FUNCTION SPACES AND POINTS IN ČECH-STONE REMAINDERS

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Abstract. Let X and Y be locally compact, σ-compact spaces and let $u \in X^*$ and $v \in Y^*$. In this paper we will show that if X_u and Y_v are l_p-equivalent then u is ω-near if and only if v is. This result does not necessarily hold for spaces that are not locally compact and σ-compact. We will also show that if in addition X and Y are metrizable and u and v are ω-near, then if X_u and Y_v are l_p-equivalent, $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic for some 'unique' $\widehat{u}, \widehat{v} \in \omega^*$ 'good' for u and v. These results allow us to find an isomorphic classification of function spaces $C_p(\alpha_u)$, where $\alpha < \omega^\omega$ is a limit ordinal and $u \in \alpha^*$. This extends a result due to Gul'ko for $\alpha = \omega$. We will also indicate that the proof for this isomorphic classification can only partly be extended for $\alpha \geq \omega^\omega$.

1. Introduction

All topological spaces in this paper are Tychonoff spaces.

For a space X we let $C(X)$ denote the set of real-valued continuous functions on X. The set $C(X)$ endowed with the topology of point-wise convergence will be denoted by $C_p(X)$. This function space is a topological vector space and is a dense subspace of $\mathbb{R}^X$. Function spaces with the topology of point-wise convergence have been widely investigated. An extensive overview of what has been achieved can be found in the works of Arhangel'skii, [2] and Tkachuk [18], [19], [20], [21]. In this paper the focus will be on linear homeomorphisms between function spaces. Following Arhangel'skii, we define spaces X and Y to be l_p-*equivalent* if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. A topological property $\mathscr{P}$ is defined to be l_p-*invariant* if for l_p-equivalent spaces X and Y we have that X has property $\mathscr{P}$ if and only if Y has property $\mathscr{P}$. For an overview of l_p-invariant properties and conditions for spaces to be l_p-equivalent we refer to [6], [17] and [21].

Isomorphic classification results in linear spaces have a long history. Miljutin [15] proved that all Banach spaces $C(X)$, where X is any compact metrizable uncountable space, are linearly homeomorphic. And Bessaga and Pelczyński [7] found an isomorphic classification of all Banach spaces $C(K)$, where K is any countable compact space. The aim of this paper is to continue this line of investigation.

For a space X, let βX be its Čech-Stone compactifaction; put $X^* = \beta X \setminus X$, the (Čech-Stone) remainder of X. For every $u \in X^*$ let X_u be the subspace $X \cup \{u\}$ of βX. In this paper we will be particularly interested in ordinal spaces $\alpha < \omega_1$ with the order topology, and for every $u \in \alpha^*$, in $\alpha_u = \alpha \cup \{u\} \subseteq \beta\alpha$.

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If $\alpha < \omega_1$ and $u, v \in \alpha^*$, then a 'trivial' sufficient condition for $C_p(\alpha_u)$ and $C_p(\alpha_v)$ to be linearly homeomorphic is that α_u and α_v are homeomorphic. In [12], it was shown by Gul'ko that this 'trivial' condition is also necessary for $\alpha = \omega$.

Theorem 1.1. *Let $u, v \in \omega^*$. Then $C_p(\omega_u)$ and $C_p(\omega_v)$ are linearly homeomorphic if and only if ω_u and ω_v are homeomorphic.*

Homeomorphy between ω_u and ω_v can clearly be translated into purely combinatorial properties of the ultrafilters u and v. Homeomorphy gives us a permutation of ω taking one ultrafilter onto the other, and vice versa.

A natural question is whether a version of Gul'ko's isomorphic classification can be proved more generally for function spaces $C_p(X_u)$. In particular for spaces $C_p(\alpha_u)$, where $\alpha < \omega_1$ and $u \in \alpha^*$. In Section 5, we will obtain such a classification for all limit ordinals $\alpha < \omega^\omega$. This isomorphic classification links to the results found in [4] for zero dimensional locally compact separable metric spaces which obviously includes all limit ordinal spaces $\alpha < \omega_1$.

Our method of proof uses the fact that all points in α^* are *near* α when $\alpha < \omega^\omega$ is a limit ordinal (we will make this precise later). But this is not true if $\alpha \geq \omega^\omega$. This suggests the following problem: if X and Y are spaces (not necessarily ordinal numbers), $u \in X^*$, $v \in Y^*$, X_u and Y_v are l_p-equivalent, then is u is 'near' X if and only if v is 'near' Y? We will demonstrate that this is true for locally compact σ-compact spaces, but in general it is not true. We will also show that if in addition X and Y are metrizable and u is 'near' X and v is 'near' Y, then if X_u and Y_v are l_p-equivalent, $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic for some unique $\widehat{u}, \widehat{v} \in \omega^*$ 'good' for u and v.

It will turn out that an isomorphic classification of function spaces $C_p(\alpha_u)$, where $\omega^\omega \leq \alpha < \omega_1$ is out of reach, let alone a general isomorphic classification of function spaces $C_p(X_u)$. We will discuss partial results for limit ordinals greater than or equal to ω^ω and we show that we run into all sorts of problems trying to generalize our results for these ordinals.

The results presented here depend partly on a generalized version of a lemma by Gul'ko in [12] (see Section 3). In addition we will make use of the Rudin-Frolík (pre-)order of ultrafilters, and a result in [13] by Hasumi on selections of upper semicontinuous functions on extremally disconnected spaces.

2. Terminology

2.1. **Generalities.** By $X \approx Y$ we mean that X and Y are homeomorphic spaces.

We use standard notion with respect to ordinal exponentiation.

The symbol $\oplus$ denotes topological sum.

A subset A of a space X is *bounded* if for every $f \in C(X)$, $f(A)$ is a bounded subset of $\mathbb{R}$. If X is normal, then by the Tietze Extension Theorem, for closed $A \subseteq X$, A is bounded if and only if A is pseudocompact if and only if A is countably compact. If X is not normal, this need not hold (Engelking [11, 3.10.29]).

For a space X and $A \subseteq X$ we put $C_{p,A}(X) = \{f \in C_p(X) : f(A) = \{0\}\}$. We define A to be C_p-*zero embedded* in X if $C_p(X)$ is linearly homeomorphic to $C_{p,A}(X) \times C_p(A)$. The following result is well-known (see V.370 in [21] and Proposition 2.3.2 in [6]).

Lemma 2.1. *Let X be a normal space and let A be closed subset of X. Then if either*

(a) A is a retract of X, or

(b) X is metrizable

then A is C_p-zero embedded and C^*-embedded in X.

2.2. **Linear spaces.** For convenience, we let $E \sim F$ denote that the linear spaces E and F are linearly homeomorphic.

Let X and Y be spaces and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. For $y \in Y$, the map $\psi_y : C_p(X) \to \mathbb{R}$ defined by $\psi_y(f) = \phi(f)(y)$ is continuous and linear. This means $\psi_y \in L(X)$, the dual space of $C_p(X)$. Since the evaluation mappings ξ_x ($x \in X$) defined by $\xi_x(f) = f(x)$ for $f \in C_p(X)$ form a Hamel basis for $L(X)$ (a proof of this statement is not trivial), there are $x_1, \ldots, x_n \in X$ and $\lambda^y_{x_1}, \ldots, \lambda^y_{x_n} \in \mathbb{R} \setminus \{0\}$ such that $\psi_y = \sum_{i=1}^n \lambda^y_{x_i} \xi_{x_i}$. This means that for every $f \in C_p(X)$, $\phi(f)(y) = \sum_{i=1}^n \lambda^y_{x_i} f(x_i)$.We define the *support* $\mathrm{supp}_\phi(y)$ of y to be $\{x_1, \ldots, x_n\}$. For $B \subseteq Y$, we denote $\bigcup_{y \in B} \mathrm{supp}_\phi(y)$ by $\mathrm{supp}_\phi(B)$ or $\mathrm{supp}(B)$. Note that if $f, g \in C_p(X)$ coincide on $\mathrm{supp}(B)$, then $\phi(f)$ and $\phi(g)$ coincide on B.

If ϕ is a linear homeomorphism, for every $y \in Y$ we define $\theta_\phi(y) = \{x \in \mathrm{supp}_\phi(y) : y \in \mathrm{supp}_{\phi^{-1}}(x)\}$. Since $y \in \mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(y))$ (again, the proof of this is not trivial) we have $\theta_\phi(y) \neq \emptyset$. Note that $x \in \theta_\phi(y)$ if and only if $y \in \theta_{\phi^{-1}}(x)$. For $B \subseteq Y$, we define $\theta_\phi(B) = \bigcup_{y \in B} \theta_\phi(y)$.

For more information on the support function and proofs of the above statements, we refer to [6] or [17, Chapter 6].

In [1], Arhangel'skii proved the following basic result:

Lemma 2.2. *Let X and Y be Tychonoff spaces and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. If $A \subseteq Y$ is bounded, then supp$(A) \subseteq X$ is bounded.*

2.3. **Čech-Stone compactifications.** As we stated in §1, if X is a space, then βX and X^* denote its Čech-Stone compactification and remainder, respectively. If $A \subseteq X$ is closed and C^*-embedded in X, then the closure $\mathrm{cl}_{\beta X} A$ of A in βX is a compactification of A which is equivalent to its Čech-Stone compactification βA (Engelking [11, 3.6.7]). This is true for example if A is clopen in X or a retract of X. Moreover, if X is normal, then by the Tietze Extention Theorem, there are no restrictions (Engelking [11, 3.6.8]). In these cases, we will always identify βA and $\mathrm{cl}_{\beta X} A$.

As we stated in §1, for every $u \in X^*$ let X_u be the subspace $X \cup \{u\}$ of βX. Assume X is a space containing a point u such that $X \setminus \{u\}$ is dense in X and C^*-embedded. Then $\beta X = \beta(X \setminus \{u\})$ by Engelking [11, 3.6.1]. Hence $(X \setminus \{u\})_u = X$. As a consequence, the topology of X_u can be characterized internally without referring to Čech-Stone compactifications.

For $f : \omega \to \omega$, let $\beta f : \beta\omega \to \beta\omega$ be the Stone extension of f. That is, for every ultrafilter $u \in \beta\omega$, $\beta f(u)$ is the ultrafilter generated by $\{f(U) : U \in u\}$. The *Rudin-Keisler (pre-)order* $\leq_{\mathrm{RK}}$ on $\beta\omega$ is defined as follows: for $u, v \in \beta\omega$, $u \leq_{\mathrm{RK}} v$ if and only if there is a function $f : \omega \to \omega$ such that $\beta f(v) = u$. The following theorem is well-known, see [8, Corollary 9.3] or [16, Theorem 3.1.1].

Theorem 2.3. *Let $u, v \in \omega^*$. Then $u \leq_{\mathrm{RK}} v$ and $v \leq_{\mathrm{RK}} u$ if and only if ω_u and ω_v are homeomorphic.*

Let X be a space and let $D = \{x_i : i \in \omega\} \subseteq X$ be countably infinite, closed, discrete and C^*-embedded in X. Then the closure $\mathrm{cl}_{\beta X} D$ of D in βX is βD which is canonically homeomorphic to $\beta\omega$. In fact, if $u \in \mathrm{cl}_{\beta X} D$, then $\{A \subseteq \omega : u \in \mathrm{cl}_{\beta X}\{x_i : i \in A\}\}$ is an ultrafilter on ω and hence a point in $\beta\omega$. We denote this point by $\tilde{u}$. Now consider $Y = X \oplus \omega$. Since ω is clopen in Y, for every $u \in \mathrm{cl}_{\beta X} D$, the point $\tilde{u} \in \beta\omega$ can be seen as a point in βY. In fact, the map $u \mapsto \tilde{u}$ is a homeomorphism $\mathrm{cl}_{\beta X} D \to \beta\omega \subseteq \beta Y$.

A point $p \in X^*$ is an *ω-far point* (of X) if for every countable closed discrete $D \subseteq X$, $p \notin \mathrm{cl}_{\beta X} D$. This notion is due to van Douwen [9] who showed that every non-compact metrizable space without isolated points has such a point. Observe that if $X = \omega$, then no point of X^* is far, hence the condition that the space has no isolated points is essential in van Douwen's result. A point $p \in X^*$ that is not ω-far is called *ω-near.*

3. On a result by Gul'ko and ω-near points

The next lemma is a generalization of a result by Gul'ko [12, Lemma 5] for $X = Y = \omega$. Gul'ko's version of this lemma was used to prove his main result Theorem 1.1 in [12].

Lemma 3.1. *Let X and Y be spaces with X normal. Let $\phi : C_p(X) \to C_p(Y)$ be a linear homeomorphism. Let $\mathscr{B}$ be a countably infinite discrete family of bounded subsets of X, and let $B = \bigcup \mathscr{B}$. Let $A \subseteq Y$ be countable such that*

(a) For every $y \in A$ we have $\theta_\phi(y) \cap B \neq \emptyset$,

(b) Every $F \subseteq A$ that is bounded in Y, is finite.

Then A is closed in Y.

Proof. Suppose that A is not closed in Y. Then there is $v \in Y$ such that $v \in \overline{A} \setminus A$.

Claim. If $\mathscr{G} \subseteq \mathscr{B}$ is finite and $G = \bigcup \mathscr{G}$ then

(i) $F = \{y \in A : \theta_\phi(y) \cap G \neq \emptyset\}$ is finite.

(ii) $\theta_\phi(A) \cap G$ is finite.

Since G is bounded we have by Lemma 2.2 that $\mathrm{supp}_{\phi^{-1}}(G)$ is a bounded subset of Y. For (i), let $y \in F$ and let $x_y \in \theta_\phi(y) \cap G$. Then $y \in \mathrm{supp}_{\phi^{-1}}(x_y) \subseteq \mathrm{supp}_{\phi^{-1}}(G)$. Hence $F \subseteq \mathrm{supp}_{\phi^{-1}}(G)$ and so F is a bounded subset of Y. Since $F \subseteq A$, we have by (b) that F is finite.

Since F is finite by (i), for (ii) it suffices to observe that $\theta_\phi(A) \cap G \subseteq \mathrm{supp}_\phi(F)$.

Since $\mathrm{supp}_\phi(v)$ is finite, there is a finite $\mathscr{G} \subseteq \mathscr{B}$ such that $\mathrm{supp}(v) \cap B \subseteq G = \bigcup \mathscr{G}$. By (i) of the Claim, $F = \{y \in A : \theta_\phi(y) \cap G \neq \emptyset\}$ is finite. Therefore we may assume that $\mathrm{supp}(v) \cap B = \emptyset$ (replace $\mathscr{B}$ by $\mathscr{B} \setminus \mathscr{G}$ and A by $A \setminus F$).

Since A is countable it follows that $\theta_\phi(A) \subseteq \mathrm{supp}_\phi(A)$ is countable. If $\theta_\phi(A) \cap B$ is finite, then there is a finite $\mathscr{G} \subseteq \mathscr{B}$ such that $\theta_\phi(A) \cap B = \theta_\phi(A) \cap G$, where $G = \bigcup \mathscr{G}$. Since A

is infinite we have a contradiction by (a) and (i) of the Claim. So $\theta_\phi(A) \cap B$ is countably infinite.

Enumerate $\theta_\phi(A) \cap B$ as $\{x_j : j < \omega\}$, where $x_i \neq x_j$ for $i \neq j$. Since $\mathscr{B}$ is a discrete family of subsets of X we have by (ii) of the Claim that $\{x_j : j < \omega\}$ is closed and discrete in X. Therefore, since X is normal, for every $j < \omega$ there exists an open neighborhood V_j of x_j such that the collection $\{V_j : j < \omega\}$ is discrete and $\mathrm{supp}_\phi(v) \cap \bigcup_{j<\omega} V_j = \emptyset$ (use e.g. the Tietze Extension Theorem).

Put $A_0 = B_0 = \emptyset$. For every $1 \leq j < \omega$ define

$$B_j = \{x_i : i \leq j\} \text{ and } A_j = \{y \in A : \mathrm{supp}_\phi(y) \cap B \subseteq B_j\}.$$

Then $B_j \subseteq B_{j+1}$ and so $A_j \subseteq A_{j+1}$, $\bigcup_{j<\omega} B_j = B$ and $\bigcup_{j<\omega} A_j = A$. Since by (a), $A_j \subseteq \{y \in A : \theta_\phi(y) \cap B_j \neq \emptyset\}$ we have by (i) of the Claim that A_j is finite.

For $j < \omega$ we will define $h_j \in C_p(X)$ such that

(1) $h_j(X \setminus V_j) \subseteq \{0\}$,
(2) $h_j(\mathrm{supp}_\phi(A_j) \setminus \{x_j\}) \subseteq \{0\}$,
(3) for $j > 0$, $\phi(h_j)(A_{j-1}) \subseteq \{0\}$,
(4) if $A_j \neq \emptyset$, then for every $y \in A_j$, $|\phi(\sum_{i\leq j} h_i)(y)| > 1$.

Put $h_0 = 0$. For some $j > 0$, assume that for $k < j$, h_k satisfies (1) through (4). If $A_j \setminus A_{j-1} = \emptyset$, let $h_j = 0$. If $A_j \setminus A_{j-1} \neq \emptyset$, then for $y \in A_j \setminus A_{j-1}$, $\mathrm{supp}_\phi(y) \cap B \subseteq B_j$ and $x_j \in \mathrm{supp}_\phi(y)$. Put $C = \mathrm{supp}_\phi(A_j) \setminus \{x_j\}$, and observe that C is finite since A_j is. Since $\lambda^y_{x_j} \neq 0$ for every $y \in A_j \setminus A_{j-1}$, there clearly exists $\alpha \in \mathbb{R}$ such that for every y in the finite set $A_j \setminus A_{j-1}$,

$$|\phi(\sum_{i\leq j-1} h_i)(y) + \lambda^y_{x_j}\alpha| > 1.$$

Let $h_j \in C_p(X)$ be such that $h_j(C \cup (X \setminus V_j)) \subseteq \{0\}$ and $h_j(x_j) = \alpha$. Then for $y \in A_j \setminus A_{j-1}$ we have

$$\phi(h_j)(y) = \sum\{\lambda^y_x h_j(x) : x \in \mathrm{supp}_\phi(y)\} = \lambda^y_{x_j} h_j(x_j) = \lambda^y_{x_j}\alpha$$

and hence

$$|\phi(\sum_{i\leq j} h_i)(y)| = |\phi(\sum_{i\leq j-1} h_i)(y) + \phi(h_j)(y)| = |\phi(\sum_{i\leq j-1} h_i)(y) + \lambda^y_{x_j}\alpha| > 1.$$

Moreover, if $y \in A_{j-1}$, then $\mathrm{supp}_\phi(y) \subseteq C$, hence $h_j(y) = 0$. Hence by (4) we have

$$|\phi(\sum_{i\leq j} h_i)(y)| = |\phi(\sum_{i\leq j-1} h_i)(y)| > 1.$$

and so h_j satisfies (1) through (4).

Now define $h : X \to \mathbb{R}$ by $h = \sum_{j<\omega} h_j$. Then $h \in C_p(X)$ since the collection $\{V_j : j < \omega\}$ is discrete. Since $\mathrm{supp}_\phi(v) \cap \bigcup_{j<\omega} V_j = \emptyset$ we have by (2), $h(\mathrm{supp}_\phi(v)) = 0$ and hence $\phi(h)(v) = 0$. For $y \in A_j$ we have by (3) and (4) that $|\phi(h)(y)| = |\phi(\sum_{i\leq j} h_i)(y)| > 1$. Since $\bigcup_{j<\omega} A_j = A$ and $v \in \overline{A} \setminus A$, $\phi(h)$ is not continuous at v, which is a contradiction. $\square$

In this section we address the natural question whether 'being ω-near' is l_p-invariant. In some cases, it is. The proof makes use of both Lemma 2.2 and Lemma 3.1.

Proposition 3.2. *Let X and Y be a normal spaces, and suppose $Y = \bigcup_{i<\omega} Y_i$, where each Y_i is pseudocompact and $Y_i \subseteq \operatorname{int}(Y_{i+1})$ for $i < \omega$. If $u \in X^*$ is ω-near, $v \in Y^*$, and X_u and Y_v are l_p-equivalent, then v is ω-near.*

Proof. Assume that $\phi\colon C_p(X_u) \to C_p(Y_v)$ is a linear homeomorphism. Let D be a countable closed discrete subset of X such that $u \in \mathrm{cl}_{\beta X} D$. For $i < \omega$, let $F_i = \mathrm{supp}_\phi(Y_i) \cap D$ and $G_i = \theta_{\phi^{-1}}(D) \cap Y_i$. Note that $F_i \subseteq F_{i+1}$ and $G_i \subseteq G_{i+1}$ for $i < \omega$. Assume that some F_i is infinite. By normality of X, there is a continuous function $f\colon X \to \mathbb{R}$ such that $f(F_i)$ is unbounded. Hence $\mathrm{supp}_\phi(Y_i)$ is unbounded in X, which violates Lemma 2.2. Hence each F_i is finite and so each $\mathrm{supp}_{\phi^{-1}}(F_i)$ is finite as well. We will show that $G_i \subseteq \mathrm{supp}_{\phi^{-1}}(F_i)$. To this end, take an arbitrary $y \in G_i$. Pick $d \in D$ such that $y \in \theta_{\phi^{-1}}(d)$. Then $d \in \theta_\phi(y) \cap D \subseteq F_i$, so $y \in \mathrm{supp}_{\phi^{-1}}(d) \subseteq \mathrm{supp}_{\phi^{-1}}(F_i)$, as required. From this we conclude that each G_i is finite as well. Hence $G = \bigcup_{i<\omega} G_i$ is countable. Pick an arbitrary $y \in Y$, and $i < \omega$ such that $y \in Y_i \subseteq \operatorname{int}(Y_{i+1})$. Note that $\operatorname{int}(Y_{i+1}) \cap G$ is contained in the finite set G_{i+1} and hence y is not an accumulation point of G. So G is a countable closed and discrete subset of Y.

For $A \subseteq D$ infinite we have that A is not bounded in X_u. To see this, split A into two disjoint infinite subsets, say A_0 and A_1. Since A_0 and A_1 are disjoint closed subsets of X we have by normality of X that either u is not in the closure of A_0, or not in the closure of A_1. So we may assume without loss of generality that A is closed in X_u. Let C be a closed neighborhood of u in X_u which misses A. Again by normality of X, there is a continuous function $f\colon X \to \mathbb{R}$ such that $f(C \cap X) \subseteq \{0\}$ and $f(A)$ is unbounded. This function can be extended to a continuous function $g\colon X_u \to \mathbb{R}$ by $g \equiv f$ on X and $g(u) = 0$. This implies that A is not bounded in X_u.

Since for every $x \in D$, $\theta_{\phi^{-1}}(x) \neq \emptyset$, there is $i < \omega$ such that $\theta_{\phi^{-1}}(x) \cap Y_i \neq \emptyset$ and hence $\theta_{\phi^{-1}}(x) \cap G \neq \emptyset$. Since D is not closed in X_u and Y is normal, it now follows by Lemma 3.1 that $\{G_i : i < \omega\}$ is not discrete in Y_v, i.e., $v \in \mathrm{cl}_{\beta Y} G$. ☐

Clearly every locally compact and σ-compact space X can be written as $X = \bigcup_{i<\omega} X_i$, where each X_i is compact and $X_i \subseteq \operatorname{int}(X_{i+1})$ for $i < \omega$. Moreover since every σ-compact space is Lindelöf and hence normal, we now get:

Theorem 3.3. *Let X and Y be locally compact, σ-compact spaces. Let $u \in X^*$ and $v \in Y^*$ be such that X_u and Y_v are l_p-equivalent. Then u is ω-near if and only if v is.*

In particular, Theorem 3.3 holds for the real line $\mathbb{R}$. Since every countable ordinal number is locally compact and σ-compact, we also have:

Theorem 3.4. *Let $\alpha, \gamma < \omega_1$ be limit ordinals. Let $u \in \alpha^*$ and $v \in \gamma^*$ be such that α_u and γ_v are l_p-equivalent. Then u is ω-near if and only if v is.*

Recall that for a space X a point in X^* is a *remote point* of X if it is not in the closure $\mathrm{cl}_{\beta X} D$ of any nowhere dense subset D of X. Clearly every remote point is ω-far if X is dense in itself. In light of Theorem 3.3, this suggests:

Question 3.5. *Let X and Y be locally compact, σ-compact spaces. Let $u \in X^*$ and $v \in Y^*$ be such that X_u and Y_v are l_p-equivalent. Is then u a remote point of X if and only if v is a remote point of Y?*

Theorem 3.3 and Question 3.5 suggest the question whether similar results hold for all spaces, or perhaps for all normal spaces. We will show that this unfortunately is not true.

Consider the space $\mathbb{Q}$ consisting of all rational numbers, and let $\mathbb{Z}$ denote the set of integers. Let $Y = \mathbb{Q} \times (\mathbb{Z} \times \mathbb{Z})$, and let $\sigma \colon \mathbb{Z} \to \mathbb{Z}$ be the shift $\sigma(n) = n{+}1$. Take $x \in \mathbb{Q}$, and let p be a limit point of $\{(x, 0, n) : n \in \mathbb{Z}\}$ in βY. Then $p \in Y^*$ and is ω-near. Define $\tau \colon Y \to Y$ by $\tau(x, n, m) = (x, n{+}1, m)$. Then τ is a homeomorphism, hence $f = \beta\tau \colon \beta Y \to \beta Y$ is a homeomorphism. Put

$$Z = Y \cup \{f^k(p) : k \in \mathbb{Z}, k \leq 0\}.$$

Observe that if $k_0 \neq k_1$ are in $\mathbb{Z}$, then $f^{k_0}(p) \neq f^{k_1}(p)$. Hence, $f^{-1}(Z) = Z \setminus \{p\}$.

The space Z has the following properties:

(1) Z contains a dense copy of Y which is a topological copy of $\mathbb{Q}$ and so Z has no isolated points and has countable π-weight.
(2) Z contains a point p such that $Z \setminus \{p\}$ is C^*-embedded in Z, p is an ω-near point of $Z \setminus \{p\}$, and, finally $Z \setminus \{p\}$ is homeomorphic to Z.

Now we repeat a slightly different version of this construction with Z as building block instead of $\mathbb{Q}$. To this end, let $S = Z \times (\mathbb{Z} \times \mathbb{Z})$. Consider the clopen subspace $T = Z \times \{0\} \times \mathbb{Z}$ of S. This subspace is dense in itself and clearly has countable π-weight. By van Douwen [10], there is a remote point $q \in T^*$. Observe that $q \in S^*$. Define $\eta \colon S \to S$ by $\eta(x, n, m) = (x, n{+}1, m)$. Then η is a homeomorphism, hence $g = \beta\eta \colon \beta S \to \beta S$ is a homeomorphism. Put

$$T = S \cup \{g^k(q) : k \in \mathbb{Z}, k \leq 0\}.$$

Observe that if $k_0 \neq k_1$ are in $\mathbb{Z}$, then $g^{k_0}(p) \neq g^{k_1}(p)$. Hence, $g^{-1}(T) = T \setminus \{q\}$.

The space T has the following properties:

(3) T contains the clopen copy of $Z \times \{0\} \times \{0\}$ of Z, which we identify with Z,
(4) T contains a point q such that $T \setminus \{q\}$ is C^*-embedded in T, q is not in the closure of any nowhere dense subset of T (hence is ω-far), and, finally $T \setminus \{q\}$ is homeomorphic to T.

Now put $P = T \setminus \{p, q\}$. Then P is C^*-embedded in $P \cup \{p\}$, hence $P_p = T \setminus \{q\} \approx T$. Also, p is an ω-near point of P. Similarly, P is C^*-embedded in $P \cup \{q\}$, hence $P_q = T \setminus \{p\} \approx T$ (this is so since Z is clopen in T, does not contain q and $Z \setminus \{p\} \approx Z$). Also, q is an ω-far point of P. This means that $P_p \approx T \approx P_q$, and so $C_p(P_p)$ is linearly homeomorphic to $C_p(P_q)$, and p is an ω-near point of P while q is not.

4. More on ω-near points

As mentioned in the introduction, we are interested in classes of spaces X for which Gul'ko's Theorem 1.1 can be generalized. In this section we will present results for the class of locally compact σ-compact metrizable spaces amongst which are the real line $\mathbb{R}$ and all limit ordinals $\alpha < \omega_1$. We start with the following:

Lemma 4.1. *Let $X = \bigcup_{i<\omega} X_i \oplus \omega$ and $Y = Z \oplus \omega$, where X and Z are normal spaces and where each X_i is pseudocompact and $X_i \subseteq \operatorname{int}(X_{i+1})$ for $i < \omega$. Let $u \in \omega^* \subseteq X^*$ and $v \in \omega^* \subseteq Y^*$. Let $\phi : C_p(X_u) \to C_p(Y_v)$ be a linear homeomorphism. Then $F : \omega_v \to \mathscr{P}(\omega_u)$, defined by $F(n) = \theta_\phi(n) \cap \omega$ for $n < \omega$ and $F(v) = \{u\}$, is upper semicontinuous. Moreover, $\{n < \omega : F(n) \neq \emptyset\} \approx \omega_v$.*

Proof. Let $U \subseteq \omega \cup \{u\}$ be open and let $Z = \{y \in \omega_v : F(y) \subseteq U\}$. Our aim is to show that Z is open. If $u \notin U$ then $v \notin Z$ and hence there is nothing to prove. So assume that $u \in U$, from which it follows that $v \in Z$. Put $A = \omega_v \setminus Z$ and observe that it is contained in ω. If A is finite then Z is open so we may assume without loss of generality that A is infinite. Note that every subset of A that is bounded in Y_v is finite.

For $i < \omega$ let $A_i = \operatorname{supp}_{\phi^{-1}}(X_i) \cap A$ and $B_i = \theta_\phi(A) \cap X_i$. Since X_i is bounded in X_u we have that $\operatorname{supp}_{\phi^{-1}}(X_i)$ is by Lemma 2.2 bounded in Y_v and hence A_i is finite. For $x \in B_i$ there is $y \in A$ such that $x \in \theta_\phi(y)$. But then $y \in \operatorname{supp}_{\phi^{-1}}(x) \cap A \subseteq A_i$ and hence $x \in \operatorname{supp}_\phi(A_i)$. Therefore $B_i \subseteq \operatorname{supp}_\phi(A_i)$ and hence B_i is finite.

Let $\mathscr{B} = \{B_i : i < \omega\} \cup \{\{n\} : n \in \omega_u \setminus U\}$ and let $B = \bigcup \mathscr{B}$. Then $\mathscr{B}$ is a countably infinite family of finite subsets of X. Pick an arbitrary $x \in \bigcup_{i<\omega} X_i$, and $i < \omega$ such that $x \in X_i \subseteq \operatorname{int}(X_{i+1})$. Note that $\operatorname{int}(X_{i+1}) \cap B$ is contained in the finite set B_{i+1} and hence x is not an accumulation point of B. So $\mathscr{B}$ is a countably infinite closed and discrete family in X_u. Note that for $n \in A$ we have $F(n) \cap B \neq \emptyset$. Since $F(n) \subseteq \theta_\phi(n)$ we have $\theta_\phi(n) \cap B \neq \emptyset$. From Lemma 3.1 it now follows that A is closed in Y_v and hence $Z = \omega_v \setminus A$ is open.

Let $Z = \{n < \omega : F(n) \neq \emptyset\}$ and assume that Z is closed and discrete. Then $v \in \overline{\omega \setminus Z}$. Let $\mathscr{B} = \{\theta_\phi(Z) \cap X_i : i < \omega\}$ and let $B = \bigcup \mathscr{B}$. As above we can show that $\mathscr{B}$ is a countably infinite closed and discrete family in X_u. For $n \in \omega \setminus Z$ we have $F(n) = \emptyset$ hence $\theta_\phi(n) \cap B \neq \emptyset$. From Lemma 3.1 it now follows that $v \notin \overline{\omega \setminus Z}$, which is a contradiction. $\square$

We need a result by Hasumi [13] on selections of certain upper semicontinuous functions on extremally disconnected spaces and a well-known theorem on the Rudin-Keisler (pre-) order on $\beta\omega$. A space X is *extremally disconnected* if disjoint open subsets of X have disjoint closures. Note that for every $u \in \omega^*$, ω_u is extremally disconnected.

Theorem 4.2. (Hasumi [13]) *Let X and Y be spaces with X extremely disconnected. Let $F : X \to \mathscr{P}(Y)$ be an upper semicontinuous function such that for every $x \in X$, $F(x)$ is non-empty and compact. Then there exists a continuous function $f : X \to Y$ such that for every $x \in X$, $f(x) \in F(x)$ (i.e., f is a continuous selection for F).*

We now come to the following generalisation of Theorem 1.1.

Theorem 4.3. *Let $X = \bigcup_{i<\omega} X_i \oplus \omega$ and $Y = \bigcup_{i<\omega} Y_i \oplus \omega$, where X and Y are normal spaces, X_i and Y_i are pseudocompact, $X_i \subseteq \operatorname{int}(X_{i+1})$ and $Y_i \subseteq \operatorname{int}(Y_{i+1})$ for $i < \omega$. Let $u \in \omega^* \subseteq X^*$ and $v \in \omega^* \subseteq Y^*$. If $\phi : C_p(X_u) \to C_p(Y_v)$ is a linear homeomorphism, then $\omega_v \approx \omega_u$.*

Proof. By Lemma 4.1 we have that $F : \omega_v \to \mathscr{P}(\omega_u)$ defined by $F(n) = \theta(n) \cap \omega$ and $F(v) = \{u\}$ is upper semicontinuous. Moreover by Lemma 4.1 we may assume that $F(n) \neq \emptyset$ for all $n < \omega$. Since ω_v is extremely disconnected, by Theorem 4.2, there is a continuous $f : \omega_v \to \omega_u$ such that $f(v) = u$. Hence $u \leq_{\mathrm{RK}} v$. Similarly, by applying the above for ϕ^{-1} instead of ϕ, we have $v \leq_{\mathrm{RK}} u$. And so by Theorem 2.3, ω_u and ω_v are homeomorphic. □

Our next result will be an important tool in the rest of this section.

Proposition 4.4. *Let X be a normal space and let $Y = X \oplus \omega$. Let $u \in D^*$, where $D \subseteq X$ is countably infinite, closed, discrete, C_p-zero embedded and C^*-embedded in X. Then $C_p(X_u) \sim C_p(Y_{\tilde{u}})$.*

Proof. Since $D \approx \omega$ and D is C_p-zero embedded in X we have

$$C_p(X) \sim C_{p,D}(X) \times C_p(D) \sim C_{p,D}(Y).$$

Note that $\omega_{\tilde{u}} \approx \omega \oplus \omega_{\tilde{u}}$, and so,

$$\begin{aligned} C_p(Y_{\tilde{u}}) &= C_p(X \oplus \omega_{\tilde{u}}) \sim C_p(X) \times C_p(\omega_{\tilde{u}}) \sim C_{p,D}(Y) \times C_p(\omega_{\tilde{u}}) \\ &\sim C_{p,D}(X \oplus \omega) \times C_p(\omega_{\tilde{u}}) \sim C_{p,D}(X \oplus \omega \oplus \omega_{\tilde{u}}) \\ &\sim C_{p,D}(X \oplus \omega_{\tilde{u}}) \sim C_{p,D}(Y_{\tilde{u}}). \end{aligned}$$

Since X is normal, there is a discrete family $\{O_i : i < \omega\}$ of open subsets of X such that $x_i \in O_i$ for $i < \omega$. Let $h : X \to [0,1]$ be such that $h \equiv 1$ on D and $h \equiv 0$ on $X \setminus \bigcup_{i<\omega} O_i$.

Let $\pi : D \to \omega$ be the bijection $x_i \mapsto i$. Define $\phi : C_{p,D}(Y_{\tilde{u}}) \to C_p(X_u)$ by

$$\phi(f)(x) = \begin{cases} f(\tilde{u}) & (x = u), \\ f(x) + f(\pi(x_i)) \cdot h(x) & (x \in O_i \text{ for } i < \omega), \\ f(x) & (x \in X \setminus \bigcup_{i<\omega} O_i). \end{cases}$$

It is clear that $\phi(f)$ is continuous on $(\bigcup_{i<\omega} O_i) \cup (X \setminus \bigcup_{i<\omega} \overline{O_i})$ and that ϕ is linear. Let $\varepsilon > 0$. For $i < \omega$ and $x \in \overline{O_i} \setminus O_i$ we have $\phi(f)(x) = f(x)$ and $h(x) = 0$. Let U be an open neighborhood of $x \in X$ such that for every $z \in U$ we have $|f(z) - f(x)| < \varepsilon/2$ and $|f(\pi(x_i)) \cdot h(z)| < \varepsilon/2$. For $z \notin O_i$ we have $|\phi(f)(z) - \phi(f)(x)| = |f(z) - f(x)| < \varepsilon/2$. For $z \in O_i$ we have

$$\begin{aligned} |\phi(f)(z) - \phi(f)(x)| &= |f(z) + f(\pi(x_i)) \cdot h(z) - f(x)| \\ &\leq |f(z) - f(x)| + |f(\pi(x_i)) \cdot h(z)| < \varepsilon \end{aligned}$$

and hence $\phi(f)$ is continuous at x. To prove that $\phi(f)$ is continuous at u note that there is $V \subseteq \omega$ such that $V \cup \{\tilde{u}\}$ is a neighborhood of $\tilde{u}$ and for every $i \in V$ we have $|f(i) - f(\tilde{u})| = |f(\pi(x_i)) - f(\tilde{u})| < \varepsilon/4$. Then $\pi^{-1}(V) \cup \{u\}$ is a neighborhood of u in $D \cup \{u\}$. For each $i \in V$ there is, since $f(x_i) = 0$ and $h(x_i) = 1$, a neighborhood $U_i \subseteq O_i$ of x_i in X such that for every $z \in U_i$ we have $|f(z)| < \varepsilon/2$ and $|f(\pi(x_i)) \cdot (1 - h(z))| < \varepsilon/4$. Let $U = \bigcup_{i \in V} U_i$. Then $U \cup \{u\}$ is a neighborhood of u in X_u. For every $z \in U_i$ we have

$$\begin{aligned} |\phi(f)(z) - \phi(f)(u)| &\leq |f(z)| + |f(\pi(x_i)) \cdot h(z) - f(\tilde{u})| \\ &\leq |f(z)| + |f(\pi(x_i)) - f(\tilde{u})| + |f(\pi(x_i)) \cdot (1 - h(z))| \\ &< \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon. \end{aligned}$$

Hence for every $x \in U$, $|\phi(f)(x) - \phi(f)(u)| < \varepsilon$, so $\phi(f)$ is continuous at u.

Define $\psi : C_p(X_u) \to C_{p,D}(Y_{\tilde{u}})$ by

$$\psi(g)(y) = \begin{cases} g(u) & (y = \tilde{u}), \\ g(\pi^{-1}(i)) & (y = i \text{ for } i < \omega), \\ g(y) - g(\pi^{-1}(i)) \cdot h(y) & (y \in O_i, i < \omega), \\ g(y) & (y \in X \setminus \bigcup_{i<\omega} O_i). \end{cases}$$

By a similar reasoning as the one above, we find that ψ is continuous. Finally, $\psi = \phi^{-1}$ and so ϕ is a linear homeomorphism. Hence $C_p(X_u) \sim C_{p,D}(Y_{\tilde{u}}) \sim C_p(Y_{\tilde{u}})$, as required. □

Let X be a space. If there exists $v \in \omega^*$ such that X_u and $X \oplus \omega_v$ are l_p-equivalent, then we say that v is *good* for u.

Proposition 4.5. *Let X be metrizable and let $u \in X^*$. Then*

(a) If u is ω-near, there is $v \in \omega^*$ such that v is good for u.

(b) If X is locally compact and σ-compact and $v_1, v_2 \in \omega^*$ are both good for u, then ω_{v_1} and ω_{v_2} are homeomorphic.

Proof. For (a), let D be a countable closed and discrete subset of X such that $u \in \mathrm{cl}_{\beta X} D$. By Lemma 2.1 we have that D is C_p-zero embedded and C^*-embedded in X. So (a) follows from Proposition 4.4.

For (b), suppose $v_1 \in \omega^*$ and $v_2 \in \omega^*$ are both good for u. Then $X \oplus \omega_{v_1}$ and $X \oplus \omega_{v_2}$ are l_p-equivalent and hence by Theorem 4.3, ω_{v_1} is homeomorphic to ω_{v_2}. □

Hence this proposition shows that if X is a locally compact, σ-compact metrizable space and $u \in X^*$ is ω-near, then there is $v \in \omega^*$ such that v is good for u, and that v is 'unique'. We denote this unique v by $\widehat{u}$. Clearly every locally compact and σ-compact space X can be written as $X = \bigcup_{i<\omega} X_i$, where each X_i is compact and $X_i \subseteq \mathrm{int}(X_{i+1})$ for $i < \omega$. Moreover every σ-compact space is Lindelöf and hence normal. Note that $\mathbb{R}$ and all limit ordinals $\alpha < \omega_1$ are locally compact, σ-compact and metrizable.

We now come to the main results in this section.

Theorem 4.6. *Let X and Y be a locally compact σ-compact metrizable spaces and let $u \in X^*$ and $v \in Y^*$ be ω-near. If X_u and Y_v are l_p-equivalent, then $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic.*

Proof. By Proposition 4.5 we have that X_u and $X \oplus \omega_{\widehat{u}}$ are l_p-equivalent and that Y_v and $Y \oplus \omega_{\widehat{v}}$ are l_p-equivalent. But then $X \oplus \omega_{\widehat{u}}$ and $Y \oplus \omega_{\widehat{v}}$ are l_p-equivalent and hence by Theorem 4.3 it follows that $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic. □

Corollary 4.7. *Let X be a locally compact σ-compact metrizable space and let $u, v \in X^*$ be ω-near. Then X_u and X_v are l_p-equivalent if and only if $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic.*

Corollary 4.8. *Let $u, v \in \mathbb{R}^*$ be ω-near. Then $\mathbb{R}_u$ and $\mathbb{R}_v$ are l_p-equivalent if and only if $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic.*

Corollary 4.9. *Let $\alpha < \omega_1$ be a limit ordinal and let $u, v \in \alpha^*$ be ω-near. Then α_u and α_v are l_p-equivalent if and only if $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic.*

5. Points in Čech-Stone remainders of ordinals

In this section we are particularly interested in spaces that are (homeomorphic to) countable limit ordinals. Hence such spaces are locally compact and countable (hence zero-dimensional and Lindelöf) and scattered. As we will show, some of these spaces have ω-far points (observe that van Douwen's result just quoted does not apply for this class of spaces). We will concentrate on those $\alpha < \omega_1$ having the property that all $p \in \alpha^*$ are ω-near.

The following trivial result 'characterizes' ω-near points in the situation we are interested in.

Lemma 5.1. *Let X be locally compact, zero-dimensional and Lindelöf. Then $p \in X^*$ is ω-near iff X can be written as $X = \bigoplus_{i<\omega} X_i$, where each X_i is compact and for every $i < \omega$ there exists $x_i \in X_i$ such that p is in the βX-closure of $\{x_i : i < \omega\}$.*

Our aim is now to show that each point of α^*, where α is any limit ordinal smaller than ω^ω, is ω-near. As usual, if X is a space, then X' denotes $X \setminus \{x : \{x\} \text{ is isolated in } X\}$.

Lemma 5.2. *Let X be a normal space such that each point of $(X')^*$ is ω-near. If $X \setminus X'$ is countable, then each point of X^* is ω-near as well.*

Proof. Let $p \in X^*$ be arbitrary. If $p \in \mathrm{cl}_{\beta X} X' = \beta X'$, then there is nothing to prove since X' is closed in X. Hence we may assume that there is a closed neighborhood V of p in βX such that V misses $\mathrm{cl}_{\beta X} X'$. Then $V \cap X$ is a closed, discrete and countable subset of X that has p in its closure. ☐

Since each point of ω^* is ω-near, by a simple inductive argument, the following is now obvious.

Corollary 5.3. *Let X be α, where $\alpha < \omega^\omega$ is a limit ordinal. Then each point in X^* is ω-near.*

This result will allow us to derive a complete isomorphic classication of the function spaces $C_p(\alpha_u)$, where $\alpha < \omega^\omega$ and $u \in \alpha^*$.

Things are not so easy if $\alpha \geq \omega^\omega$, as we will now demonstrate. We will show that all such α have $2^\mathfrak{c}$ ω-far points. Our result does not follow from van Douwen's result quoted earlier since ordinal numbers are not dense in themselves.

To start the construction, let $K = \omega + 1$. For every $n \in \mathbb{N}$, let K_n be the topological sum of two copies of K^n. That is, $K_n = K^n \times \{0, 1\}$. Let $X = \bigoplus_{n\in\mathbb{N}} K_n$. Observe that X is homeomorphic to $\alpha = \omega^\omega$. Let $Z = \{u \in X^* : u \text{ is } \omega\text{-far}\}$. Note that Z has cardinality at most $2^\mathfrak{c}$ since βX has weight $\mathfrak{c}$.

Let $\pi\colon X \to \mathbb{N}$ be the function that sends K_n onto $\{n\}$ for every n. Since π is a perfect map, its Stone extension $g = \beta\pi\colon \beta X \to \beta\omega$ maps X^* onto $\mathbb{N}^*$.

For every $x \in K$ and $n \in \mathbb{N}$ define $K_{x,n} \subseteq K_n$ by

$$K_{x,n} = \{(p, i) \in K_n : (i \in \{0, 1\}) \wedge \big((\exists\, j \leq n)(p_j = x)\big)\}.$$

Note that $K_{x,n}$ is a closed subset of K_n and hence of X.

Consider the following collection $\mathscr{F}$ of closed subsets of X, where $F \in \mathscr{F}$ if and only if

$$(\forall\, n \in \mathbb{N})(\exists\, x \in K)(K_{x,n} \subseteq F).$$

Let $p \in \mathbb{N}^*$, $A \in p$ and put $X_A = \bigcup_{n \in A} X_n \subseteq X$. Finally, fix $f \colon \mathbb{N} \to \{0,1\}$ for the time being. For every $F \in \mathscr{F}$, define $F_f = \bigcup_{n\in\mathbb{N}}(F \cap (K^n \times \{f(n)\})) \subseteq F$. We define the collection $\mathscr{F}(f,p)$ of closed subsets of X by

$$\{F_f : F \in \mathscr{F}\} \cup \{X_A : A \in p\}.$$

Lemma 5.4. *$\mathscr{F}(f,p)$ has the finite intersection property and $\bigcap \mathscr{F}(f,p) = \emptyset$.*

Proof. Let $n, m \in \mathbb{N}$, $F_1, \ldots, F_n \in \mathscr{F}$ and $A_1, \ldots, A_m \in p$. If $A = \bigcap_{j=1}^m A_j$, then $A \in p$ and $X_A = \bigcap_{j=1}^m X_{A_j}$. Let $k \geq n$ be such that $k \in A$. For each $i \leq n$, pick $x_i \in K$ such that $K_{x_i,k} \subseteq F_i \cap K_k$. Let

$$x = (\underbrace{0, 0, \ldots, 0}_{k-n}, x_1, \ldots, x_n) \in K^k.$$

Then clearly $(x, f(k)) \in K_k \subseteq X$, and

$$(x, f(k)) \in (K^k \times \{f(k)\}) \cap \bigcap_{i=1}^n K_{x_i,k} \cap X_A \subseteq \bigcap_{i=1}^n F_{i,f} \cap \bigcap_{j=1}^m X_{A_j},$$

which proves the first part of the lemma.

Let $n \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in K^n$ and $j \in \{0,1\}$. Pick $x \in K \setminus \{x_1, \ldots, x_n\}$. Let $F = K_{x,n} \cup \bigcup_{m \neq n} X_m$. Then F is closed in X and $F \in \mathscr{F}$. Note that $(x,j) \notin K_{x,n}$ and hence $(x,j) \notin F_f$, which proves the second part of the lemma. $\square$

Let $Z(f,p) = \{u \in X^* : \mathscr{F}(f,p) \subseteq u\}$. Then $Z(f,p)$ is a closed subset of X^*. By Lemma 5.4, $Z(f,p) \neq \emptyset$. Note that for every $A \in p$, $\pi(X_A) = A$ and for every $F \in \mathscr{F}$, $\pi(F_f) = \mathbb{N}$. This implies $\pi(\mathscr{F}(f,p)) = \{A : A \in p\}$, hence for every $u \in Z(f,p)$ we have $g(u) = p$. Therefore if $p \neq q$, then $Z(f,p) \cap Z(f,q) = \emptyset$.

Lemma 5.5. *Every $u \in Z(f,p)$ is ω-far.*

Proof. For every $n \geq 1$ let $G_n \subseteq K_n$ be a finite subset. For $i \leq n$, let $\pi_i^n : K_n \to K$ be the projection onto the i-th factor of K^n. That is, if $p = (p_1, \ldots, p_n) \in K^n$ and $j \in \{0,1\}$, then $\pi_i^n(p,j) = p_i$. Since $\bigcup_{i\leq n} \pi_i^n(G_n)$ is finite and K is infinite, there is $x_n \in K$ such that $x_n \notin \bigcup_{i \leq n} \pi_i^n(G_n)$. Let $F = \bigcup_{n\in\mathbb{N}} K_{x_n,n}$. Then F is a closed subset of X and $F \in \mathscr{F}$.

If $F \cap \bigcup_{n\in\mathbb{N}} G_n \neq \emptyset$, then there is $n \in \mathbb{N}$ such that $F \cap G_n = K_{x_n,n} \cap G_n \neq \emptyset$. Let $p = (p_1, \ldots, p_n) \in K^n$ and $j \in \{0,1\}$ be such that $(p,j) \in K_{x_n,n} \cap G_n$. Then there is $i \leq n$ such that $p_i = x_n$. But then $x_n = p_i \in \pi_i^n(G_n)$, which is a contradiction. So $F \cap \bigcup_{n\in\mathbb{N}} G_n = \emptyset$ and hence $F_f \cap \bigcup_{n\in\mathbb{N}} G_n = \emptyset$ which proves the lemma. $\square$

Split ω into countably many pairwise disjoint infinite sets, say $\{E_n : n \in \mathbb{N}\}$, and let f_n be the characteristic function of E_n. Then, clearly, $Z(f_n,p) \cap Z(f_m,p) = \emptyset$ provided that $n \neq m$. For every $n \in \mathbb{N}$ there is, by Lemma 5.4, $u_n \in Z(f_n,p)$. Let $Z(p) = \bigcup_{n\in\mathbb{N}} Z(f_n,p)$ and $D = \{u_n : n \in \mathbb{N}\}$. Then $Z(p)$ is a closed subset of X^* and D is a countable discrete subset of $Z(p)$. By Lemma 5.5 it follows that $Z(p) \subseteq Z$, hence $Z(p)$ has cardinality at most $2^{\mathfrak{c}}$.

Since X is a countable locally compact noncompact space, we have by Theorem 1.2.5 in [16] that X^* is an F-space. Since $Z(p)$ is a closed subspace of X^* we have by Lemma 1.2.2(d) in [16] that $Z(p)$ is an infinite compact F-space. The countable discrete subspace D of $Z(p)$ is C^*-embedded in it (see the proof of Corollary 3.4.2 in [16]) and hence its closure $\overline{D} \subseteq Z(p)$ is homeomorphic to $\beta\mathbb{N}$. Hence $Z(p)$ has cardinality at least $2^{\mathfrak{c}}$. We conclude that $Z(p)$ and Z have cardinality $2^{\mathfrak{c}}$.

Corollary 5.6. *Let $q \in \omega^*$ and $f : \mathbb{N} \to \{0,1\}$. Then for every $u \in Z(f,q)$ there is $v \in Z(f,q)$ such that $C_p(X_u) \not\approx C_p(X_v)$.*

Proof. Observe that X_u being countable, $C_p(X_u)$ is homeomorphic to a linear subspace of $\mathbb{R}^\infty$. Hence, as in the proof of Proposition 1 in [14], it follows that

$$\{v \in Z(f,q) : C_p(X_u) \approx C_p(X_v)\}$$

has cardinality at most $\mathfrak{c}$. Hence the corollary follows by the above remarks. □

Corollary 5.7. *Let $\alpha \geq \omega^\omega$ be a limit ordinal. Then the set of ω-far points in α^* has size $2^{\mathfrak{c}}$.*

Proof. Since $\alpha \geq \omega^\omega$, we can write $\alpha = \bigoplus_{i=1}^\infty X_i$, where each X_i is a compact ordinal that contains a closed copy Y_i of $\omega^i + 1$. Let $Y = \bigoplus_{i=1}^\infty Y_i$. Then Y is a closed copy of ω^ω in α. If $u \in Y^*$ is ω-far then, since $Y^* \subseteq \alpha^*$, u is also an ω-far point in α^*. □

Now we relate the previously discussed concepts to function spaces of ordinals.

Theorem 5.8. *Let $\alpha, \gamma < \omega^\omega$ be limit ordinals, and let $u \in \alpha^*$ and $v \in \gamma^*$. Then α_u and γ_v are l_p-equivalent if and only if α and γ are l_p-equivalent while moreover $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic.*

Proof. Since u and v are both ω-near by Corollary 5.3, we have by Proposition 4.5 that α_u and $\alpha \oplus \omega_{\widehat{u}}$ are l_p-equivalent and, similarly, that γ_v and $\gamma \oplus \omega_{\widehat{v}}$ are l_p-equivalent.

Suppose α_u and γ_v are l_p-equivalent. Then $\alpha \oplus \omega_{\widehat{u}}$ and $\gamma \oplus \omega_{\widehat{v}}$ are l_p-equivalent and so by Theorem 4.3, $\omega_{\widehat{u}}$ is homeomorphic to $\omega_{\widehat{v}}$. We will now check that α and γ are l_p-equivalent. Indeed, suppose $\phi : C_p(\alpha_u) \to C_p(\gamma_v)$ is a linear homeomorphism. First, assume that $\alpha = \omega$. If $\gamma > \omega$, there is a closed copy K of $\omega + 1$ in γ. By Lemma 2.2, $\mathrm{supp}_\phi(K)$ is bounded in γ_v hence $\mathrm{supp}_\phi(K)$ finite. But then $K \subseteq \mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(K))$ is finite. Contradiction. So $\gamma = \alpha = \omega$ and hence α and γ are l_p-equivalent. Second, assume that $\omega < \alpha < \omega^\omega$. By the above we have $\omega < \gamma < \omega^\omega$. But it then follows directly from the classification results in [4] that α and γ are l_p-equivalent.

Conversely, if α and γ are l_p-equivalent and $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic, then $\alpha \oplus \omega_{\widehat{u}}$ and $\gamma \oplus \omega_{\widehat{v}}$ are l_p-equivalent. It consequently follows that α_u and γ_v are l_p-equivalent as well. □

Of course this result does not say much if we do not know when limit ordinals below ω^ω are l_p-equivalent. Fortunately, there is a combinatorial characterization of when this happens.

Theorem 5.9 (Baars and de Groot [4]). *Let $\omega \leq \alpha, \gamma < \omega_1$. Then*

a) $\alpha + 1$ and $\gamma + 1$ are l_p-equivalent if and only if $\max(\alpha, \gamma) < [\min(\alpha, \gamma)]^\omega$.

b) If α and γ are limit ordinals, then α and γ are l_p-equivalent if and only if there are sequences of ordinals $(\alpha_i)_{i \in \mathbb{N}}$ and $(\gamma_i)_{i \in \mathbb{N}}$ such that $\alpha_i \to \alpha$, $\gamma_i \to \gamma$ and for every $i \in \mathbb{N}$, $\alpha_i + 1$ and $\gamma_i + 1$ are l_p-equivalent.

Hence Theorems 5.8 and 5.9 yield a complete isomorphic classification of the function spaces $C_p(\alpha_u)$, where $\alpha < \omega^\omega$ and $u \in \alpha^*$.

Although Theorem 5.8 looks like a generalization of Gul'ko's result (Theorem 1.1), it actually is not.

Corollary 5.10. *Let $\omega < \alpha < \omega_1$ be a limit ordinal. Then there are $u, v \in \alpha^*$ such that α_u and α_v are l_p-equivalent but α_u and α_v are not homeomorphic.*

Proof. Write $\alpha = \bigoplus_{i<\omega} X_i$ where each X_i is infinite and compact. For each $i < \omega$, let $x_i \in X_i$ be a non-isolated point. Pick $u \in \overline{\{x_0, x_1, \dots\}} \setminus \{x_0, x_1, \dots\} \subseteq \alpha^*$, and let $v = \widehat{u}$. By Proposition 4.5, α_u and $\alpha \oplus \omega_v$ are l_p-equivalent. Observe that α and $\alpha \oplus \omega$ are homeomorphic, but α_u and $\alpha \oplus \omega_v$ are not. Hence we are done. $\square$

The results in this section suggest the question what happens if we consider ordinals greater than or equal to ω^ω as well. We settle this in the following special case.

Theorem 5.11. *Let $\alpha < \omega^\omega$ and $\gamma \geq \omega^\omega$ be limit ordinals. Then for every $u \in \alpha^*$ and $v \in \gamma^*$ we have that α_u and γ_v are not l_p-equivalent.*

Proof. Suppose $\phi : C_p(\alpha_u) \to C_p(\gamma_v)$ is a linear homeomorphism. Let K be a closed copy of $\omega^\omega + 1 \subseteq \gamma$. By Lemma 2.2, $\mathrm{supp}_\phi(K)$ is bounded in α_u, hence there is $n < \omega$ such that $\mathrm{supp}_\phi(K) \subseteq (\omega^n + 1) \cup \{u\}$. Let $L = (\omega^n + 1) \cup \{u\}$. Clearly K is a retract of γ_v and L is a retract of α_u. If $r : \gamma_v \to K$ and $s : \alpha_u \to L$ are retractions then if we define $\theta : C_p(K) \to C_p(L)$ by $\theta(g) = \phi^{-1}(g \circ r)|_L$ and $\psi : C_p(L) \to C_p(K)$ by $\psi(f) = \phi(f \circ s)|_K$ we have $\psi \circ \theta = \mathrm{id}$ and hence θ is a linear embedding. Indeed for $g \in C_p(K)$ we have $(\theta(g) \circ s)|_L = \theta(g) = \phi^{-1}(g \circ r)|_L$. Since $\mathrm{supp}_\phi(K) \subseteq L$ we then have $\phi(\theta(g) \circ s)|_K = (g \circ r)|_K$ and hence $\psi(\theta(g)) = g$. Since $L \approx \omega^n + 1$ we then have by Proposition 2 in [1] that there exists a linear embedding from $C_0(\omega^\omega + 1)$ to $C_0(\omega^n + 1)$, where the function spaces are endowed with the compact-open topology. This contradicts Theorem 1 in [7]. $\square$

There is more to say, but the results are not conclusive. By making use of the results and techniques developed in [4] and [7] and combining them with enhanced versions of the proofs of Theorem 5.8 and Theorem 5.11, the following can be derived:

Theorem 5.12 (Baars [3]). *Let $\alpha, \gamma < \omega_1$ be limit ordinals. Let $u \in \alpha^*$ and $v \in \gamma^*$ be such that α_u and γ_v are l_p-equivalent. Then α and γ are l_p-equivalent.*

Hence, by exactly the same argument as in the proof of Theorem 5.8, we obtain:

Theorem 5.13. *Let $\alpha, \gamma < \omega_1$ be limit ordinals, and let $u \in \alpha^*$ and $v \in \gamma^*$ be ω-near. Then α_u and γ_v are l_p-equivalent if and only if α and γ are l_p-equivalent while moreover $\omega_{\widehat{u}}$ and $\omega_{\widehat{v}}$ are homeomorphic.*

In light of Corollary 5.7, this is a rather limited result. In fact, we believe that an isomorphic classification of the function spaces $C_p(\alpha_u)$, where $\alpha < \omega_1$ is a limit ordinal and $u \in \alpha^*$, will turn out to be rather complicated, if impossible.

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