0.

The set $N = \{a_n : n \in \mathbb{N}\}$ is closed and C-embedded in X. It is closed as a locally finite set of points. If $g : N \to \mathbb{R}$ is given, then we can take a continuous function $h : (0, 1] \to \mathbb{R}$ such that $h(f(a_n)) = g(a_n)$ for all n. Then $h \circ f$ is a continuous extension of g. □

The space in section 5 illustrates Proposition 2.1 quite well. One can point out very many infinite C-embedded subsets of the non-C-embedded copy of $\mathbb{N}$ explicitly. This proposition also shows why the initial planks in section 3 are not realcompact: there are not enough C-embedded copies of $\mathbb{N}$.

2.1. Closed copies of $\mathbb{N}$ in other spaces.

Here, we collect a few natural questions that arise when one considers C^*- and C-embedding of closed copies of $\mathbb{N}$.

Suppose one has two closed copies, $\mathbb{N}_1$ and $\mathbb{N}_2$ say, of the space of natural numbers in a Tychonoff space X.

(1) If $\mathbb{N}_1$ and $\mathbb{N}_2$ are C-embedded, is their union C-embedded?
(2) If $\mathbb{N}_1$ and $\mathbb{N}_2$ are C^*-embedded, is their union C^*-embedded?
(3) If $\mathbb{N}_1$ is C-embedded and $\mathbb{N}_2$ is C^*-embedded, is their union C^*-embedded?

Questions (1) and (3) have positive answers.

For question (3), one uses a continuous extension $f : X \to \mathbb{R}$ of a bijection between $\mathbb{N}_1$ and $\mathbb{N}$ to obtain a discrete family $\{O_x : x \in \mathbb{N}_1\}$ of open sets with $x \in O_x$ for all $x \in \mathbb{N}_1$. Then, given a bounded function $g : \mathbb{N}_1 \cup \mathbb{N}_2 \to \mathbb{R}$, one first takes a bounded extension $\bar{g} : X \to \mathbb{R}$ of $g \restriction \mathbb{N}_2$ and then modifies $\bar{g}$ on each O_x to obtain an extension of g.

The argument for question (1) is similar but easier because one can find a single discrete family of open sets that separates the points of $\mathbb{N}_1 \cup \mathbb{N}_2$.

A counterexample to question (2) can be obtained by taking M. Katětov's example [6, p. 88] of a pseudocompact space with a closed C^*-embedded copy of $\mathbb{N}$ or see [4, Example 3.10.29]. The example is $\mathbb{K} = \beta\mathbb{R} \setminus \mathbb{N}^*$, and the copy of $\mathbb{N}$ is just $\mathbb{N}$ itself. Take the sum of two copies of this space, $\mathbb{K} \times \{0, 1\}$, and for every $x \in \mathbb{K} \setminus \mathbb{R}$, identify the points $\langle x, 0\rangle$

and $\langle x, 1\rangle$. The copies $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ are both C^*-embedded in the resulting quotient, but their union is not.

Below, we shall show that question (2) also has a negative answer in the class of realcompact spaces.

2.2. Closed copies of $\mathbb{N}$ in powers of $\mathbb{R}$.

The discrete space $\mathbb{N}$ is realcompact; hence, it admits many embeddings into powers of $\mathbb{R}$ as a closed and C-embedded set.

The specific question from the introduction is equivalent to the question whether there is a closed copy of $\mathbb{N}$ in some power of $\mathbb{R}$ that is C^*-embedded but not C-embedded. Indeed, the latter is a special case of the former, and a positive answer to the former answers the latter by embedding the example as a closed C-embedded copy into some power of $\mathbb{R}$; the copy of $\mathbb{N}$ is then not C-embedded in that power.

The difference between C- and C^*-embedding manifests itself also in the way certain maps can be factored through partial products.

Assume first that $\mathbb{N}$ is C-embedded in a power of $\mathbb{R}$, say $\mathbb{R}^\kappa$. Then there is a continuous function $f : \mathbb{R}^\kappa \to \mathbb{R}$ such that $f(n) = n$ for all $n \in \mathbb{N}$. It is well-known that f factors through a countable subset of κ: There are a countable subset I of κ and a continuous function $g : \mathbb{R}^I \to \mathbb{R}$ such that $f = g \circ \pi$ where π is the projection onto $\mathbb{R}^I$; see [4, Problem 2.7.12]. Then the projection $\pi[\mathbb{N}]$ of $\mathbb{N}$ in $\mathbb{R}^I$ is C-embedded and we see that every function from $\mathbb{N}$ to $\mathbb{R}$ has an extension that factors through the partial power $\mathbb{R}^I$.

Now assume $\mathbb{N}$ is C^*-embedded but not C-embedded in $\mathbb{R}^\kappa$. Then every bounded function from $\mathbb{N}$ to $[0, 1]$ has a continuous extension to $\mathbb{R}^\kappa$. Such a continuous extension will then factor through a partial product with countably many factors, but the set of factors will vary with the function.

Indeed, assume that there is a single countable set I such that every bounded function $f : \mathbb{N} \to [0, 1]$ has a continuous extension that factors through $\mathbb{R}^I$. Apply this to the function defined by $f(n) = 2^{-n}$ and take a function $g : \mathbb{R}^I \to \mathbb{R}$ such that $\bar{f} = g \circ \pi$ is a continuous extension, where π is the projection onto $\mathbb{R}^I$. Then π is injective on $\mathbb{N}$ and $\pi[\mathbb{N}]$ is relatively discrete in $\mathbb{R}^I$.

We also find that $\pi[\mathbb{N}]$ is C^*-embedded in the metric space $\mathbb{R}^I$ and, hence, closed. But then $\pi[\mathbb{N}]$ is C-embedded in $\mathbb{R}^I$ and $\mathbb{N}$ is C-embedded in $\mathbb{R}^\kappa$.

Using the plank $\mathbb{A}$ from section 5, we obtain such a copy of $\mathbb{N}$ in a power of $\mathbb{R}$. The standard embedding of $\mathbb{A}$ in the power $\mathbb{R}^{C(\mathbb{A})}$ yields a closed C-embedded copy of $\mathbb{A}$. The right-hand side R is a closed copy of $\mathbb{N}$ that is C^*-embedded in $\mathbb{A}$ and, hence, in $\mathbb{R}^{C(\mathbb{A})}$, but not C-embedded in $\mathbb{R}^{C(\mathbb{A})}$. This then suggests the following question.

Question 2.2. What is the minimum cardinal κ such that $\mathbb{R}^\kappa$ contains a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded?

Since $\mathbb{R}^{\omega_0}$ is metrizable and, as we shall see, $\left|C(\mathbb{A})\right| = \mathfrak{c}$, we know that $\aleph_0 < \kappa \leq \mathfrak{c}$. This means that the continuum hypothesis settles this question, but there may be some variation under other assumptions.

Our answer to question (2) on page 207 produces, in the same way, a closed copy of $\mathbb{N}$ in $\mathbb{R}^\mathfrak{c}$ that is not C^*-embedded. After we submitted this paper, we were able to answer the analogue of Question 2.2: The smallest cardinal κ such that $\mathbb{R}^\kappa$ contains a closed copy of $\mathbb{N}$ that is not C^*-embedded is $\aleph_1$. See [3] for a surprising (to us) variety of closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that are not C^*-embedded.

3. The Plank $\mathbb{J}$ and a Variation

In our third variation of the idea of the plank, the topology on $\omega_0 + 1$ remains as it is and we let, from now on, $\omega_1 + 1$ carry the topology of the one-point compactification of the discrete space ω_1, with ω_1 the point at infinity.

In this case, we denote the resulting space by $\mathbb{J}$. It is a minor variation of [4, Example 2.3.36]; in the terminology of that book, $\mathbb{J} = A(\aleph_1) \times A(\aleph_0) \setminus \{\langle x_0, y_0\rangle\}$, where we have specified the underlying sets of the factors explicitly.

As in the case of $\mathbb{T}$ and $\mathbb{D}$, the top line $T = \omega_1 \times \{\omega_0\}$ and the right-hand side $R = \{\omega_1\} \times \omega_0$ cannot be separated by open sets in $\mathbb{J}$. Hence, their union is not C^*-embedded in the space $\mathbb{J}$.

A more careful analysis of the continuous functions on $\mathbb{J}$ will reveal that neither T nor R is C^*-embedded.

Indeed, let $f : \mathbb{J} \to \mathbb{R}$ be continuous. For each $n \in \omega_0$, the set $\{\alpha \in \omega_1 : f(\alpha, n) \neq f(\omega_1, n)\}$ is countable. It follows that there is an α in ω_1 such that $f(\beta, n) = f(\omega_1, n)$ for all n and all $\beta \geq \alpha$. By continuity, this implies that $f(\beta, \omega_0) = f(\alpha, \omega_0)$ for all $\beta \geq \alpha$. This shows that the function $\langle \alpha, \omega_0\rangle \mapsto \alpha \bmod 2$ (the characteristic function of the odd ordinals), which is continuous on T, has no continuous extension to $\mathbb{J}$.

If we let $r = f(\alpha, \omega_0)$, then it follows that $\lim_{n\to\infty} f(\omega_1, n) = r$. We see that the function $\langle \omega_1, n\rangle \mapsto n \bmod 2$, which is continuous on R, has no continuous extension to $\mathbb{J}$ either.

This argument also shows that $\mathbb{J}$ is not realcompact; the co-countable sets on the top line form a zero-set ultrafilter with the countable intersection property that has an empty intersection. Alternatively, use Proposition 2.1: No infinite subset of R is C^*-embedded.

The space $\mathbb{J}$ is not pseudocompact either; the diagonal $\{\langle n, n\rangle : n \in \omega_0\}$ is a clopen discrete subset.

3.1. Ensuring C^*-embeddedness.

To ensure that R is C^*-embedded, we change the second factor in our product.

We let $X = (\omega_1 + 1) \times \beta\omega_0$ and $P = X \setminus (\{\omega_1\} \times \omega_0^*)$. The right-hand side R remains unchanged, but the top line T now becomes $\omega_1 \times \omega_0^*$.

To see why this makes the right-hand side C^*-embedded, let $f : R \to [0,1]$ be continuous. Take the unique continuous extension of $n \mapsto f(\omega_1, n)$ to $\beta\omega_0$ and it on every vertical line $\{\alpha\} \times \beta\omega_0$ to get an extension of f to the plank P.

This does not make the right-hand side C-embedded; the analysis of the continuous functions on $\mathbb{J}$ shows that, for any extendable function f, the function $n \mapsto f(\omega_1, n)$ should be extendable from ω_0 to $\beta\omega_0$ and, hence, should be bounded.

When we adapt the analysis of continuous functions on $\mathbb{J}$ to continuous functions on P, we obtain that the intersection of a zero-set with the top line T contains a set of the form $A(\alpha, Z) = [\alpha, \omega_1) \times Z$, where $\alpha \in \omega_1$ and Z is a zero-set of ω_0^* (and Z could be empty, of course).

Now take any point u in ω_0^* and let $\mathcal{Z}_u$ be the family of zero-sets of ω_0^* that contain u. Then $\{A(\alpha, Z) : \alpha \in \omega_1, Z \in \mathcal{Z}_u\}$ generates a zero-set ultrafilter with the countable intersection property that has an empty intersection. Thus, the present plank is not realcompact. Again, Proposition 2.1 applies as well: No closed copy of $\mathbb{N}$ (and there are many) in R is C-embedded.

4. The Plank $\mathbb{V}$

It should be clear that the fact that continuous functions on $\omega_1 + 1$ are constant on co-countable sets is the main cause that the two previous examples are not realcompact. To alleviate that, we replace $\omega_1 + 1$ by $\beta\omega_1$, where ω_1 still has the discrete topology. We take the product $\Pi = \beta\omega_1 \times \beta\omega_0$; our example is $\mathbb{V} = \Pi \setminus (\omega_1^* \times \omega_0^*)$. The top line and the right-hand side now become $T = \omega_1 \times \omega_0^*$ and $R = \omega_1^* \times \omega_0$.

4.1. The right-hand side R is C^*-embedded in $\mathbb{V}$.

This is proved almost as in the case of the plank P.

Let $f : R \to [0,1]$ be continuous. Apply the Tietze-Urysohn extension theorem to each horizontal line H_n to obtain a continuous extension $f_n : H_n \to [0,1]$ of the restriction of f to $\omega_1^* \times \{n\}$.

Next, for each $\alpha \in \omega_1$, take the unique extension g_α of the map $\langle \alpha, n\rangle \mapsto f_n(\alpha, n)$ to $\{\alpha\} \times \beta\omega_0$. The union of the maps g_α and f_n is an extension of f to $\mathbb{V}$.

4.2. The right-hand side R is not C-embedded in $\mathbb{V}$.

Define $f : R \to \mathbb{R}$ by $f(x, n) = n$. Assume $g : \mathbb{V} \to \mathbb{R}$ is a continuous extension of f. For each n and k, the set

$$\{\alpha \in \omega_1 : \left|g(\alpha, n) - n\right| \geq 2^{-k}\}$$

is finite; hence, for each n, the set $\{\alpha : g(\alpha, n) \neq n\}$ is countable. It follows that there are co-countably many $\alpha \in \omega_1$ such that $g(\alpha, n) = n$ for all n. For each such α, the restriction of g to the compact set $\{\alpha\} \times \beta\omega_0$ would be unbounded, which is a contradiction.

4.3. The space $\mathbb{V}$ is realcompact.

Let $\mathcal{Z}$ be a zero-set ultrafilter with the countable intersection property. We show that its intersection is nonempty.

To begin, if for some n the clopen "horizontal line" $H_n = \beta\omega_1 \times \{n\}$ belongs to $\mathcal{Z}$, then the compactness of this line implies that $\bigcap \mathcal{Z}$ is nonempty.

In the opposite case, the complements of the H_n belong to $\mathcal{Z}$; the intersection of these complements is equal to the top line T. By the countable intersection property, we find that every member of $\mathcal{Z}$ intersects T; hence, $T \in \mathcal{Z}$.

For every subset A of ω_1, the partial top line $T_A = A \times \omega_0^*$ is a zero-set as it is the intersection of T with the clopen subset $\operatorname{cl} A \times \beta\omega_0$ of Π. It follows that the family $u = \{A : T_A \in \mathcal{Z}\}$ is an ultrafilter on ω_1 that has the countable intersection property. Because ω_1 is not a measurable cardinal, it is a principal ultrafilter. Let $\alpha \in \omega_1$ be such that $u = \{A \subseteq \omega_1 : \alpha \in A\}$. Then the compact set $\{\alpha\} \times \omega_0^*$ belongs to $\mathcal{Z}$ and so $\bigcap \mathcal{Z} \neq \emptyset$.

4.4. Comments.

The natural maps from $\beta\omega_1$ onto $\omega_1 + 1$ and from $\beta\omega_0$ onto $\omega_0 + 1$-as-the-one-point-compactification are perfect and irreducible. Hence, so is the product map from Π onto $(\omega_1 + 1) \times (\omega_0 + 1)$. It follows that the restriction of this map to $\mathbb{V}$ is perfect as well because $\mathbb{V}$ is the preimage of $\mathbb{J}$.

We have seen that $\mathbb{J}$ is not realcompact, so we have here a very simple perfect map that does not preserve realcompactness.

We also note that $\mathbb{V}$ is extremally disconnected, and it is, in fact, the absolute of $\mathbb{J}$.

5. Another Plank

In this section, we construct a realcompact space with a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded.

We let D be the tree $2^{<\omega}$ with the discrete topology and we topologize $D \cup 2^\omega$ so as to obtain a natural compactification cD of D. If $x \in 2^\omega$, then its n^{th} neighbourhood $U(x,n)$ will be the "wedge" above $x \restriction n$:

$$U(x,n) = \{s \in cD : x \restriction n \subseteq s\}.$$

Let $e : \beta D \to cD$ be extension of the identity map. This yields a partition of D^* into closed sets, indexed by 2^ω; let $K_x = \{u \in D^* : e(u) = x\}$.

To construct our plank, we take a point ∞ not in 2^ω and topologize $\mathfrak{C} = 2^\omega \cup \{\infty\}$ by making every point of 2^ω isolated and letting

$$\{U : \infty \in U \wedge |\mathfrak{C} \setminus U| \leq \aleph_0\}$$

be a local base at ∞.

Let us note that $\mathfrak{C}$ has a property in common with the horizontal lines in our planks above: For every continuous function $f : \mathfrak{C} \to \mathbb{R}$, there is a neighbourhood of ∞ (a co-countable set) on which f is constant.

We let $\mathbb{A}$ be the following subspace of $\mathfrak{C} \times \beta D$:

$$\mathbb{A} = (\mathfrak{C} \times D) \cup \bigcup_{x \in 2^\omega} \{x\} \times K_x.$$

We let $R = \{\infty\} \times D$ denote the right-hand side of the plank. The top line $T = \bigcup_{x \in 2^\omega} \{x\} \times K_x$ is not as smooth as in the other examples; every point u of D^* occurs just once in the top line when $e(u) = x$.

5.1. R is C^*-embedded.

This is as in the previous examples: R is even C^*-embedded in $R \cup (2^\omega \times \beta D)$. Given $f : R \to [0,1]$, let $g : \beta D \to [0,1]$ be the Čech-Stone extension of $s \mapsto f(\infty, s)$ and then define $\bar{f} : \mathbb{A} \setminus R \to [0,1]$ by $\bar{f}(x,u) = g(u)$ (replicate g on each vertical line, but restrict it to $\{x\} \times (\omega_0 \cup K_x)$ each time). Then $f \cup \bar{f}$ is a continuous extension of f.

5.2. R is not C-embedded.

Below, we show that $\mathbb{A}$ is realcompact, so Proposition 2.1 implies that R has many infinite C-embedded subsets. Therefore, the unbounded function without continuous extension must be chosen with some care.

Define $f(\infty, s) = |s|$ (the length of s). Assume $g : \mathbb{A} \to \mathbb{R}$ is a continuous extension. As noted before, there is a neighbourhood U of ∞ such that g is constant on $U \times \{s\}$ for every $s \in D$. But, then, for every $x \in U \setminus \{\infty\}$ and $n \in \omega_0$, we have $g(x, x \restriction n) = g(\infty, x \restriction n) = f(\infty, x \restriction n) = n$. Since

$K_x = \bigcap_n \mathrm{cl}_{\beta D}\{x \restriction i : i \geq n\}$, this would imply that $g(x,u) \geq n$ for all n when $u \in K_x$.

5.3. $\mathbb{A}$ IS REALCOMPACT.

In the plank P in section 3, we used ω_0^* everywhere in the top line. Combined with the fact that continuous functions were constant on a tail on each horizontal line, this implied that P is not realcompact, mainly because unbounded (to the right) zero-sets in the top line contain sets of the form $[\alpha, \omega_1) \times Z$, where Z is a zero-set of ω_0^*. In the present example, the disjointness of the K_x will provide us with a richer supply of zero-sets; these will help ensure realcompactness of $\mathbb{A}$.

Let $\mathcal{Z}$ be a zero-set ultrafilter on $\mathbb{A}$ with the countable intersection property.

For each $s \in D$, the horizontal $\mathfrak{C} \times \{s\}$ is clopen, hence, a zero-set.

The continuous function $f : \mathbb{A} \to [0,1]$, determined by setting $f(x,s) = 2^{-|s|}$ for all $\langle x, s\rangle \in \mathfrak{C} \times D$, has the top line T as its zero-set. Thus, we obtain a partition of $\mathbb{A}$ into countably many zero-sets. It follows that one of these sets must belong to $\mathcal{Z}$.

If $\mathfrak{C} \times \{s\} \in \mathcal{Z}$, then either $\langle \infty, s\rangle \in \bigcap \mathcal{Z}$ or there is a $Z \in \mathcal{Z}$ such that $\infty \notin Z$. But then Z is discrete and countable because $\{x \in \mathfrak{C} : \langle x, s\rangle \notin Z\}$ is open in $\mathfrak{C}$ and contains ∞. Then $\mathcal{Z}$ determines a countably complete ultrafilter on Z, which is fixed because $|Z|$ is countable.

We are left with the case that $T \in \mathcal{Z}$. Here is where we use the partition $\{K_x : x \in 2^\omega\}$ of D^* to show that T may be split into zero-sets in many ways.

We show that whenever A is clopen in the Cantor set 2^ω, the union $Z(A) = \bigcup_{x \in A}\{x\} \times K_x$ is a zero-set in $\mathbb{A}$.

By compactness and zero-dimensionality of cD, we know there is a continuous function $f : cD \to \{0,1\}$ such that $f[A] = \{0\}$ and $f[2^\omega \setminus A] = \{1\}$. (We assume both A and its complement are non-empty.)

We use f to define $F : \mathbb{A} \to \{0,1\}$ by $F(x,s) = f(s)$ if $\langle x, s\rangle \in \mathfrak{C} \times D$ and $F(x,u) = f(x)$ if $u \in K_x$.

The function F is continuous on $\mathbb{A}$ and we have $Z(A) = T \cap Z_F$, so $Z(A)$ is a zero-set of $\mathbb{A}$. Using this, we build countably many pairs of complementary zero sets in T. For every $n \in \omega$, we let $A_n = \{x \in 2^\omega : x(n) = 0\}$ and $B_n = \{x \in 2^\omega : x(n) = 1\}$; these clopen sets determine the zero-sets $Z(n,0) = \bigcup_{x \in A_n}\{x\} \times K_x$ and $Z(n,1) = \bigcup_{x \in B_n}\{x\} \times K_x$, respectively.

Since $\mathcal{Z}$ is a zero-set ultrafilter and $T \in \mathcal{Z}$, we deduce that, for every n, there is an element $x(n)$ of $\{0,1\}$ such that $Z(n, x(n)) \in \mathcal{Z}$. Thus, we get an $x \in 2^\omega$ such that $\{Z(n, x(n)) : n \in \omega\}$ is a subfamily of $\mathcal{Z}$.

Its intersection is equal to $\{x\} \times K_x$ and, because $\mathcal{Z}$ has the countable intersection property, this compact set belongs to $\mathcal{Z}$, and so $\bigcap \mathcal{Z} \neq \emptyset$.

As mentioned before, Proposition 2.1 implies that R has many infinite C-embedded subsets. A lot of these can be pointed out explicitly.

For every $x \in 2^\omega$, the set $N_x = \{\langle \infty, x \restriction n\rangle : n \in \omega\}$ is C-embedded in $\mathbb{A}$. Given a function $f : N_x \to \mathbb{R}$, we extend it to R, first by setting $\bar{f}(\infty, s) = 0$ for all other s. Then we extend $\bar{f}$ horizontally: $\bar{f}(y, s) = \bar{f}(\infty, s)$ for all y and s, *except* for $y = x$; we set $\bar{f}(x, s) = 0$ for all s. Now we can set $\bar{f}(t) = 0$ for all t in the top line to get our continuous extension to all of $\mathbb{A}$.

In a similar fashion, every infinite antichain in $2^{<\omega}$ yields an infinite C-embedded subset as well.

5.4. More answers.

We can use $\mathbb{A}$ and some variations to answer some of the questions raised earlier in this paper.

5.4.1. *The smallest power of* $\mathbb{R}$. The set $\mathfrak{C} \times D$ is dense in $\mathbb{A}$, so every member of $C(\mathbb{A})$ is determined by its restriction to this set. Using the fact that continuous functions on $\mathfrak{C}$ are constant on co-countable sets, we see that there are $\mathfrak{c}$ many such restrictions. We conclude that $C(\mathbb{A})$ has cardinality $\mathfrak{c}$, as claimed in the discussion of Question 2.2.

5.4.2. *The union of two closed C^*-embedded copies of* $\mathbb{N}$. We can use $\mathbb{A}$ much like we used $\mathbb{K}$ to create a realcompact space with two closed C^*-embedded copies of $\mathbb{N}$ whose union is not C^*-embedded. Take $\mathbb{A} \times \{0, 1\}$ and identify the points $\langle t, 0\rangle$ and $\langle t, 1\rangle$ for all t in the top line T. Then $R \times \{0\}$ and $R \times \{1\}$ are still C^*-embedded in the resulting quotient space, but their union is not; mapping $\langle r, i\rangle$ to i results in a bounded function without a continuous extension. The proof that the quotient space is realcompact is almost verbatim that of the realcompactness of $\mathbb{A}$. Note that the $R \times \{0\}$ and $R \times \{1\}$ are separated (neither intersects the closure of the other), so their union is a closed copy of $\mathbb{N}$ that is not C^*-embedded. The quotient space also has $\mathfrak{c}$ many real-valued continuous functions; hence, we also obtain a closed copy of $\mathbb{N}$ in $\mathbb{R}^{\mathfrak{c}}$ that is not C^*-embedded. This copy is quite unlike the closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that are constructed in [3].

5.4.3. *Another closed copy of* $\mathbb{N}$ *that is not C^*-embedded.* If we replace βD by cD in $\mathbb{A}$, then we obtain a realcompact plank where the right-hand side is a closed copy of $\mathbb{N}$ that is not C^*-embedded.

The analogue of $\mathbb{A}$ is the following subspace of $\mathfrak{C} \times cD$:

$$(\mathfrak{C} \times D) \cup \{\langle x, x\rangle : x \in 2^\omega\}.$$

That this space is realcompact is shown exactly as for $\mathbb{A}$. However, in this space, the right-hand side R is not C^*-embedded.

Since 2^ω is homeomorphic to its own square, it is relatively easy to produce two disjoint open sets U and V in 2^ω with a dense union and whose common boundary F is homeomorphic to 2^ω itself.

Via the map $e : \beta D \to cD$, we can find a subset C of D such that $\operatorname{cl} U \subseteq \operatorname{cl} C$ and $\operatorname{cl} V \subseteq \operatorname{cl}(D \setminus C)$.

Define $f : R \to [0,1]$ by $f(\infty, s) = \chi(s)$, where χ is the characteristic function of C. As before, given a continuous extension $\bar{f}$ of f, we would have a countable set B such that $\bar{f}(x,s) = f(\infty,s)$ for all $x \in 2^\omega \setminus B$ and all $s \in D$. But then $\bar{f}$ would not be continuous at $\langle x, x\rangle$ whenever $x \in F \setminus B$.

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