A NEW PROOF OF THE MCKINSEY-TARSKI THEOREM

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Abstract. It is a landmark theorem of McKinsey and Tarski that if we interpret modal diamond as closure (and hence modal box as interior), then S4 is the logic of any dense-in-itself metrizable space [14, 17]. The McKinsey-Tarski Theorem relies heavily on a metric that gives rise to the topology. We give a new and more topological proof of the theorem, utilizing Bing’s Metrization Theorem [8, 10].

1. Introduction

It is a famous result of McKinsey and Tarski [14] that the modal system S4 is the logic of any dense-in-itself separable metrizable space when interpreting ◊ as closure (and hence □ as interior). Rasiowa and Sikorski [17] proved that the McKinsey-Tarski Theorem remains true for an arbitrary dense-in-itself metrizable space (that is, the separability condition can be dropped harmlessly). On the other hand, dropping the dense-in-itself condition results in new logics, and a complete classification of them can be found in [6].

Both the original proof of McKinsey and Tarski [14, Sec. 3] and the proof of Rasiowa and Sikorski [17, Sec. III.7 and III.8] rely heavily on a metric generating the topology of a given dense-in-itself (separable) metrizable space X to show that every finite subdirectly irreducible closure algebra is embeddable in the closure algebra of X. The result follows since S4 has the finite model property.

In the recent literature many simplified proofs of the McKinsey-Tarski Theorem have been produced for specific dense-in-itself metrizable spaces such as the real line [1, 7, 16], the rational line [3], and the Cantor discontinuum [15, 1]. These new proofs utilize relational semantics of S4 that was not available to McKinsey and Tarski. The proof technique of [7] produces an interior mapping of the real line onto any finite quasi-tree (such mappings correspond to the isomorphic embeddings used in the original proofs of the McKinsey-Tarski Theorem), which is obtained by iteratively removing a copy of the Cantor discontinuum from the corresponding real intervals. This technique does not utilize the usual metric of the real line.

The aim of the present paper is to give a new proof of the McKinsey-Tarski Theorem, which makes the aforementioned idea work for an arbitrary dense-in-itself metrizable space. The new proof is more topological in that a metric is never used explicitly. Such is possible because of Bing’s Metrization Theorem which characterizes metrizable spaces as exactly those spaces that admit a σ-discrete basis [8, 10]. It is this σ-discrete basis that encodes a blueprint for the interior mapping onto any finite rooted S4-frame.

We conclude by giving an example of a dense-in-itself hereditarily paracompact space whose logic is stronger than S4. This indicates that the McKinsey-Tarski Theorem does not generalize to the setting of hereditarily paracompact spaces.

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2. Classical approach

We start by outlining the original proof that S4 is the logic of any dense-in-itself metrizable space. We recall that S4 is the least set of formulas in the basic propositional modal language (with □) that contains the classical tautologies, the axioms:

- □(p → q) → (□p → □q),
- □p → p,
- □□p → □p,

and is closed under Modus Ponens \( \varphi, \varphi \rightarrow \psi \), substitution \( \varphi(p_1, \ldots, p_n) \), and necessitation \( \diamond \varphi \). As usual, we use the standard abbreviation \( \diamond \varphi := \neg \Box \neg \varphi \).

An S4-algebra is a pair \( \mathfrak{A} = (B, \Box) \), where \( B \) is a Boolean algebra and \( \Box : B \rightarrow B \) satisfies Kuratowski’s axioms for interior:

- \( \Box(a \land b) = \Box a \land \Box b \),
- \( \Box 1 = 1 \),
- \( \Box a \leq a \),
- \( \Box a \leq \Box \Box a \).

Every such interior operator has its dual closure operator \( \Diamond : B \rightarrow B \), defined by \( \Diamond a = \neg \Box \neg a \). Fixpoints of \( \Box \) are called open elements and fixpoints of \( \Diamond \) are called closed elements of \( \mathfrak{A} \).

**Remark 2.1.** S4-algebras were introduced by McKinsey and Tarski [14] in the \( \Diamond \)-signature under the name of closure algebras. Rasiowa and Sikorski [17] call them topological Boolean algebras, and Blok [9] calls them interior algebras. In the modern modal logic literature it is common to call them S4-algebras. We follow McKinsey and Tarski in working with S4-algebras in the \( \Diamond \)-signature.

Typical examples of S4-algebras come from topology: If \( X \) is a topological space, then \( \mathfrak{A}_X := (\wp(X), c) \) is an S4-algebra, where \( \wp(X) \) is the powerset of \( X \) and \( c \) is the closure operator of \( X \). By the McKinsey-Tarski Representation Theorem [14, Thm. 2.4], each S4-algebra is isomorphic to a subalgebra of \( \mathfrak{A}_X \) for some topological space \( X \).

The modal language is interpreted in an S4-algebra \( \mathfrak{A} = (B, \diamond) \) by evaluating propositional letters as elements of \( B \), the classical connectives as the corresponding Boolean operations, and the modal box as the interior operator and hence modal diamond as the closure operator of \( \mathfrak{A} \). A formula \( \varphi \) is valid in \( \mathfrak{A} \), written \( \mathfrak{A} \models \varphi \), provided it evaluates to 1 under all interpretations. It is well known (see, e.g., [17, Sec. XI.7]) that S4 \( \vdash \varphi \iff \varphi \) is valid in every S4-algebra. In this notation, the McKinsey-Tarski Theorem can be stated as follows:

**Theorem 2.2 (McKinsey and Tarski).** S4 \( \vdash \varphi \iff \mathfrak{A}_X \models \varphi \) for every dense-in-itself metrizable space \( X \).

**Proof.** The left to right implication is obvious. For the right to left implication, if S4 \( \not\vdash \varphi \), then we must find a valuation on \( \mathfrak{A}_X \) refuting \( \varphi \). This can be done in three steps. We recall (see [14, Def. 1.10]) that an S4-algebra is well-connected if \( \Diamond a \land \Diamond b = 0 \) implies \( a = 0 \) or \( b = 0 \).

**Step 1 (Finite Model Property):** If S4 \( \not\vdash \varphi \), then there is a finite well-connected S4-algebra \( \mathfrak{A} \) refuting \( \varphi \) (see [14, Thm. 4.16]).

For Step 2, we require the key notion of a dissectable S4-algebra. For two elements \( x, y \) of a Boolean algebra \( B \), write \( a - b := a \land \neg b \), and say that \( x_1, \ldots, x_k \in B \) are disjoint provided \( x_i \land x_j = 0 \) for each \( i \neq j \).

**Definition 2.3.** [14, Def. 3.4] An S4-algebra \( \mathfrak{A} = (B, \diamond) \) is dissectable if for every open \( a \in B \setminus \{0\} \) and every pair of integers \( n \geq 0 \) and \( m > 0 \), there are disjoint \( u_1, \ldots, u_n, a_1, \ldots, a_m \in B \setminus \{0\} \) such that
The elements \( u_1, \ldots, u_n \) are open;
• \( \Diamond a_1 = \cdots = \Diamond a_m \);
• \( u_1 \lor \cdots \lor u_n \lor a_1 \lor \cdots \lor a_m = a \);
• \( \Diamond a - a \leq \Diamond a_i \leq \Diamond u_j \) for each \( i \leq m \) and \( j \leq n \).

**Step 2 (Dissection Lemma):** If \( X \) is a dense-in-itself metrizable space, then \( \mathfrak{A}_X \) is dissectable (see [14, Thm. 3.5] for the separable case, and [17, III.7.1] for the general case).

**Step 3 (Embedding Lemma):** Every finite well-connected \( S4 \)-algebra is embedded into every dissectable \( S4 \)-algebra (see [14, Thm. 3.7]).

Now, suppose \( S4 \not\vdash \phi \). By Step 1, there is a finite well-connected \( S4 \)-algebra \( \mathfrak{A} \) refuting \( \phi \). By Step 2, \( \mathfrak{A}_X \) is dissectable. Therefore, by Step 3, \( \mathfrak{A} \) is isomorphic to a subalgebra of \( \mathfrak{A}_X \). Thus, since \( \phi \) is refuted on \( \mathfrak{A} \), it is refuted on \( \mathfrak{A}_X \), and the proof of the McKinsey-Tarski Theorem is complete. \( \square \)

The proof of the Dissection Lemma makes nontrivial use of a metric that generates the topology on \( X \). In the next section, we will discuss how this can be avoided using the modern approach.

3. Modern approach

The modern approach utilizes the relational semantics of modal logic. This semantics has its roots in the work of Jónsson and Tarski [12], and became the dominant semantics after the work of Kripke [13].

An \( S4 \)-frame is a pair \( \mathfrak{F} = (W, R) \), where \( W \) is a nonempty set and \( R \) is a reflexive and transitive binary relation on \( W \). As usual, for \( w \in W \) and \( A \subseteq W \), we write:

• \( R[w] := \{ v \in W \mid wRv \} \) and \( R^{-1}[w] := \{ v \in W \mid vRw \} \);
• \( R[A] := \{ v \in W \mid \exists w \in A : wRv \} \) and \( R^{-1}[A] := \{ v \in W \mid \exists w \in A : vRw \} \).

Each \( S4 \)-frame \( \mathfrak{F} \) gives rise to the \( S4 \)-algebra \( \mathfrak{A}_\mathfrak{F} := (\mathfrak{F}(W), R^{-1}) \). By the Jónsson-Tarski Representation Theorem [12, Thm. 3.14], every \( S4 \)-algebra \( \mathfrak{A} \) is isomorphic to a subalgebra of \( \mathfrak{A}_\mathfrak{F} \) for some \( S4 \)-frame \( \mathfrak{F} \). In fact, if \( \mathfrak{A} \) is finite, then \( \mathfrak{A} \) is isomorphic to \( \mathfrak{A}_\mathfrak{F} \).

The modal language is interpreted in \( \mathfrak{F} \) by interpreting formulas in \( \mathfrak{A}_\mathfrak{F} \). A formula \( \phi \) is valid in \( \mathfrak{F} \), written \( \mathfrak{F} \models \phi \), provided \( \mathfrak{A}_\mathfrak{F} \models \phi \). The completeness of \( S4 \) with respect to the algebraic semantics together with the Jónsson-Tarski Representation Theorem yields that \( S4 \vdash \phi \) iff \( \phi \) is valid in every \( S4 \)-frame.

The relational semantics of \( S4 \) is a particular case of its topological semantics (see, e.g., [2, Sec. 2.4.1]). For an \( S4 \)-frame \( \mathfrak{F} = (W, R) \), call \( A \subseteq W \) an \( R \)-cone if \( A = R[A] \), and let \( \tau_R \) be the set of all \( R \)-cones. Then \( \tau_R \) is a topology on \( W \) such that \( R^{-1} \) is the closure operator, and each \( w \in W \) has the least open neighborhood \( R[w] \). Such spaces are usually referred to as Alexandroff spaces.

Let \( \mathfrak{F} = (W, R) \) be an \( S4 \)-frame. We call \( \mathfrak{F} \) rooted if there is \( r \in W \) such that \( R[r] = W \); such \( r \) is called a root of \( \mathfrak{F} \). For finite \( \mathfrak{F} \), it is well known (and easy to see) that \( \mathfrak{A}_\mathfrak{F} \) is well-connected iff \( \mathfrak{F} \) is rooted.

A map \( f : X \to Y \) between topological spaces is called interior if it is continuous (\( V \) open in \( Y \) implies \( f^{-1}(V) \) is open in \( X \)) and open (\( U \) open in \( X \) implies \( f(U) \) is open in \( Y \)). Equivalently, \( f \) is interior iff \( cf^{-1}(A) = f^{-1}(cA) \) for each \( A \subseteq Y \).

**Remark 3.1.** It is well known (and easy to see) that if \( X \) and \( Y \) are Alexandroff spaces, then \( f : X \to Y \) is an interior map iff it is a p-morphism (\( R^{-1}[f^{-1}(x)] = f^{-1}(R^{-1}[x]) \) for each \( x \in X \)).
We say $Y$ is an **interior image** of $X$ provided there is an interior mapping from $X$ onto $Y$. Interior images will play an important role in our story since $\mathfrak{A}_X$ is isomorphic to a subalgebra of $\mathfrak{A}_Y$ iff $Y$ is an interior image of $X$. In particular, $\mathfrak{A}_X$ is isomorphic to a subalgebra of $\mathfrak{A}_Y$ iff $\mathfrak{F}$ viewed as an Alexandroff space is an interior image of $X$. Thus, proving the McKinsey-Tarski Theorem amounts to showing that every finite rooted $S4$-frame $\mathfrak{F}$ is an interior image of every dense-in-itself metrizable space $X$.

We can further restrict the class of finite rooted $S4$-frames. Let $\mathfrak{F} = (W, R)$ be an $S4$-frame. The equivalence classes of the equivalence relation $\{(w, v) \mid wRv \text{ and } vRw\}$ on $W$ are called **clusters**. A quasi-chain is a subset $Q$ of $W$ such that $wRv$ or $vRw$ for $w, v \in Q$. We call $\mathfrak{F}$ a **quasi-tree** if $\mathfrak{F}$ is rooted and $R^{-1}[w]$ is a quasi-chain for each $w \in W$.

**Remark 3.2.** The relation $R$ induces a partial ordering on the set of clusters such that quasi-chains in $\mathfrak{F}$ correspond to chains in the poset $P$ of clusters, and $\mathfrak{F}$ is a quasi-tree iff $P$ is a tree (see Figure 1).

![Figure 1. A quasi-tree $\mathfrak{F}$ and its poset of clusters $P$.](image)

It is well known (see, e.g., [7, Cor. 6]) that $S4$ is complete with respect to the class of finite quasi-trees. Therefore, if $S4 \not\vdash \varphi$, then there is a finite quasi-tree $\mathcal{T}$ refuting $\varphi$. Thus, to prove the McKinsey-Tarski Theorem, it is sufficient to show that every finite quasi-tree $\mathcal{T}$ is an interior image of every dense-in-itself metrizable space $X$.

Let us examine what it takes for an onto interior map $f : X \to \mathcal{T}$ to exist. We recall that the **depth** of $\mathcal{T}$, denoted $\text{depth}(\mathcal{T})$, is the greatest $n \geq 1$ such that there are $w_1, \ldots, w_n \in W$ satisfying $w_i R w_{i+1}$ but not $w_{i+1} R w_i$ for each $i \in \{1, \ldots, n-1\}$.

If $\text{depth}(\mathcal{T}) = 1$, then $\mathcal{T}$ is a single cluster, consisting of $m$ points (see Figure 2).

![Figure 2. A single cluster quasi-tree $\mathcal{T}$.](image)

We recall that a space $X$ is **$m$-resolvable** provided there is a partition $\{A_1, \ldots, A_m\}$ of $X$ such that each $A_i$ is dense in $X$; such partitions are called **dense**. By [5, Lem. 5.9], $\mathcal{T}$ is an interior image of $X$ iff $X$ is $m$-resolvable. It follows from Hewitt’s theory of resolvability (see [11]) that every dense-in-itself metrizable space is $m$-resolvable. Therefore, if $\text{depth}(\mathcal{T}) = 1$, then it is a consequence of Hewitt’s theory of resolvability that $\mathcal{T}$ is an interior image of $X$.

Suppose $\text{depth}(\mathcal{T}) > 1$ and $C$ is the root cluster of $\mathcal{T} = (W, R)$ consisting of $m$ points. Then $W \setminus C \neq \emptyset$, and there are $w_1, \ldots, w_n \in W$ such that $\{C, R[w_1], \ldots, R[w_n]\}$ is a partition of $W$ (see Figure 3).

If an onto interior map $f : X \to \mathcal{T}$ exists, then set $G = f^{-1}(C)$ and $U_i = f^{-1}(R[w_i])$ for each $i \in \{1, \ldots, n\}$. A direct calculation shows that $\{G, U_1, \ldots, U_n\}$ is a partition of $X$ such that $G$ is $m$-resolvable and nowhere dense ($\text{ic}G = \emptyset$), $U_i$ is open, and $G \subseteq cU_i$ for each $i \in \{1, \ldots, n\}$.
Thus, the existence of $f$ amounts to the existence of such a partition of $X$, which is the simplified version of the dissectability of $X$. How can we build such a partition without using a metric generating the topology? We will see in the next section that this is achievable using Bing’s Metrization Theorem. As our guiding example, we consider the case of the real line $\mathbb{R}$ as described in [7].

Since $\mathbb{R}$ is homeomorphic to any nonempty bounded open interval, it is sufficient to show that there is an onto interior map $f : (a, b) \to T$, where $(a, b)$ is an arbitrary nonempty bounded open interval. The proof is by induction on depth($T$), and we only discuss the inductive step in which depth($T$) > 1.

Construct the Cantor set $C$ inside $(a, b)$ by the usual process of taking away open “middle thirds”. Let the root cluster $C$ of $T$ consist of $m$ points. Since $C$ is $m$-resolvable, there is an interior map $f : C \to \mathbb{R}$. Our aim is to extend $f$ to the entire $(a, b)$. As depth($T$) > 1, there are $w_1, \ldots, w_n \in W$ such that $\{C, R[w_1], \ldots, R[w_n]\}$ is a partition of $W$. By the inductive hypothesis, we may let $f$ map each removed open “middle third” onto one of the quasi-trees, say $T_i$ whose underlying set is $R[w_i]$. For the sake of illustration, suppose $(c, d)$ is a removed open “middle third” and $f$ sends it to $T_1$. Then we construct the Cantor set inside $(c, d)$ and proceed by induction (see Figure 4). Therefore, the mapping $f$ is defined iteratively by moving “upward” through $T$ and sending appropriately chosen “copies” of the Cantor set to the “lower” parts of $T$. Notice that the role of $G$ is played by the initial copy of the Cantor set in $(a, b)$, and the roles of the $U_i$ are played by the removed open middle thirds, which themselves contain a copy of the Cantor set containing $f^{-1}(w_i)$.

Our goal for the remainder of the paper is to mimic this proof in the setting of an arbitrary dense-in-itself metrizable space.

4. THE NEW PROOF

In this section we present a new proof of the McKinsey-Tarski Theorem, in which the Embedding Lemma is replaced by the Mapping Lemma, and the key Dissection Lemma by the simpler Partition Lemma. To prove these two lemmas, we require some preparation. The section is divided into four subsections. The first subsection presents the auxiliary lemmas, culminating in Lemma 4.5; the second subsection proves the Partition Lemma (Lemma 4.13); the third subsection the Mapping Lemma (Lemma 4.22); and the fourth subsection shows that the McKinsey-Tarski Theorem does not generalize to the hereditarily paracompact setting.
4.1. **Auxiliary lemmas.** We start by recalling some basic definitions; see, e.g., [10]. For a topological space $X$, we recall that $i$ and $c$ stand for the interior and closure operators of $X$. As usual, we call $U \subseteq X$ regular open if $U = icU$.

**Definition 4.1.** Let $X$ be a space and $\mathcal{A}$ a family of subsets of $X$.

1. Call $\mathcal{A}$ discrete if each $x \in X$ has an open neighborhood $U$ such that $\{A \in \mathcal{A} \mid A \cap U \neq \emptyset\}$ consists of at most one element.
2. Call $\mathcal{A}$ $\sigma$-discrete if $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n$ and each $\mathcal{A}_n$ is discrete.
3. Call $\mathcal{A}$ closure preserving if $c \bigcup B = \bigcup \{cB \mid B \in \mathcal{B}\}$ for each $\mathcal{B} \subseteq \mathcal{A}$.

**Remark 4.2.** It is easy to see that if $\mathcal{A}$ is discrete, then $\mathcal{A}$ is pairwise disjoint. Moreover, if $\mathcal{A}$ is discrete and $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B}$ is discrete. Furthermore, if $\mathcal{A}$ is finite, then $\mathcal{A}$ is closure preserving.

**Lemma 4.3.** Let $X$ be a space, $B$ an open subset of $X$, and $\mathcal{U}$ a closure preserving family of nonempty regular open subsets of $X$ such that $\{cU \mid U \in \mathcal{U}\}$ is pairwise disjoint. Then $B \subseteq \bigcup \mathcal{U}$ iff $B \subseteq c \bigcup \mathcal{U}$.

**Proof.** We only need to prove the right to left implication. Suppose $B \subseteq c \bigcup \mathcal{U}$. Let $U \in \mathcal{U}$ and set $V = \mathcal{U} \setminus \{U\}$. Then $c(U) \cap \bigcup \{cV \mid V \in V\} = \emptyset$ and $B \subseteq c \bigcup \mathcal{U} = c(U) \bigcup \bigcup \{cV \mid V \in V\}$. Therefore, $B \cap cU = B \setminus \bigcup \{cV \mid V \in V\} = B \setminus c \bigcup V$ is open in $X$, so $B \cap cU \subseteq icU = U$. Thus, $B = \bigcup \{B \cap cU \mid U \in \mathcal{U}\} \subseteq \bigcup \mathcal{U}$. $\square$

**Lemma 4.4.** Let $X$ be a nonempty dense-in-itself regular space and $Y$ a nonempty open subspace of $X$.

1. There is a nonempty regular open subset $U$ of $X$ such that $cU \subset Y$.
2. For each $n \geq 1$, there is a family $\mathcal{U}$ consisting of $n$ nonempty regular open subsets of $X$ such that $\{cU \mid U \in \mathcal{U}\}$ is pairwise disjoint and $c \bigcup \mathcal{U} \subset Y$.

**Proof.** (1) Let $x \in Y$. Since $X$ is a dense-in-itself $T_1$-space, $Y \setminus \{x\}$ is a nonempty open subset of $X$. As $X$ is regular, there is a nonempty open subset $V$ of $X$ such that $cV \subseteq Y \setminus \{x\}$. Thus, $U := icV$ is as required.

(2) Induction on $n \geq 1$. Applying (1) renders the base case $n = 1$. Suppose $n \geq 1$ and there is a family $\mathcal{V}$ consisting of $n$ nonempty regular open subsets of $X$ such that $\{cV \mid V \in \mathcal{V}\}$ is pairwise disjoint and $c \bigcup \mathcal{V} \subset Y$. Then $Y \setminus c \bigcup \mathcal{V}$ is a nonempty open subset of $X$. Applying (1) yields a nonempty regular open subset $W$ of $X$ such that $cW \subset Y \setminus c \bigcup \mathcal{V}$. The family $\mathcal{U} := \mathcal{V} \cup \{W\}$ is as required. $\square$
Lemma 4.5. Let $X$ be a dense-in-itself regular space, $F$ a closed discrete subspace of $X$, and $\mathcal{U}_1, \ldots, \mathcal{U}_n$ families of subsets of $X$ such that $\mathcal{U} := \bigcup_{i=1}^n \mathcal{U}_i$ is a closure preserving family of nonempty regular open subsets of $X$ satisfying $\{cU \mid U \in \mathcal{U}\}$ is pairwise disjoint and $c(U) \cap F = \emptyset$ for each $U \in \mathcal{U}$.

If $\mathcal{B}$ is a discrete family of open subsets of $X$, then there are families $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of subsets of $X$ and a closed discrete subspace $D$ of $X$ such that:

1. $\mathcal{U}_i \subseteq \mathcal{V}_i$ for each $i \in \{1, \ldots, n\}$.
2. The family $\mathcal{V} := \bigcup_{i=1}^n \mathcal{V}_i$ is a closure preserving family of nonempty regular open subsets of $X$ such that $\{cV \mid V \in \mathcal{V}\}$ is pairwise disjoint.
3. $c(V) \cap (F \cup D) = \emptyset$ for each $V \in \mathcal{V}$.
4. If $B \in \mathcal{B}$ and $B \nsubseteq \bigcup \mathcal{V}$, then:
   (a) For each $i \in \{1, \ldots, n\}$ there is $V_i \in \mathcal{V}_i$ such that $cV_i \subseteq B$.
   (b) The set $B \cap D$ contains at least two elements.

Proof. Let $\mathcal{C} = \{B \in \mathcal{B} \mid B \nsubseteq \bigcup \mathcal{U}\}$. Suppose that $\mathcal{C} = \emptyset$. Set $\mathcal{V}_i = \mathcal{U}_i$ for $i \in \{1, \ldots, n\}$ and $D = \emptyset$. Then $D$ is a closed discrete subspace of $X$ and $\mathcal{V} = \mathcal{U}$. Therefore, conditions (1)–(4) are satisfied trivially.

Suppose that $\mathcal{C} \neq \emptyset$. Let $B \in \mathcal{C}$. Then $B \nsubseteq \bigcup \mathcal{U}$. By Lemma 4.3, $B \nsubseteq c \bigcup \mathcal{U}$, so $B \setminus c \bigcup \mathcal{U}$ is a nonempty open subset of $X$. Because $F$ is a closed discrete subspace of $X$ and $X$ is dense-in-itself, $B \setminus (F \cup c \bigcup \mathcal{U})$ is a nonempty open subset of $X$. Lemma 4.4(2) delivers a family $\{B_1, \ldots, B_n\}$ of nonempty regular open subsets of $X$ such that $\{cB_1, \ldots, cB_n\}$ is pairwise disjoint and $\bigcup_{i=1}^n cB_i = c \bigcup_{i=1}^n B_i \setminus (F \cup c \bigcup \mathcal{U})$. Let $D_B$ consist of any two points in the nonempty open subset $(B \setminus (F \cup c \bigcup \mathcal{U})) \setminus \bigcup_{i=1}^n cB_i$ of $X$. Set $\mathcal{V}_i = \mathcal{U}_i \cup \{B_i \mid B \in \mathcal{C}\}$ and $D = \bigcup \{D_B \mid B \in \mathcal{C}\}$.

Claim 4.6. $D$ is a closed discrete subspace of $X$.

Proof. Let $x \in cD$. Since $\mathcal{B}$ is a discrete family, there is an open neighborhood $U$ of $x$ such that $\{B \in \mathcal{B} \mid U \cap B \neq \emptyset\}$ consists of at most one element. Because $\emptyset \neq U \cap D \subseteq U \cap \bigcup \mathcal{C} \subseteq \bigcup \{U \cap B \mid B \in \mathcal{B}\}$, there is $B' \in \mathcal{C}$ such that $\{B \in \mathcal{B} \mid U \cap B \neq \emptyset\} = \{B'\}$. Note that $D_{B'} \setminus \{x\}$ is finite and hence closed since $D_{B'}$ consists of two points. Therefore, $U \setminus (D_{B'} \setminus \{x\})$ is an open neighborhood of $x$, and so

$$
\emptyset \neq \left(U \setminus (D_{B'} \setminus \{x\})\right) \cap D = (U \setminus (D_{B'} \setminus \{x\})) \cap \bigcup \{D_B \mid B \in \mathcal{C}\}
$$

$$
= \left((U \setminus (D_{B'} \setminus \{x\})) \cap D_B \mid B \in \mathcal{C}\right) = (U \setminus (D_{B'} \setminus \{x\})) \cap D_{B'} \subseteq \{x\},
$$

giving that $(U \setminus (D_{B'} \setminus \{x\})) \cap D = \{x\}$. This shows that $D$ is both closed and discrete. \(\square\)

We now verify that conditions (1)–(4) hold. Clearly condition (1) holds by the definition of the $\mathcal{V}_i$. That condition (2) holds follows from Claims 4.7, 4.8, and 4.10 below.

Claim 4.7. Each $V \in \mathcal{V}$ is a nonempty regular open subset of $X$.

Proof. Since for each $B \in \mathcal{C}$ and $i \in \{1, \ldots, n\}$, the families $\{B_1, \ldots, B_n\}$ and $\mathcal{U}_i$ consist of nonempty regular open subsets of $X$, each $V \in \mathcal{V}_i = \mathcal{U}_i \cup \{B_i \mid B \in \mathcal{C}\}$ is a nonempty regular open subset of $X$. The result follows since $\mathcal{V} = \bigcup_{i=1}^n \mathcal{V}_i$. \(\square\)

Claim 4.8. The family $\{cV \mid V \in \mathcal{V}\}$ is pairwise disjoint.

Proof. Suppose $V, W \in \mathcal{V}$ are distinct. If $V, W \in \mathcal{U}$, then $cV \cap cW = \emptyset$ since $\{cU \mid U \in \mathcal{U}\}$ is pairwise disjoint. If $V \in \mathcal{U}$ and $W \notin \mathcal{U}$, then $W \in \{B_1, \ldots, B_n\}$ for some $B \in \mathcal{C}$. Therefore, $cW \subseteq \bigcup_{i=1}^n cB_i \subseteq B \setminus (F \cup c \bigcup \mathcal{U}) \subseteq B \setminus cV \subseteq X \setminus cV$, which gives $cV \cap cW = \emptyset$. The case $W \in \mathcal{U}$ and $V \notin \mathcal{U}$ is similar. If $V, W \notin \mathcal{U}$, then
\(V \in \{B_1, \ldots, B_n\}\) and \(W \in \{B_1', \ldots, B_n'\}\) for some \(B, B' \in \mathcal{C}\). If \(B = B'\), then \(cV \cap cW = \emptyset\) since \(\{B_1, \ldots, B_n\} = \{B_1', \ldots, B_n'\}\) and \(\{cB_1, \ldots, cB_n\}\) is pairwise disjoint. If \(B \neq B'\), then \(B \cap B' = \emptyset\) since \(\mathcal{B}\) is discrete, and hence pairwise disjoint. Thus,

\[
cV \cap cW \subseteq \left( \bigcup_{i=1}^{n} cB_i \right) \cap \left( \bigcup_{i=1}^{n} cB'_i \right)
\]

\[
\subseteq \left[ B \setminus \left( F \cup c \bigcup \mathcal{W} \right) \right] \cap \left[ B' \setminus \left( F \cup c \bigcup \mathcal{W} \right) \right] \subseteq B \cap B' = \emptyset,
\]

and hence \(\{cV \mid V \in \mathcal{V}\}\) is pairwise disjoint. 

\(\square\)

**Claim 4.9.** The family \(\mathcal{D} := \{B_i \mid B \in \mathcal{C} \text{ and } i \in \{1, \ldots, n\}\}\) is discrete.

**Proof.** Let \(x \in X\). Since \(\mathcal{B}\) is discrete, there is an open neighborhood \(N_x\) of \(x\) such that \(\{B \in \mathcal{B} \mid B \cap N_x \neq \emptyset\}\) consists of at most one element. If \(\{B \in \mathcal{B} \mid B \cap N_x \neq \emptyset\}\) is empty, then \(\{B_i \mid B \in \mathcal{C}, i \in \{1, \ldots, n\}, B_i \cap N_x \neq \emptyset\}\) is empty because \(N_x \cap B_i \subseteq N_x \cap B\) for each \(B \in \mathcal{C}\) and \(i \in \{1, \ldots, n\}\).

Suppose \(B' \in \mathcal{B}\) is the unique element of \(\{B \in \mathcal{B} \mid B \cap N_x \neq \emptyset\}\). If \(B' \notin \mathcal{C}\), then \(\{B_i \mid B \in \mathcal{C}, i \in \{1, \ldots, n\}, N_x \cap B_i \neq \emptyset\}\) is empty. Suppose \(B' \in \mathcal{C}\). If \(x \notin c \bigcup_{i=1}^{n} B_i\), then \(U := N_x \cap c \bigcup_{i=1}^{n} B_i\) is an open neighborhood of \(x\) and \(\{B_i \mid B \in \mathcal{C}, i \in \{1, \ldots, n\}\}\) is empty.

If \(x \in c \bigcup_{i=1}^{n} B_i\), then since \(\{cB_1, \ldots, cB_n\}\) is pairwise disjoint, there is a unique \(j \in \{1, \ldots, n\}\) such that \(x \in cB_j\). Therefore, \(U := N_x \setminus \{cB_i \mid i \neq j\}\) is an open neighborhood of \(x\) and \(\{B_i \mid B \in \mathcal{C}, i \in \{1, \ldots, n\}\}\) is empty. Since \(x \in cB_j\) and \(U\) is an open neighborhood of \(x\), we have \(U \setminus B_j' \neq \emptyset\). If \(U \setminus B_i \neq \emptyset\) for some \(B \in \mathcal{C}\) and \(i \in \{1, \ldots, n\}\), then \(\emptyset \neq U \setminus B_i \subseteq N_x \cap B\) gives \(B = B'\) and \(\emptyset \neq U \setminus B_i = (N_x \setminus \{cB_i \mid i \neq j\}) \cap B_i' = \emptyset\) for \(i \neq j\). Thus, \(\mathcal{D}\) is discrete. 

\(\square\)

**Claim 4.10.** The family \(\mathcal{V}\) is closure preserving.

**Proof.** Let \(\mathcal{W} \subseteq \mathcal{V}\). Using \(\mathcal{D}\) as defined in Claim 4.9, we have that \(\mathcal{V} = \mathcal{W} \cup \mathcal{D}\) and \(\mathcal{W} = (\mathcal{W} \cap \mathcal{D}) \cup (\mathcal{W} \cap \mathcal{D})\). Since \(\mathcal{D}\) is discrete, so is \(\mathcal{W} \cap \mathcal{D}\). Therefore, \(\mathcal{W} \cap \mathcal{D}\) is closure preserving (which follows from [10, Thm. 1.1.11] since a discrete family is locally finite). Because \(\mathcal{W}\) is closure preserving, so is \(\mathcal{W} \cap \mathcal{D}\). Therefore,

\[
c \bigcup \mathcal{W} = c \bigcup ((\mathcal{W} \cap \mathcal{D}) \cup (\mathcal{W} \cap \mathcal{D})) = c \left( \bigcup (\mathcal{W} \cap \mathcal{D}) \cup (\mathcal{W} \cap \mathcal{D}) \right)
\]

\[
= c \left( \bigcup (\mathcal{W} \cap \mathcal{D}) \cup c \bigcup (\mathcal{W} \cap \mathcal{D}) \right)
\]

\[
= \bigcup \{cV \mid V \in \mathcal{W} \cap \mathcal{D}\} \cup \{cV \mid V \in \mathcal{W} \cap \mathcal{D}\}
\]

\[
= \bigcup \{cV \mid V \in (\mathcal{W} \cap \mathcal{D}) \cup (\mathcal{W} \cap \mathcal{D})\} = \bigcup \{cV \mid V \in \mathcal{W}\}.
\]

Thus, \(\mathcal{V}\) is closure preserving. 

\(\square\)

**Claim 4.11.** Condition (3) holds.

**Proof.** Let \(V \in \mathcal{V}\). If \(V \notin \mathcal{W}\), then \(cV \cap F = \emptyset\) by assumption. Moreover, \(cV \cap D = \emptyset\) because for each \(B \in \mathcal{C}\), from

\[
D_B \subseteq \left( B \setminus (F \cup c \bigcup \mathcal{W}) \right) \cup \bigcup_{i=1}^{n} cB_i \subseteq B \setminus (F \cup c \bigcup \mathcal{W}) \subseteq X \setminus c \bigcup \mathcal{W} \subseteq X \setminus cV
\]

it follows that \(D = \bigcup \{D_B \mid B \in \mathcal{C}\} \subseteq X \setminus cV\). Thus, \(cV \cap (F \cup D) = \emptyset\).

If \(V \notin \mathcal{W}\), then \(V = B_j'\) for some \(B' \in \mathcal{C}\) and \(j \in \{1, \ldots, n\}\). From

\[
cB_j' \subseteq \bigcup_{i=1}^{n} cB_i \subseteq B' \setminus (F \cup c \bigcup \mathcal{W}) \subseteq B' \setminus F \subseteq X \setminus F
\]
it follows that \( c(B'_j) \cap F = \emptyset \). Also, from
\[
D_{B'} \subseteq \left( B' \setminus (F \cup c(\mathcal{U})) \right) \setminus \bigcup_{i=1}^{n} cB'_i \subseteq X \setminus \bigcup_{i=1}^{n} cB'_i \subseteq X \setminus cB'_j
\]
it follows that \( c(B'_j) \cap D_{B'} = \emptyset \). Since \( \mathcal{B} \) is pairwise disjoint, we have
\[
c(B'_j) \cap D = c(B'_j) \cap \bigcup \{ D_B \mid B \in \mathcal{C} \} = \bigcup \{ c(B'_j) \cap D_B \mid B \in \mathcal{C} \}
= \left( c(B'_j) \cap D_{B'} \right) \cup \bigcup \{ c(B'_j) \cap D_B \mid B \in \mathcal{C} \setminus \{ B' \} \}
\subseteq \emptyset \cup \bigcup \{ c(B'_j) \cap B \mid B \in \mathcal{C} \setminus \{ B' \} \}
\subseteq \bigcup \{ B' \cap B \mid B \in \mathcal{C} \setminus \{ B' \} \} = \emptyset.
\]
Therefore, \( cV \cap (F \cup D) = \emptyset \), and hence condition (3) holds.

**Claim 4.12.** Condition (4) holds.

*Proof.* Suppose \( B \in \mathcal{B} \) and \( B \not\subseteq \bigcup \mathcal{U} \). Then \( B \not\subseteq \bigcup \mathcal{U} \) since \( \mathcal{U} \not\subseteq \mathcal{U} \). Therefore, \( B \in \mathcal{C} \).

Let \( i \in \{ 1, \ldots, n \} \). Then \( cB_i \subseteq \bigcup_{j=1}^{n} cB_j \subseteq B \setminus (F \cup c(\mathcal{U})) \subseteq B \). Since \( B_i \in \mathcal{V}_i \), condition (4a) holds. Because \( \mathcal{B} \) is pairwise disjoint and \( D_{B'} \subseteq B' \) for each \( B' \in \mathcal{C} \), we have
\[
B \cap D = B \cap \bigcup \{ D_{B'} \mid B' \in \mathcal{C} \} = \bigcup \{ B \cap D_{B'} \mid B' \in \mathcal{C} \} = B \cap D = D.
\]
Thus, \( B \cap D_B \) consists of two elements, and hence condition (4b) holds.

This completes the proof of Lemma 4.5.

With these preliminary results established we are ready to prove the Partition Lemma.

### 4.2. The Partition Lemma

This subsection is dedicated to proving the Partition Lemma, and it is exactly here where Bing’s Metrization Theorem will be utilized.

**Lemma 4.13 (Partition Lemma).** Let \( X \) be a dense-in-itself metrizable space, \( F \) a non-empty closed discrete subspace of \( X \), and \( n \geq 1 \). Then there is a partition \( \{ G, U_1, \ldots, U_n \} \) of \( X \) such that

1. \( G \) is a dense-in-itself closed nowhere dense subspace of \( X \) containing \( F \).
2. Each \( U_i \) is an open subspace of \( X \) such that there is a discrete subspace \( F_i \) of \( U_i \) with \( cF_i = F_i \cup G \).

*Proof.* By Bing’s Metrization Theorem (see, e.g., [10, Thm. 4.4.8]), \( X \) has a \( \sigma \)-discrete basis \( \mathcal{B} = \bigcup \{ \mathcal{B}_m \mid m \geq 1 \} \), where each \( \mathcal{B}_m \) is a discrete family of open subsets of \( X \). By Lemma 4.4(2), there is a family \( \mathcal{V}_0 = \{ W_1, \ldots, W_n \} \) of nonempty regular open subsets of \( X \) such that \( \{ cW_1, \ldots, cW_n \} \) is pairwise disjoint and \( c \bigcup \mathcal{V}_0 \subseteq X \setminus F \). Put \( \mathcal{V}_0 = \{ W_i \} \) for each \( i \in \{ 1, \ldots, n \} \) and \( D_0 = F \). Then \( \mathcal{V}_0, \ldots, \mathcal{V}_0, \mathcal{V}_0 = \bigcup_{i=1}^{n} \mathcal{V}_0 \), and \( D_0 \) satisfy the conditions of Lemma 4.5.

For each \( m \geq 1 \), define recursively families \( \mathcal{V}_1^m, \ldots, \mathcal{V}_n^m \) of subsets of \( X \) and a closed discrete subspace \( D_m \) of \( X \) as follows. Suppose for some \( m \geq 1 \) the families \( \mathcal{V}_1^{m-1}, \ldots, \mathcal{V}_n^{m-1} \) and the closed discrete subspace \( D_{m-1} \) are already defined so that \( \mathcal{V}_1^{m-1} := \bigcup_{i=1}^{n} \mathcal{V}_1^{m-1} \) is a closure preserving family of nonempty regular open subsets of \( X \) satisfying \( \{ cV \mid V \in \mathcal{V}_1^{m-1} \} \) is pairwise disjoint and \( cV \cap D_{m-1} = \emptyset \) for each \( V \in \mathcal{V}_1^{m-1} \). Then Lemma 4.5 applied to \( \mathcal{V}_i = \mathcal{V}_1^{m-1}, F = D_{m-1}, \) and \( \mathcal{B} = \mathcal{B}_m \) yields families \( \mathcal{V}_1^m, \ldots, \mathcal{V}_n^m \) and a closed discrete subspace \( D_m^i \) such that:

1. \( \mathcal{V}_i^{m-1} \subseteq \mathcal{V}_i^m \) for each \( i \in \{ 1, \ldots, n \} \).
2. The family \( \mathcal{V}^m := \bigcup_{i=1}^{n} \mathcal{V}_i^m \) is a closure preserving family of nonempty regular open subsets of \( X \) such that \( \{ cV \mid V \in \mathcal{V}^m \} \) is pairwise disjoint.
Let $\mathcal{V} \cap (D_{m-1} \cup D'_m) = \emptyset$ for each $V \in \mathcal{V}^m$.

(4) If $B \in \mathcal{B}_m$ and $B \not\subseteq \bigcup \mathcal{V}^m$, then:
   (a) For each $i \in \{1, \ldots, n\}$ there is $V_i \in \mathcal{V}_i^m$ such that $cV_i \subseteq B$.
   (b) The set $B \cap D'_m$ contains at least two elements.

Set $D_m = D_{m-1} \cup D'_m$. Then $D_m$ is a closed discrete subset of $X$ since a finite union of closed discrete subsets of any space is closed and discrete. Therefore, we have:

1. $\mathcal{V}_i^m \subseteq \mathcal{V}_i^{m+1}$ and $D_m \subseteq D_{m+1}$ for each $i \in \{1, \ldots, n\}$.
2. The family $\mathcal{V}^m := \bigcup_{i=1}^n \mathcal{V}_i^m$ is a closure preserving family of nonempty regular open subsets of $X$ such that \{$cV \mid V \in \mathcal{V}^m$\} is pairwise disjoint.
3. $\mathcal{V} \cap D_m = \emptyset$ for each $V \in \mathcal{V}^m$.
4. If $B \in \mathcal{B}_m$ and $B \not\subseteq \bigcup \mathcal{V}^m$, then:
   (a) For each $i \in \{1, \ldots, n\}$ there is $V_i \in \mathcal{V}_i^m$ such that $cV_i \subseteq B$.
   (b) The set $B \cap D'_m$ contains at least two elements.

For each $i \in \{1, \ldots, n\}$, set $\mathcal{V}_i = \bigcup_{m \in \omega} \mathcal{V}_i^m$, $U_i = \bigcup \mathcal{V}_i$, and $G = X \setminus \bigcup_{i=1}^n U_i$. It remains to prove that \{$G, U_1, \ldots, U_n$\} is as desired.

**Claim 4.14.** \{$\mathcal{V}_i^m \mid i \in \{1, \ldots, n\}$\} is pairwise disjoint for all $m \in \omega$.

**Proof.** By induction on $m \in \omega$. For $m = 0$, the family \{$W_1, \ldots, W_n$\} is chosen so that \{$cW_1, \ldots, cW_n$\} is pairwise disjoint. Therefore, \{$\mathcal{V}_i^0 \mid i \in \{1, \ldots, n\}$\} = \{$\{W_1\}, \ldots, \{W_n\}$\} is pairwise disjoint, and hence the base case holds.

Let $m \geq 1$ and \{$\mathcal{V}_i^{m-1} \mid i \in \{1, \ldots, n\}$\} be pairwise disjoint. Observe that for each $i \in \{1, \ldots, n\}$ we have

$$\mathcal{V}_i^m = \mathcal{V}_i^{m-1} \cup \left\{B_i \mid B \in \mathcal{B}_{m-1} \text{ and } B \not\subseteq \bigcup \mathcal{V}^{m-1}\right\}.$$ 

Also, for any $i, k \in \{1, \ldots, n\}$, $V \in \mathcal{V}_i^{m-1}$, and $B \in \mathcal{B}_{m-1}$ such that $B \not\subseteq \bigcup \mathcal{V}^{m-1}$, we have that $V \cap B_k = \emptyset$ because

$$B_k \subseteq c\bigcup_{j=1}^n B_j \subseteq B \setminus (D_{m-1} \cup c\bigcup \mathcal{V}^{m-1}) \subseteq X \setminus \bigcup \mathcal{V}^{m-1} \subseteq X \setminus V.$$ 

Let $V \in \mathcal{V}_i^m \cap \mathcal{V}_j^m$ for some $i, j \in \{1, \ldots, n\}$. Then $V \in \mathcal{V}_i^m$. If $V \in \mathcal{V}_i^{m-1}$, then the above observations yield that $V \in \mathcal{V}_j^{m-1}$. So $i = j$ by the inductive hypothesis. Suppose $V \not\in \mathcal{V}_j^{m-1}$. Then $V \not\in \mathcal{V}_j^{m-1}$. Therefore, $V = B_i$, and $V = B'_j$ for some $B, B' \in \mathcal{B}_{m-1}$ such that $B, B' \not\subseteq \bigcup \mathcal{V}^{m-1}$. Thus, $\emptyset \neq V = B_i \cap B'_j \subseteq B \cap B'$. Consequently, $B = B'$, so $B_i = B'_j$, and hence $i = j$, yielding that \{$\mathcal{V}_i^m \mid i \in \{1, \ldots, n\}$\} is pairwise disjoint.

**Claim 4.15.** $\mathcal{V}_i$ is pairwise disjoint for each $i \in \{1, \ldots, n\}$.

**Proof.** Let $V, W \in \mathcal{V}_i = \bigcup_{m \in \omega} \mathcal{V}_i^m$ be such that $V \cap W \neq \emptyset$. Then there are $m', m'' \in \omega$ such that $V \in \mathcal{V}_i^{m'}$ and $W \in \mathcal{V}_i^{m''}$. Let $m = \max\{m', m''\}$. Then $V, W \in \mathcal{V}_i^m \subseteq \mathcal{V}^m$. Since $\mathcal{V}^m$ is pairwise disjoint, $V = W$. Thus, $\mathcal{V}_i$ is pairwise disjoint.

**Claim 4.16.** For any $m \geq 1$ and $B \in \mathcal{B}_m$, if $B \cap G \neq \emptyset$, then $B \not\subseteq \bigcup \mathcal{V}^m$.

**Proof.** Let $m \geq 1$, $B \in \mathcal{B}_m$, and $x \in B \cap G$. Then $x \in G$, giving that $x \not\in \bigcup_{i=1}^n U_i$. Since $\mathcal{V}_i^m \subseteq \bigcup_{m \in \omega} \mathcal{V}_i^m = \mathcal{V}_i$ for each $i \in \{1, \ldots, n\}$, we have $\bigcup \mathcal{V}^m \subseteq \bigcup \mathcal{V}_i = U_i$ for each $i \in \{1, \ldots, n\}$, and hence $\bigcup \mathcal{V}^m = \bigcup_{i=1}^n \mathcal{V}_i^m = \bigcup_{i=1}^m \mathcal{V}_i^m \subseteq \bigcup_{i=1}^n U_i$. Therefore, $x \not\in \bigcup \mathcal{V}^m$, and hence $B \not\subseteq \bigcup \mathcal{V}^m$.

**Claim 4.17.** Let $D = \bigcup\{D_m \mid m \in \omega\}$. Then $D \cap \bigcup_{i=1}^n U_i = \emptyset$.
Proof. Since $D \cap \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (D \cap U_i)$, it is sufficient to show $D \cap U_i = \emptyset$ for all $i \in \{1, \ldots, n\}$. Since $D \cap U_i = \bigcup_{m \in \omega} (D_m \cap U_i) = \bigcup_{m \in \omega} (D_m \cap U_i)$, it is sufficient to show that $D_m \cap U_i = \emptyset$ for each $m \in \omega$. Since $D_m \cap U_i = D_m \cap U_i$, we have that $D_m \cap U_i = \emptyset$ for each $m \in \omega$ and $V \subseteq \mathcal{V}_m'$. But it follows from conditions (1) and (3) that $D_m \cap V \subseteq D_{m,m'} \cap V \subseteq D_{m,m'} \cap cF = \emptyset$ since $V \subseteq \mathcal{V}_m' \subseteq \mathcal{V}_m^\prime$ completing the proof.

Claim 4.18. The family $\{G,U_1,\ldots,U_n\}$ is a partition of $X$ such that $U_i$ is an open subset of $X$ for each $i \in \{1,\ldots,n\}$ and $G$ is a closed subset of $X$ containing $F$.

Proof. Let $i \in \{1,\ldots,n\}$. Because $U_i = \bigcup \mathcal{V}_i$ and each $V \subseteq \mathcal{V}_i$ is a (regular) open subset of $X$, $U_i$ is an open subset of $X$. Also $U_i \supseteq W_i \neq \emptyset$ since $\mathcal{V}_i \supseteq \mathcal{V}_i^m \supseteq \mathcal{V}_i^0 = \{W_i\}$.

To see that $\{U_1,\ldots,U_n\}$ is pairwise disjoint, let $x \in U_i \cap U_j$. Then there are $m_i, m_j \in \omega$, $V \subseteq \mathcal{V}_i^m$, and $W \subseteq \mathcal{V}_j^m$ such that $x \in V$ and $x \in W$. Let $m = \max\{m_i,m_j\}$. Then $V \subseteq \mathcal{V}_i^m \cap \mathcal{V}_j^m$, giving $\mathcal{V}_i^m \cap \mathcal{V}_j^m \neq \emptyset$. Claim 4.14 then yields $i = j$, and so $\{U_1,\ldots,U_n\}$ is pairwise disjoint.

Clearly $G = X \setminus \bigcup_{i=1}^{n} U_i$ is a closed subset of $X$. Because $\{U_1,\ldots,U_n\}$ is a pairwise disjoint family of nonempty sets and $G = X \setminus \bigcup_{i=1}^{n} U_i$, we only need to verify that $G$ is nonempty to conclude that $\{G,U_1,\ldots,U_n\}$ is a partition. But, by Claim 4.17, $\emptyset \neq F = D_0 \subseteq \bigcup\{D_m \mid m \in \omega\} = D \subseteq X \setminus \bigcup_{i=1}^{n} U_i = G$, completing the proof.

Claim 4.19. $G$ is a nowhere dense and dense-in-itself subspace of $X$.

Proof. Since $G$ is closed, to see that $G$ is nowhere dense, let $iG \neq \emptyset$. Then there are $m \geq 1$ and a nonempty $B \in \mathcal{B}_m$ such that $B \subseteq G$. By Claim 4.16, $B \subset \bigcup \mathcal{V}_m$. By condition (4a), there is (a nonempty) $V_1 \in \mathcal{V}_1^m$ such that $cV_1 \subseteq B$. But then

$$\emptyset \neq V_1 = B \cap V_1 \subseteq G \cap V_1 \subseteq G \cap \bigcup \mathcal{V}_1 \subseteq G \cap U_1 = \emptyset,$$

which is a contradiction.

To see that $G$ is dense-in-itself, let $m \geq 1$ and $B \in \mathcal{B}_m$ be such that $B \cap G \neq \emptyset$. By Claim 4.16, $B \supset \bigcup \mathcal{V}_m$. By condition (4b), $B \cap G \supset B \cap D \supset B \cap D_m$ contains at least two points.

Claim 4.20. For each $i \in \{1,\ldots,n\}$, there is $F_i \subseteq U_i$ that is discrete and $cF_i = F_i \cup G$.

Proof. Let $i \in \{1,\ldots,n\}$. Each $V \subseteq \mathcal{V}_i$ is nonempty, and hence we may choose $x_V \in V$. Set $F_i = \{x_V \mid V \subseteq \mathcal{V}_i\}$. Since $U_i = \bigcup \mathcal{V}_i$, we clearly have that $F_i \subseteq U_i$. By Claim 4.15, $\mathcal{V}_i$ is pairwise disjoint, and so $\{x_V\} = V \cap F_i$ for each $V \subseteq \mathcal{V}_i$. As each $V \subseteq \mathcal{V}_i$ is an open subset of $X$, we have that $F_i$ is discrete.

Let $x \in G$, $m \geq 1$, and $B \in \mathcal{B}_m$ be arbitrary with $x \in B$. Then $x \in B \cap G$, and so $B \supset \bigcup \mathcal{V}_m$ by Claim 4.16. By condition (4a), there is $V \in \mathcal{V}_1^m$ such that $cV \subseteq B$. Therefore, $V \in \mathcal{V}_i$, and so $B \cap F_i \subseteq V \cap F_i = \{x_V\} \neq \emptyset$, giving $x \in cF_i$. Thus, $G \subseteq cF_i$, and hence $F_i \cup G \subseteq cF_i$. 

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For the reverse inclusion, suppose \( x \notin F_i \cup G \). Then \( x \notin G \), so \( x \in \bigcup_{j=1}^{m} U_j \). If \( x \notin U_i \), then \( \bigcup \{ U_j \mid j \neq i \} \) is an open neighborhood of \( x \) that is disjoint from \( F_i \). If \( x \in U_i \), then \( x \in V \) for some \( V \in \mathcal{V}_i \) (such \( V \) is unique by Claim 4.15). Noting that \( x \notin x_V \) (since \( x \notin F_i \)) gives that \( V \setminus \{ x_V \} \) is an open neighborhood of \( x \) that is disjoint from \( F_i \) (since \( V \cap F_i = \{ x_V \} \)). In both cases, \( x \notin cF_i \). Thus, \( cF_i \subseteq F_i \cup G \), and the equality follows. □

This completes the proof of the Partition Lemma. □

4.3. The Mapping Lemma. In this subsection we prove the Mapping Lemma, which yields a new proof of the McKinsey-Tarski Theorem. This requires the following lemma.

Lemma 4.21. A dense-in-itself metrizable space \( X \) is \( m \)-resolvable for every \( m \geq 1 \).

Proof. For each \( m \geq 1 \), we recursively construct a pairwise disjoint family \( \{ A_1, \ldots, A_m \} \) of dense subsets of \( X \). Put \( X_0 = X \). Suppose \( X_n \) is a dense subset of \( X \) for some \( n \in \omega \). Then \( X_n \) is a dense-in-itself metrizable space. By [11, Thm. 41], \( X_n \) is resolvable. So there is \( A_{n+1} \subseteq X_n \) such that \( A_{n+1} \) and \( X_{n+1} := X_n \setminus A_{n+1} \) are both dense in \( X_n \). Therefore, \( X_n = cX_n A_{n+1} = c(A_{n+1}) \cap X_n \subseteq cA_{n+1} \), and similarly \( X_n \subseteq cX_{n+1} \). Thus, both \( A_{n+1} \) and \( X_{n+1} \) are dense in \( X_n \). An easy inductive argument gives that \( X_m \subseteq X_n \) whenever \( m \geq n \) since by definition \( X_{n+1} \subseteq X_n \). To see that \( \{ A_1, \ldots, A_m \} \) is pairwise disjoint, without loss of generality let \( i > j \geq 1 \). Then

\[
A_i \cap A_j \subseteq X_{i-1} \cap A_j \subseteq X_i \cap A_j = (X_{i-1} \setminus A_j) \cap A_j = \emptyset.
\]

Clearly \( \{ A_1, \ldots, A_{m-1}, X \setminus \bigcup_{i=1}^{m-1} A_i \} \) is a dense partition of \( X \) of cardinality \( m \geq 1 \). □

Lemma 4.22 (Mapping Lemma). Let \( X \) be a dense-in-itself metrizable space and \( F \) a nonempty closed discrete subspace of \( X \). Then there is an interior mapping of \( X \) onto every finite quasi-tree \( T \) such that the image of \( F \) is contained in the root cluster of \( T \).

Proof. Let the root cluster \( C \) of \( T = (W, R) \) consist of \( m \) elements, say \( C = \{ r_1, \ldots, r_m \} \). The proof is by induction on \( \text{depth}(T) \).

First suppose \( \text{depth}(T) = 1 \). Then \( W = C \). By Lemma 4.21, \( X \) is \( m \)-resolvable. Let \( \{ A_1, \ldots, A_m \} \) be a dense partition of \( X \). Define \( f : X \to W \) by \( f(x) = r_i \) when \( x \in A_i \). By [5, Lem. 5.9], \( f \) is a well-defined onto interior map.

Next suppose \( \text{depth}(T) \geq 2 \). By the inductive hypothesis, for every dense-in-itself metrizable space \( Y \), a nonempty closed discrete subspace \( Z \) of \( Y \), and a finite quasi-tree \( S \) of depth \( \text{depth}(T) \), there is an interior mapping \( g : Y \to S \) such that \( g(Z) \subseteq \text{root cluster of} \ S \). Let \( w_1, \ldots, w_n \in W \) be such that \( \{ C, R[w_1], \ldots, R[w_n] \} \) is a partition of \( W \) as depicted in Figure 3. For \( i \in \{ 1, \ldots, n \} \), let \( T_i = (W_i, R_i) \) be the generated subframe of \( T \) such that \( W_i = R[w_i] \). Then \( T_i \) is a finite quasi-tree such that \( \text{depth}(T_i) < \text{depth}(T) \) and \( C_i := R^{-1}[w_i] \) is the root cluster of \( T_i \).

By Lemma 4.13, there is a partition \( \{ G, U_1, \ldots, U_n \} \) of \( X \) such that \( G \) is a dense-in-itself closed nowhere dense subspace of \( X \) containing \( F \) and each \( U_i \) is an open subspace of \( X \) containing a nonempty discrete subspace \( F_i \) such that \( cF_i = F_i \cup G \). Since \( G \) is a dense-in-itself metrizable space, Lemma 4.21 yields a dense partition \( \{ A_1, \ldots, A_m \} \) of \( G \). Also, each \( U_i \) is a dense-in-itself metrizable space and \( F_i \) is closed relative to \( U_i \) because

\[
c_{U_i} F_i = c(F_i) \cap U_i = (F_i \cup G) \cap U_i = F_i.
\]

By the inductive hypothesis, there is an interior map \( f_i \) of \( U_i \) onto \( T_i \) such that \( f_i(F_i) \subseteq C_i \). Define \( f : X \to W \) by

\[
f(x) = \begin{cases} r_i & \text{if } x \in A_i \text{ for } i \in \{ 1, \ldots, m \}, \\ f_j(x) & \text{if } x \in U_j \text{ for } j \in \{ 1, \ldots, n \}. \end{cases}
\]
Then \( f \) is well defined since \( \{A_1, \ldots, A_m, U_1, \ldots, U_m\} \) is a partition of \( X \). It is onto because \( W = C \cup \bigcup_{i=1}^n W_i, f(G) = \bigcup_{i=1}^n f(A_i) = \bigcup_{i=1}^m \{r_i\} = C \), and \( f_i \) maps \( U_i \) onto \( W_i \). It is also clear that \( f(F) \subseteq f(G) = C \) (see Figure 5).

![Figure 5. Depiction of \( f : X \to \mathcal{T} \).](image)

To see that \( f \) is continuous, it is sufficient to show that \( f^{-1}(R[w]) \) is open in \( X \) for each \( w \in W \). If \( w \in C \), then \( R[w] = W \), and so \( f^{-1}(R[w]) = f^{-1}(W) = X \). If \( w \notin C \), then \( w \in W_i \) for a unique \( i \in \{1, \ldots, n\} \), and hence \( R[w] = R_i[w] \subseteq W_i \). Because \( f_i \) is an interior mapping, \( f_i^{-1}(R_i[w]) \) is an open subset of \( U_i \). Since \( U_i \) is open in \( X \), we conclude that \( f^{-1}(R[w]) = f_i^{-1}(R_i[w]) \) is an open subset of \( X \). Thus, \( f \) is continuous.

To see that \( f \) is open, let \( U \) be an open subset of \( X \). Recalling that \( \{G, U_1, \ldots, U_n\} \) is a partition of \( X \), we have

\[
f(U) = f\left( U \cap \left( G \cup \bigcup_{i=1}^n U_i \right) \right) = f\left( (U \cap G) \cup \bigcup_{i=1}^n (U \cap U_i) \right) = f(U \cap G) \cup \bigcup_{i=1}^n f(U \cap U_i) = f(U \cap G) \cup \bigcup_{i=1}^n f_i(U \cap U_i).
\]

Each \( U \cap U_i \) is an open subset of \( U_i \). Since \( f_i \) is interior, \( f_i(U \cap U_i) \) is an \( R_i \)-cone of \( \mathcal{T}_i \), and hence an \( R \)-cone of \( \mathcal{T} \). If \( U \cap G = \emptyset \), then \( f(U) = \bigcup_{i=1}^n f_i(U \cap U_i) \) is a union of \( R \)-cones of \( \mathcal{T} \), so is an \( R \)-cone of \( \mathcal{T} \).

Suppose \( U \cap G \neq \emptyset \). We show that \( f(U) = W \). Since \( \{A_1, \ldots, A_m\} \) is a dense partition of \( G \), it follows that \( (U \cap G) \cap A_i \neq \emptyset \) for each \( i \in \{1, \ldots, m\} \). Therefore, \( \{r_i\} = f(U \cap G \cap A_i) \subseteq f(U \cap G) \) for each \( i \in \{1, \ldots, m\} \), yielding that \( f(U \cap G) = C \). Since \( U \cap G \neq \emptyset \), we have \( U \cap cF_j \neq \emptyset \), so \( U \cap F_j \neq \emptyset \) for each \( j \in \{1, \ldots, n\} \). Let \( x_j \in U \cap F_j \). Then \( f(x_j) = f_j(x_j) \in C_j \), which is the root cluster of \( \mathcal{T}_j \). But \( f(x_j) \in f_j(U \cap U_j) \), which is an \( R_j \)-cone of \( \mathcal{T}_j \) since \( f_j \) is interior. Thus, \( f_j(U \cap U_j) = W_j \). Consequently, \( f(U) = C = \bigcup_{i=1}^n W_i = W \), and hence \( f \) is open, completing the proof.

We conclude the section by reiterating how the above delivers a modern proof of the McKinsey-Tarski Theorem that \( S4 \) is the logic of any dense-in-itself metrizable space. Let \( X \) be a dense-in-itself metrizable space. Then \( X \models S4 \). Suppose that \( S4 \not\models \varphi \). Then there is a finite quasi-tree \( \mathcal{T} \) refuting \( \varphi \). By the Mapping Lemma, \( \mathcal{T} \) is an interior image of \( X \). Thus, \( X \not\models \varphi \).

### 4.4. The hereditarily paracompact setting.

Paracompact spaces are one of the most important generalizations of metrizable spaces (and compact spaces); see, e.g., [10, Ch. 5]. However, the McKinsey-Tarski Theorem is no longer true already for hereditarily paracompact spaces. In [4, Sec. 3] a countable dense extremally disconnected subspace \( X \) of the Gleason cover of the closed real unit interval \([0,1]\) is exhibited whose logic is \( S4.3 := S4 + \Box(\Box p \to q) \lor \Box(\Box q \to p) \). Clearly \( X \) is dense-in-itself. As a countable space, \( X \) is
hereditarily Lindelöf, and hence hereditarily paracompact (e.g., by [10, Thm. 5.1.2]). Therefore, there are dense-in-itself hereditarily paracompact spaces for which the McKinsey-Tarski Theorem is no longer true.

REFERENCES


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