UNGAR'S THEOREMS ON COUNTABLE DENSE HOMOGENEITY REVISITED

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Abstract. In this paper we introduce a slightly stronger form of countable dense homogeneity that for Polish spaces can be characterized topologically in a natural way. Along the way, we generalize theorems obtained by Bennett and Ungar on countable dense homogeneity.

1. Introduction

Unless otherwise stated, all spaces under discussion are separable, metrizable and infinite.

Recall that a space X is *countable dense homogeneous* (CDH) if given any two countable dense subsets D and E of X there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$. The first result in this area is due to Cantor, who showed that the reals are CDH. Fréchet [15] and Brouwer [5], independently, proved that the same is true for the n-dimensional Euclidean space $\mathbb{R}^n$. In 1962, Fort [14] proved that the Hilbert cube is also CDH. Systematic study of CDH-spaces was initiated by Bennett [3] in 1972. He proved that strongly locally homogeneous and locally compact spaces are CDH. This was generalized by de Groot to Polish spaces in [16] and independently, but later, in [13] and [2]. The proof of Theorem 5.3 in Anderson, Curtis and van Mill [2] shows that actually something a little stronger can be proved. The homeomorphism moving one countable dense set onto the other can be chosen in such a way that it is limited by a given open cover of the space. We call a space with this property *strongly countable dense homogeneous*, abbreviated SCDH. As far as we know, all examples in the literature of CDH-spaces are in fact SCDH (see however Example 3.8 below).

The topological sum of the 1-sphere $\mathbb{S}^1$ and the 2-sphere $\mathbb{S}^2$ is an example of a CDH-space which is not homogeneous. Bennett [3] proved that for connected spaces, countable dense homogeneity implies homogeneity (see also [21, 1.6.8]). We will show in Theorem 1.4 below that Bennett's result can be generalized substantially. For locally compact spaces this was done already in 1978 by Ungar. In fact, he obtained the following interesting characterization of countable dense homogeneity among locally compact spaces.

Theorem 1.1 (Ungar [28]). *Let X be a locally compact space such that no finite set separates X. Then the following statements are equivalent:*

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(a) *X is* CDH.
(b) *X is n-homogeneous for every n.*
(c) *X is strongly n-homogeneous for every n.*

Let us comment a little on Ungar's proof. First of all, the equivalence (b) $\Leftrightarrow$ (c) follows from Corollary 3.10 in his earlier paper [27] (the assumption on local connectivity in Corollary 3.10 in [27] is superfluous since all one needs for the proof is the existence of a Polish group which makes the space under consideration n-homogeneous for all n; here a (separable metrizable) space is called *Polish* if its topology is generated by a complete metric). His proofs of the implications (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) were both based (among other things) on the well-known Effros Theorem from [7] (see also [1] and [22]) on transitive actions of Polish groups on Polish spaces.

The main aim of this paper is to investigate whether the elegant Theorem 1.1 is optimal. The question whether one can prove a similar result with the assumption of local compactness relaxed to that of completeness is a natural one in this context. In recent years it has become clear that there are delicate topological differences in the homogeneity properties of locally compact and non-locally compact Polish spaces. It is for example a trivial result that for each homogeneous locally compact space X there exists a Polish group G acting transitively on X. For Polish spaces this need not be not true, as was shown in [24]. It turns out that a transitive action by a Polish group on a Polish space is a very strong homogeneity property of that space because the Effros Theorem can be applied in that situation. Locally compact spaces have this property and the proof of Theorem 1.1 heavily depends on it. So in the light of the example in [24] it is unclear whether Theorem 1.1 can be generalized to Polish spaces. If we consider homeomorphisms that are limited by arbitrary open covers, then there is a way around the Effros Theorem.

Theorem 1.2. *For a Polish space X, the following statements are equivalent:*
(a) *X is* SCDH.
(b) *For every open cover $\mathcal{U}$ of X, every finite subset F of X and every $x \in X \setminus F$, there is a neighborhood V of x such that for all $y \in V$ there is a homeomorphism $f\colon X \to X$ that is limited by $\mathcal{U}$, restricts to the identity on F, and sends x to y.*

One should think of (b) as a strong form of n-homogeneity for all n. It is equivalent to (a) which is a strong form of countable dense homogeneity. In order to prove (b) $\Rightarrow$ (a), it is inevitable that at a certain step in the proof one has to ensure that a sequence of homeomorphisms converges to a homeomorphism. So that we run into homeomorphisms that are limited by arbitrary open covers comes as no surprise since without control one cannot make sure that the desired limit exists and is a homeomorphism. The condition in (b) about the neighborhood V is a familiar one for Effros Theorem aficionados and is needed for the standard back-and-forth proof pushing one countable dense set onto the other. So the interesting implication in Theorem 1.2 is (a) $\Rightarrow$ (b) which requires a new idea that does not depend on the Effros Theorem; in contrast, the proof of the implication (b) $\Rightarrow$ (a) is routine.

It is a little disappointing that we were not able to characterize *countable dense homogeneity* in a similar way. That we indeed did not do that in Theorem 1.2 will be demonstrated in Example 3.8 where we describe an example of a Polish CDH-space which is not SCDH. We will show that Theorem 1.2 and the Effros Theorem imply that such an example cannot be compact. We do not know whether every locally compact CDH-space is SCDH.

Corollary 1.3. *Every compact* CDH*-space is* SCDH*.*

It is also an open problem whether every compact CDH-space is strongly locally homogeneous. If so, then Corollary 1.3 is a trivial consequence of this. Kennedy [18] proved that if a continuum is 2-homogeneous, and has a nontrivial homeomorphism that is the identity on some nonempty open set, then it is strongly locally homogeneous. Hence a CDH-continuum with such a homeomorphism is strongly locally homogeneous and therefore SCDH. Simply observe that by Theorem 1.1 such a continuum is 2-homogeneous.

Let the group G act on the space X. We say that a subset H of G *makes* X CDH provided that for all countable dense subsets D and E of X there is an element $g \in H$ such that $gD = E$. So, informally speaking, H witnesses the fact that X is CDH. Similarly, we say that H makes X *n-homogeneous* provided that for all subsets F and G of X of size n there exists $g \in H$ such that $gF = G$. We finally say that H *makes X strongly n-homogeneous* if given any two n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ of distinct points of X, there exists an element $g \in H$ such that $gx_i = y_i$ for every $i \leq n$.

As was stated above, Ungar's proof of the implications (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) in Theorem 1.1 were both based on the Effros Theorem. It turns out however that the implication (a) $\Rightarrow$ (c) holds for all spaces, in essence even without connectivity assumptions. That is the new ingredient that we need in the proof of the implication (a) $\Rightarrow$ (b) in Theorem 1.2.

Theorem 1.4. *If the group G makes the space X* CDH *and no set of size $n-1$ separates X, then G makes X strongly n-homogeneous.*

Observe that this result indeed improves Bennett's result quoted above that a connected CDH-space is homogeneous. In Remark 3.6 we describe an example of a space X with very strong connectivity properties and which is strongly n-homogeneous for all n but not CDH, hence Theorem 1.4 is sharp. This space is not Polish however. It is an open problem that seems to be delicate whether there is an example of a Polish space that is strongly n-homogeneous for all n but not CDH.

For some recent results on countable dense homogeneity, see [17], [9], [23], [25].

2. Preliminaries

(A) Topology. As usual, $\mathbb{Q}$ denotes the space of *rational numbers*. If X is any countable space, then $X \times \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$. This is a consequence of the fact due to Sierpiński [26] that $\mathbb{Q}$ is topologically the unique countable space without isolated points.

Hence $\mathbb{Q}$ contains a topological copy of any countable ordinal number. Hence $\mathbb{Q}$ contains an uncountable family $\mathcal{K}$ of pairwise nonhomeomorphic compact subspaces.

A subset of a space X is called *clopen* if it is both closed and open.

A space X is called *strongly locally homogeneous* (abbreviated SLH) if it has a base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ that is supported on B (that is, f is the identity outside B) and moves x to y.

A space is *rigid* if the identity function is its only homeomorphism.

For a space X we let $\mathcal{H}(X)$ denote its group of homeomorphisms.

We say that a subset A of a space X *separates* X provided that $X \setminus A$ is disconnected.

Lemma 2.1. *Let X be CDH-space. Then the set of isolated points E of X is clopen in X and every open subspace of X that meets $X \setminus E$ is uncountable.*

Proof. If E is the set of isolated points of X and $e \in E$, then clearly

$$E = \{h(e) : h \in \mathcal{H}(X)\}.$$

Hence by [21, 1.6.7], E is a clopen subset of X.

Observe that if $X \setminus E$ is not empty, then it has no isolated points and is CDH. Hence for the second part of the lemma we may assume without loss of generality that $E = \emptyset$.

Striving for a contradiction, assume that X contains a nonempty open countable subset U. Put $V = \bigcup\{f(U) : f \in \mathcal{H}(X)\}$. Then V is clearly invariant under $\mathcal{H}(X)$. In addition, the open cover $\{f(U) : f \in \mathcal{H}(X)\}$ of V has a countable subcover. This means that V is countable since U is.

Let D be an arbitrary countable dense subset of $X \setminus V$, and fix distinct elements $v, w \in V$. Observe that V has no isolated points, hence $V \setminus \{v\}$ and $V \setminus \{v, w\}$ are both dense in V. Hence both $D \cup (V \setminus \{v\})$ and $D \cup (V \setminus \{v, w\})$ are countable dense subsets of X. There is by assumption a homeomorphism $f\colon X \to X$ such that

$$f\big(D \cup (V \setminus \{v\})\big) = D \cup (V \setminus \{v, w\}).$$

Since V is $\mathcal{H}(X)$-invariant, it follows that $f(V) = V$, hence $f(X \setminus V) \cap V = \emptyset$. But this means that $f(\{v\}) = \{v, w\}$, a contradiction. $\square$

Let $A \subseteq X$ and let $\mathcal{U}$ be an open cover of X. The *star* of A with respect to $\mathcal{U}$ is the set

$$\mathsf{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

If A is a singleton subset of X, say $A = \{x\}$, then we denote $\mathsf{St}(A, \mathcal{U})$ by $\mathsf{St}(x, \mathcal{U})$. The cover $\{\mathsf{St}(U, \mathcal{U}) : U \in \mathcal{U}\}$ is denoted by $\mathsf{St}(\mathcal{U})$. Moreover, $\mathsf{St}^2(\mathcal{U})$ denotes $\mathsf{St}\big(\mathsf{St}(\mathcal{U})\big)$, etc. We say that an open cover $\mathcal{V}$ of X is a *star-refinement* of $\mathcal{U}$ if $\mathsf{St}(\mathcal{V}) < \mathcal{U}$, i.e., if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $\mathsf{St}(V, \mathcal{V}) \subseteq U$. Every open cover admits a star-refinement, as is well-known, [8, 5.1.12].

A cover $\mathcal{V}$ of X is a *barycentric refinement* of a cover $\mathcal{U}$ of X if $\{\mathsf{St}(x, \mathcal{V}) : x \in X\}$ refines $\mathcal{U}$.

(B) The Inductive Convergence Criterion. Let X be a space with open cover $\mathcal{U}$. We say that a map $f\colon X \to X$ is *limited by* $\mathcal{U}$ if for each $x \in X$ there is an element $U \in \mathcal{U}$ containing both x and $f(x)$.

Proposition 2.2. [2, 5.1] *Suppose that X is Polish, and $\{h_n\}_n$ is a sequence of homeomorphisms of X for which there exists a sequence of open covers $\{\mathcal{U}_n\}_n$ of X such that*

(1) *$\mathcal{U}_n$ is a barycentric refinement of $\mathcal{U}_{n-1}$,*
(2) *$\mathcal{U}_n$ has mesh less than 2^{-n},*
(3) *$(h_n \circ \cdots \circ h_1)^{-1}(\mathcal{U}_n)$ has mesh less than 2^{-n},*
(4) *h_n is limited by $\mathcal{U}_n$,*

then $\lim_{n\to\infty} h_n \circ \cdots \circ h_1$ is a homeomorphism of X. (We use a complete metric on X of course.)

This is a form of the so-called *Inductive Convergence Criterion* for Polish spaces.

(C) Set theory. A cardinal is an initial ordinal, and an ordinal is the set of smaller ordinals. We use 'countable' for 'at most countable'. If X is a set and κ is a cardinal then $[X]^{<\kappa}$ and $[X]^{\kappa}$ denote $\{A \subseteq X : |A| < \kappa\}$ and $\{A \subseteq X : |A| = \kappa\}$, respectively. Hence $[X]^{<\omega}$ abbreviates the collection of all finite subsets of X.

A *cub* in ω_1 is a closed and unbounded subset of ω_1 (endowed with the order topology). A subset S of ω_1 is called *stationary* if $S \cap C \neq \emptyset$ for every cub C in ω_1. If $S \subseteq \omega_1$ is stationary, and $S = \bigcup_{n<\omega} S_n$, then for some n, S_n is stationary. For if not, then for every n there is some cub C_n in ω_1 such that $S_n \cap C_n = \emptyset$. But then $C = \bigcap_{n<\omega} C_n$ is a cub in ω_1 missing S, a contradiction. This fact will be used without explicit reference in the forthcoming.

If $S \subseteq \omega_1$ is stationary, then a function $f\colon S \to \omega_1$ is called *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in S \setminus \{0\}$. The so-called *Pressing-Down Lemma* says that if $S \subseteq \omega_1$ is stationary, and $f\colon S \to \omega_1$ is regressive, then for some $\alpha < \omega_1$, $f^{-1}(\{\alpha\})$ is stationary. For details, see [19, 6.15].

Lemma 2.3. *Let $S \subseteq \omega_1$ be stationary. If $f\colon S \to [\omega_1]^{<\omega} \setminus \{\emptyset\}$ is such that $\max f(\alpha) < \alpha$ for every $\alpha \in S$, then for some $F \in [\omega_1]^{<\omega} \setminus \{\emptyset\}$, $\{\alpha \in S : f(\alpha) = F\}$ is stationary.*

Proof. The proof is a routine application of the Pressing-Down Lemma. Indeed, let $g(\alpha) = \max f(\alpha)$ for $\alpha \in S$. Then g is regressive, hence by the Pressing-Down Lemma there exists $\lambda < \omega_1$ such that $T = g^{-1}(\{\lambda\})$ is stationary. Since $[\lambda{+}1]^{<\omega}$ is countable, it is clear that for some $F \in [\lambda{+}1]^{<\omega}$ we have that $\{\alpha \in T : f(\alpha) = F\}$ is stationary. $\square$

(D) Actions by groups. Let $a\colon G \times X \to X$ be an action of a group G on the space X. For every $g \in G$, the function $x \mapsto a(g,x)$ is a homeomorphism of X. We use gx as an abbreviation for $a(g,x)$. This notation is sometimes slightly confusing, especially if G is a group of homeomorphisms on some space. The action is called *transitive* if for all $x, y \in X$ there exists $g \in G$ such that $gx = y$. For every $x \in X$ we let Gx denote the *orbit* of x, i.e., $Gx = \{gx : g \in G\}$. If $A \subseteq X$, then

$$G_A = \{f \in G : (\forall\, x \in A)(f(x) = x)\}.$$

That is, G_A is the *stabilizer subgroup* of A.

Lemma 2.4. *Let the group G act on the infinite space X. Then the following statements are equivalent for every $n \geq 1$:*

(a) *G makes X strongly n-homogeneous.*
(b) *For every $F \in [X]^{n-1}$, the group G_F acts transitively on $X \setminus F$.*

Proof. It is clear that (a) $\Rightarrow$ (b) is trivial. We prove $(\mathrm{b})_n \Rightarrow (\mathrm{a})_n$ by induction on n. For $n = 1$ there is nothing to prove. So assume that our statement holds for $n-1$, where $n \geq 2$, and that X satisfies $(\mathrm{b})_n$. Since X is infinite it satisfies $(\mathrm{b})_{n-1}$ and hence $(\mathrm{a})_{n-1}$ by our inductive hypothesis. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be arbitrary n-tuples of distinct points of X. By what we just observed, there is an element $g_0 \in G$ such that $g_0 x_i = y_i$ for every $i \leq n-1$. Put $F = \{y_1, \ldots, y_{n-1}\}$. By $(\mathrm{b})_n$ there is an element $g_1 \in G_F$ such that $g_1 g_0 x_n = y_n$. So we conclude that $g_1 g_0 x_i = y_i$ for every $i \leq n$. ☐

Let X be a compact space. It is well-known, and easy to prove, that $\mathcal{H}(X)$ endowed with the compact-open topology is a Polish group and that the natural action $\mathcal{H}(X) \times X \to X$ is continuous. If ϱ is an admissible metric on X, then the formula

$$\hat{\varrho}(f, g) = \max\big\{\varrho\big(f(x), g(x)\big) : x \in X\big\}$$

defines an admissible metric on $\mathcal{H}(X)$. For details, see [21, §1.3].

3. Proof of Theorem 1.4

(A) Basic tools. We will first present two preliminary results that are interesting in their own rights and will be the keys in obtaining our main results.

Proposition 3.1. *Let X be a space. Suppose that G is a subset of $\mathcal{H}(X)$ that makes X* CDH. *If $F \subseteq X$ is finite and $D, E \subseteq X \setminus F$ are countable and dense in X, then there are elements $\alpha, \beta \in G$ such that $\alpha \restriction F = \beta \restriction F$ and $(\alpha^{-1} \circ \beta)(D) \subseteq E$.*

Proof. Let h_0 be an arbitrary element in G. Suppose $\{h_\beta : \beta < \alpha\} \subseteq G$ have been constructed for some $\alpha < \omega_1$. Now by CDH, pick $h_\alpha \in G$ such that

$$h_\alpha(F \cup E) = \bigcup_{\beta<\alpha} h_\beta(D). \tag{†}$$

For $1 \leq \alpha < \omega_1$, let T_α be a nonempty finite subset of α such that $h_\alpha(F) \subseteq \bigcup_{\beta \in T_\alpha} h_\beta(D)$. By Lemma 2.3 there are a stationary subset S of $\omega_1 \setminus \{0\}$ and a finite subset T of ω_1 such that for all $\lambda, \mu \in S$, $T_\lambda = T = T_\mu$. Observe that for all $\lambda \in S$,

$$h_\lambda(F) \subseteq \bigcup_{\beta \in T} h_\beta(D).$$

Since F is finite, and $\bigcup_{\beta \in T} h_\beta(D)$ is countable, and for all $\lambda \in S$ the function $h_\lambda \restriction F \colon F \to \bigcup_{\beta \in T} h_\beta(D)$ is 1-1, there are distinct $\lambda, \mu \in S$ such that $h_\lambda \restriction F = h_\mu \restriction F$.

We may assume without loss of generality that $\lambda < \mu$. Put $\alpha = h_\mu$ and $\beta = h_\lambda$, and $g = \alpha^{-1} \circ \beta$. Then, clearly, $g \in \mathcal{H}(X)_F$ and hence $g(D) \cap F = \emptyset$. Hence by (†) we consequently get

$$g(D) = (h_\mu^{-1} \circ h_\lambda)(D) = (h_\mu^{-1} \circ h_\lambda)(D) \setminus F \subseteq (F \cup E) \setminus F = E,$$

as required. □

Proposition 3.2. *Let X be a space. Suppose that G is a subset of $\mathcal{H}(X)$ that makes X* CDH*. Then for every finite $F \subseteq X$ and uncountable collection of countable subsets $\mathcal{A}$ of $X \setminus F$ that are all dense in X there are distinct $A, B \in \mathcal{A}$ and elements $\alpha, \beta \in G$ such that $\alpha \restriction F = \beta \restriction F$ and $(\alpha^{-1} \circ \beta)(A) = B$.*

Proof. Let D be an arbitrary countable dense subset of X. For every $A \in \mathcal{A}$ pick an element $f_A \in G$ such that $f_A(A \cup F) = D$. Since $\mathcal{A}$ is uncountable and D is countable, there are distinct $B, A \in \mathcal{A}$ such that $f_B \restriction F = f_A \restriction F$. Put $\alpha = f_B$ and $\beta = f_A$. Then $g = \alpha^{-1} \circ \beta$ restricts to the identity on F, and hence $g(A) = B$. □

(B) The main results and applications. We now come to our main results. In Corollary 3.5 below we prove a more general result than the one stated in Theorem 1.4.

Proposition 3.3. *Let X be a space without isolated points. Suppose that G is a subset of $\mathcal{H}(X)$ that makes X* CDH*. Then for all finite $F \subseteq X$ and uncountable $A \subseteq X \setminus F$, the set*

$$\bigcup \{(\alpha^{-1} \circ \beta)(A) : (\alpha, \beta \in G) \,\&\, (\alpha \restriction F = \beta \restriction F)\}$$

has nonempty interior.

Proof. By the Cantor-Bendixson Theorem, [8, 1.7.11], there is a subspace E of A which is homeomorphic to $\mathbb{Q}$. Let $\mathcal{K}$ be an uncountable family of pairwise nonhomeomorphic (compact) subspaces of E.

Striving for a contradiction, assume that

$$B = \bigcup \{(\alpha^{-1} \circ \beta)(A) : (\alpha, \beta \in G) \,\&\, (\alpha \restriction F = \beta \restriction F)\}$$

has empty interior. Then $X \setminus B$ is dense in X, hence we may fix a countable dense subset D of $X \setminus (B \cup F)$ (here we use that X has no isolated points). For every $K \in \mathcal{K}$, put $D_K = D \cup K$. Then $\{D_K : K \in \mathcal{K}\}$ is an uncountable family of countable dense subsets of $X \setminus F$. There are by Proposition 3.2 distinct $K_0, K_1 \in \mathcal{K}$ and elements $\alpha, \beta \in G$ such that $\alpha \restriction F = \beta \restriction F$ and $(\alpha^{-1} \circ \beta)(D \cup K_0) = D \cup K_1$. Observe that $(\alpha^{-1} \circ \beta)(K_0) \subseteq B$ and hence $(\alpha^{-1} \circ \beta)(K_0) \cap D = \emptyset$. From this we conclude that $(\alpha^{-1} \circ \beta)(K_0) \subseteq K_1$. It follows similarly that $(\beta^{-1} \circ \alpha)(K_1) \subseteq K_0$. So we conclude that $(\alpha^{-1} \circ \beta)(K_0) = K_1$. But K_0 and K_1 are not homeomorphic and hence we reached a contradiction. □

Remark 3.4. Observe that the proof of this result is based on the fact that the uncountable subset A of X contains an uncountable family of pairwise nonhomeomorphic countable subsets. In the proof we used the well-known fact that this is true for all separable metrizable spaces. The proof however works in a much broader setting. For example, if X is hereditarily Lindelöf and first countable. Since countable dense homogeneity is of limited interest outside the class of all separable metrizable spaces, there does not seem to be a point in pursuing this.

Corollary 3.5. *Let X be a space without isolated points. Assume that the group G makes X CDH. Then for every finite subset $F \subseteq X$, every G_F-invariant subset of $X \setminus F$ is open.*

Proof. Let $A \subseteq X \setminus F$ be nonempty and a G_F-orbit. By Lemma 2.1 and Proposition 3.1 it follows that A is uncountable. Hence A has nonempty interior by Proposition 3.3, and so is open being an orbit. □

Proof of Theorem 1.4. Assume that G makes the space X CDH and no set of size $n-1$ separates X. By Lemma 2.4 all we need to show is that for every $F \in [X]^{n-1}$ the group G_F acts transitively on $X \setminus F$. By Corollary 3.5 every orbit $G_F x$ for $x \in X \setminus F$ is open. Since orbits are disjoint, they are clopen. So we are done by connectivity. □

Remark 3.6. It is natural to wonder about the converse of Theorem 1.4. As we remarked in the introduction, Ungar [28] showed that in a locally compact space X such that no finite set separates X, countable dense homogeneity and strong n-homogeneity for all n are equivalent notions. Local compactness is essential in this result. In [20], an example was constructed of a bi-Bernstein set X in the plane which is strongly locally homogeneous but not CDH. It is easy to see that no finite subset of X separates X. This implies that X is strongly n-homogeneous for all n. Indeed, let $F \in [X]^{n-1}$ be arbitrary, and fix $x, y \in X \setminus F$. Let the base $\mathcal{B}$ make X SLH. Since $X \setminus F$ is connected, there is by [21, 1.5.21] a simple chain $B_1, \ldots, B_m$ of elements of $\mathcal{B}$ connecting x and y such that $\bigcup_{i=1}^m B_i \subseteq X \setminus F$. For every $i = 1, \ldots, m-1$ pick an arbitrary point $z_i \in B_i \cap B_{i+1}$. Let $z_0 = x$ and $z_m = y$. There is for every $i \leq m$ a homeomorphism f_i of X such that $f_i(z_{i-1}) = z_i$ and f_i is supported on B_i. Then $f = f_m \circ \cdots \circ f_1$ is a homeomorphism of X with $f(x) = y$ and is supported on $\bigcup_{i=1}^m B_i$, and hence restricts to the identity on F. So we are done by Lemma 2.4.

Remark 3.7. It was stated as a corollary to the main results in Ungar [28] that every open dense subset of a locally compact CDH-space is itself CDH. As Ungar mentioned in private conversation, the proof for the argument for this corollary is incomplete, and it is unclear whether it is true (see also [12, p. 2]). This generated quite some activity in the literature. Fitzpatrick and Zhou [10] proved that there is a connected, locally connected, CDH, Baire Hausdorff space with a dense, open, connected subspace that is not CDH. See Watson and Simon [29] for a completely regular space with similar properties. Fitzpatrick and Zhou [10] asked whether there is such a space that is metrizable. This question was repeated in [11, Problem 2] and specified in [11, Problem 2′]. These questions were recently answered in [25]: there is a CDH-space X containing a dense connected rigid open subset. This example is Polish but not locally compact. So the question whether Ungar's result is true remains open. This seems a rather delicate problem. Kennedy [18] proved that if a continuum is 2-homogeneous, and has a nontrivial homeomorphism that is the identity on some nonempty open set, then it is strongly locally homogeneous. Hence every open subspace of such a space is CDH.

Observe that by Corollary 3.5, if X is CDH and has no isolated points, then $X \setminus F$ for any finite subset F of X has 'many' homeomorphisms, hence is not rigid. We do not know whether $X \setminus F$ is CDH. Again, this seems to be a delicate question.

(C) A counterexample. We will now answer the obvious question whether every CDH-space is SCDH in the negative. For all undefined notions, see [25].

Example 3.8. There is a Polish convex subset X of Hilbert space ℓ^2 having an open cover $\mathcal{U}$ such that

(1) If f is a homeomorphism of X that is limited by $\mathcal{U}$, then f is the identity.
(2) No finite set in X separates X.
(3) Homeomorphisms between compact subsets of X can be extended to homeomorphisms of X (with control). Hence X is strongly n-homogeneous for every n.
(4) X is CDH.

We will show that the space in [23] is the example we are looking for. Hence there are Polish CDH-spaces that are not SCDH for a very strong reason.

Let X be a nonempty compact space. We say that a countable collection of Z-sets $\mathcal{X}$ in the Hilbert cube Q is X*-dense* if

(1) $\mathcal{X}$ is pairwise disjoint and every $X' \in \mathcal{X}$ is homeomorphic to X,
(2) for every $f \in C(X, Q)$ and $\varepsilon > 0$ there are an $X' \in \mathcal{X}$ and a homeomorphism $\alpha\colon X \to X'$ such that $\hat{\varrho}(\alpha, f) < \varepsilon$.

The basic properties of X-dense collections that are important to us are listed in the following result.

Proposition 3.9 ([23])**.** *Let X be a nonempty compact space.*

(a) *There is an X-dense collection of Z-sets in Q.*
(b) *Let $\mathcal{S}$ and $\mathcal{T}$ be X-dense collections of Z-sets in Q. Then there is an arbitrarily close to the identity homeomorphism $h\colon Q \to Q$ such that $h(\bigcup \mathcal{S}) = \bigcup \mathcal{T}$.*

Now let $\mathcal{P}$ be a Q-dense collection of Z-sets in Q $\big($Proposition 3.9(a)$\big)$, and put $Y = Q \setminus \bigcup \mathcal{P}$. Then Y is a dense G_δ-subset of Q and hence is Polish. It was shown in [23] that Y is an example of a CDH-space which is not strongly locally homogeneous. Moreover, Y is homeomorphic to a convex subset of Hilbert space ℓ^2. The space Y has many other interesting properties. For example, it has a dense rigid connected open subset, as was shown in [25]. We will show here that it has the properties of the space X that were promised in Example 3.8.

That homeomorphisms between compact subsets of Y can be extended to homeomorphisms of Y (with control), is an easy consequence of the proof of Proposition 3.3 in [23]. In this paper we are only interested in the fact that X is strongly n-homogeneous, and that follows from Theorem 1.4. So we do not bother to prove the homeomorphism extension result for compacta in detail. Moreover, since Y is the complement of a σZ-set in Q, we clearly get that no finite set separates Y. So all there remains to check is that Y has the open cover promised in Example 3.8(1). This in our opinion rather unexpected result, has a surprisingly simple proof; it is inspired by the proof of Theorem 4.1 in [25].

Let $\{Q_n : n \in \mathbb{N}\}$ be a faithful enumeration of $\mathcal{P}$.

Lemma 3.10 ([25, Lemma 3.2]). *There are a compact set K in $Q \setminus Y$ and an open base $\mathcal{B}$ for $Q \setminus K$ such that*

(1) *for every $B \in \mathcal{B}$, $\overline{B} \cap K = \emptyset$,*
(2) *for all $B, B' \in \mathcal{B}$ such that $\overline{B} \cap \overline{B'} = \emptyset$, there exists n such that $Q_n \cap K \neq \emptyset$, $Q_n \cap B \neq \emptyset$ but $Q_n \cap B' = \emptyset$.*

Now let $\mathcal{V}$ be a so-called Dugundji cover for Q and K, [21, 1.2.1]. That is, $\mathcal{V}$ is a locally finite open cover of $Q \setminus K$ such that

(D) $$\text{if } V_n \in \mathcal{V} \text{ for every } n \text{ and } \lim_{n\to\infty} \varrho(V_n, K) = 0, \text{ then } \lim_{n\to\infty} \operatorname{diam}(V_n) = 0.$$

Put $\mathcal{U} = \{V \cap Y : V \in \mathcal{V}\}$.

Theorem 3.11. *If h is a homeomorphism of Y that is limited by $\mathcal{U}$, then h is the identity on Y.*

Proof. We first note that h 'permutes' $\mathcal{P}$.

Claim 1 ([23, 4.2]). There is a bijection $\alpha \colon \mathcal{P} \to \mathcal{P}$ such that for every $A \subseteq Y$ and $P \in \mathcal{P}$, if $\overline{A} \cap P \neq \emptyset$ then $\overline{h(A)} \cap \alpha(P) \neq \emptyset$ (here closure means closure in Q).

Assume that for n we have $Q_n \cap K \neq \emptyset$, and let $p \in K \cap Q_n$. There is a sequence $(x_i)_i$ in Y converging to p. For every n let $V_n \in \mathcal{V}$ be such that $\{x_n, h(x_n)\} \subseteq V_n$. Observe that $\lim_{n\to\infty} \varrho(V_n, K) \leq \lim_{n\to\infty} \varrho(x_n, K) = 0$, hence $\lim_{n\to\infty} \operatorname{diam}(V_n) = 0$ by (D) above. From this we conclude that $(h(x_i))_i$ converges to p, i.e., $\alpha(Q_n) = Q_n$ by Claim 1.

Striving for a contradiction, assume that there exists $x \in X$ such that $h(x) \neq x$. Since $\{x, h(x)\} \cap K = \emptyset$, there are $B, B' \in \mathcal{B}$ such that $x \in B$, $h(B \cap X) \subseteq B' \cap X$, and $\overline{B} \cap \overline{B'} = \emptyset$. By Lemma 3.10 there exists n such that $Q_n \cap K \neq \emptyset$, $Q_n \cap B \neq \emptyset$ but $Q_n \cap \overline{B'} = \emptyset$. Pick a sequence $(x_i)_i$ in $B \cap X$ such that it converges to an element of $Q_n \cap \overline{B}$. By Claim 1, the sequence $\big(h(x_i)\big)_i$ has a cluster point in $\alpha(Q_n)$. But as we just saw, $\alpha(Q_n) = Q_n$ and all cluster points of $\big(h(x_i)\big)_i$ are contained in $\overline{B'}$ which is disjoint from Q_n. This is a contradiction. $\square$

Question 3.12. Is there is an example of a Polish space that is strongly n-homogeneous for all n but not CDH?

Remark 3.13. That our space Y has the property stated in Theorem 3.11 was observed independently also by Dobrowolski in [6].

4. Proof of Theorem 1.2 and Corollary 1.3

It will be convenient to introduce some notation. If X is a space and $\mathcal{U}$ is an open cover of X, then put $\mathcal{H}(X; \mathcal{U}) = \{f \in \mathcal{H}(X) : f \text{ is limited by } \mathcal{U}\}$.

Lemma 4.1. *Let X be a space without isolated points. Let $\mathcal{U}$ be an open cover of X. Suppose that $\mathcal{H}(X;\mathcal{U})$ makes X CDH. Then for all (possibly empty) finite $F \subseteq X$ and $x \in X \setminus F$ there is an open neighborhood V of x such that for every $y \in V$ there is an element $f \in \mathcal{H}(X)_F$ such that $f(x) = y$ and f is limited by $\mathsf{St}^4(\mathcal{U})$.*

Proof. Denote $\mathcal{H}(X;\mathcal{U})$ by G. Put

$$H = \{\alpha^{-1} \circ \beta : (\alpha, \beta \in G) \,\&\, (\alpha{\restriction}F = \beta{\restriction}F)\}$$

and

$$A = \{h(x) : h \in H\},$$

respectively. Then A is uncountable by Lemma 2.1 and Proposition 3.1. Put

$$B = \{h(a) : a \in A, h \in H\}.$$

Then B has nonempty interior by Proposition 3.3. Let W be the interior of B, and pick an arbitrary element $p \in W$. Then are $h_0, h_1 \in H$ such that $p = h_1\big(h_0(x)\big)$. Put

$$V = (h_1 \circ h_0)^{-1}(W).$$

Then V is an open neighborhood of x, and we claim that it is as required. To this end, pick an arbitrary element y in V. There are $\xi_0, \xi_1 \in H$ such that

$$(\xi_1 \circ \xi_0)(x) = (h_1 \circ h_0)(y).$$

As a consequence, $(h_0^{-1} \circ h_1^{-1} \circ \xi_1 \circ \xi_0)(x) = y$. Observe that $f = h_0^{-1} \circ h_1^{-1} \circ \xi_1 \circ \xi_0 \in \mathcal{H}(X)_F$ and that every $h \in H$ is limited by $\mathsf{St}(\mathcal{U})$. This means that f is limited by $\mathsf{St}^4(\mathcal{U})$. ☐

Proof of Theorem 1.2. We first prove (a) $\Rightarrow$ (b). Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ such that $\mathsf{St}^4(\mathcal{V}) < \mathcal{U}$. By (a), the homeomorphisms of X that are limited by $\mathcal{V}$ make X CDH. Hence we are done by Lemma 4.1.

Finally observe that (b) $\Rightarrow$ (a) follows from Proposition 2.2 by adapting the standard back-and-forth proof that locally compact and SLH-spaces are CDH. For the convenience of the reader, we will provide the details.

To this end, let $\mathcal{U}$ be an arbitrary open cover of X. There is by [4, Theorem 4.1] an admissible complete metric ϱ on X such that the family of all open ϱ-balls of radius 1 forms a refinement of $\mathcal{U}$.

Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be faithfully indexed dense subsets of X. The hypothesis (b) implies that if E is a neighborhood of a point $x \in X$, and $F \subseteq X \setminus \{x\}$ is finite and $G \subseteq X$ is dense, then there exists for every open cover $\mathcal{V}$ of X a homeomorphism f of X which is limited by $\mathcal{V}$, restricts to the identity on F and takes x into $G \cap E$.

Let h_1 denote the identity function on X, and let $\mathcal{V}_1$ be the family of all open ϱ-balls of radius $1/8$. We now construct a sequence $(h_i)_{i\geq 2}$ of homeomorphisms of X and a sequence $\{\mathcal{V}_i\}_{i\geq 2}$ of open covers of X such that for each $i \geq 1$ the following conditions are satisfied:

(1) $h_n \circ \cdots \circ h_1(a_i) = h_{2i} \circ \cdots \circ h_1(a_i) \in B$ for each $n \geq 2i$,
(2) $(h_n \circ \cdots \circ h_1)^{-1}(b_i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(b_i) \in A$ for each $n \geq 2i + 1$,
(3) if $i \geq 2$, then $\mathcal{V}_i$ is a barycentric refinement of $\mathcal{V}_{i-1}$,

(4) $\mathcal{V}_i$ has mesh less than $2^{-(i+1)}$,
(5) $(h_i \circ \cdots \circ h_1)^{-1}(\mathcal{V}_i)$ has mesh less than $2^{-(i+1)}$,
(6) h_i is limited by $\mathcal{V}_i$.

Assume $h_1, \ldots, h_{2i-1}$ and $\mathcal{V}_1, \ldots, \mathcal{V}_{2i-1}$ have been defined for certain $i \geq 1$.

Let $\mathcal{V}_{2i}$ be an open cover of X such that

(7) $\mathcal{V}_{2i}$ has mesh less than $2^{-(2i+1)}$,
(8) $(h_{2i-1} \circ \cdots \circ h_1)^{-1}\big(\mathsf{St}(\mathcal{V}_{2i})\big)$ has mesh less than $2^{-(2i+1)}$,
(9) $\mathsf{St}(\mathcal{V}_{2i})$ refines $\mathcal{V}_{2i-1}$.

If $h_{2i-1} \circ \cdots \circ h_1(a_i) \in B$, take h_{2i} the identity function on X. Otherwise, choose a small neighborhood U_{2i} of $h_{2i-1} \circ \cdots \circ h_1(a_i)$ which is disjoint from the finite set

$$K = \{b_1, \ldots, b_{i-1}\} \cup h_{2i-1} \circ \cdots \circ h_1(\{a_1, \ldots, a_{i-1}\}).$$

Take h_{2i} to be a homeomorphism of X which is limited by $\mathcal{V}_{2i}$ such that

$$h_{2i} \circ h_{2i-1} \circ \cdots \circ h_1(a_1) \in B \cap U_{2i}$$

and restricts to the identity on K. It is clear that the only thing we need to verify is $(5)_{2i}$. Indeed, if $W \in \mathcal{V}_{2i}$, then since h_{2i} is limited by $\mathcal{V}_{2i}$ we clearly have that $h_{2i}^{-1}(W) \subseteq \mathsf{St}(W, \mathcal{V}_{2i})$. From this we get by (8) that

$$\operatorname{diam}(h_{2i} \circ \cdots \circ h_1)^{-1}(W) < 2^{-(2i+1)},$$

as required.

Let $\mathcal{V}_{2i+1}$ be an open cover of X such that

(10) $\mathcal{V}_{2i+1}$ has mesh less than $2^{-(2i+2)}$,
(11) $(h_{2i} \circ \cdots \circ h_1)^{-1}\big(\mathsf{St}(\mathcal{V}_{2i+1})\big)$ has mesh less than $2^{-(2i+2)}$,
(12) $\mathsf{St}(\mathcal{V}_{2i+1})$ refines $\mathcal{V}_{2i}$.

If $(h_{2i} \circ \cdots \circ h_1)^{-1}(b_i) \in A$, take h_{2i+1} the identity function on X. Otherwise, choose a small neighborhood U_{2i+1} of b_i which is disjoint from the finite set

$$L = \{b_1, \ldots, b_{i-1}\} \cup h_{2i} \circ \cdots \circ h_1(\{a_1, \ldots, a_i\}).$$

Take h_{2i+1} to be a homeomorphism of X which is limited on $\mathcal{V}_{2i+1}$ such that

$$h_{2i+1}^{-1}(b_i) \in (h_{2i} \circ \cdots \circ h_1)(A) \cap U_{2i+1}$$

and restricts to the identity on L. It is clear that the only thing we need to verify is $(5)_{2i+1}$. Indeed, if $W \in \mathcal{V}_{2i+1}$, then since h_{2i+1} is limited by $\mathcal{V}_{2i+1}$ we clearly have that $h_{2i+1}^{-1}(W) \subseteq \mathsf{St}(W, \mathcal{V}_{2i+1})$. From this we get by (11) that

$$\operatorname{diam}(h_{2i+1} \circ \cdots \circ h_1)^{-1}(W) < 2^{-(2i+2)},$$

as required.

By Proposition 2.2 we have that $h = \lim_{n\to\infty} h_n \circ \cdots \circ h_1$ is a homeomorphism of X. The conditions (1) and (2) insure that $h(A) = B$. Finally, if $x \in X$ is arbitrary, then by (4) and (6) we get

$$\varrho\big(x, h(x)\big) \leq \sum_{i=1}^{\infty} 2^{-(i+1)} = {}^1\!/\!_2.$$

Since x was arbitrary, this proves that h is limited by $\mathcal{U}$. □

Proof of Corollary 1.3. We may clearly assume that X is infinite. The set E of isolated points of X is clopen and discrete by Lemma 2.1. A moments reflection shows that we may assume without loss of generality that $E = \emptyset$.

We will verify that X satisfies the condition in Theorem 1.2(b). To this end, let $F \subseteq X$ be finite, take an arbitrary $x \in X \setminus F$, and let $\varepsilon > 0$. Our task is to find an open neighborhood P of x in X such that for every $a \in P$ there exists an element $h \in \mathcal{H}(X)_F$ such that $h(x) = a$ and h moves no point more than ε. As to be expected, this is a consequence of the Effros Theorem from [7].

The Polish group $G = \mathcal{H}(X)_F$ acts on the locally compact space $X \setminus F$ and its orbits are clopen subsets of $X \setminus F$ by Corollary 3.5. As a consequence, the set $U = Gx$ is an open subset of X on which G acts transitively. In addition, U is Polish being locally compact. Let O be an open neighborhood of x in X such that the compact set $\overline{O}$ is contained in U. Let $V = \{f \in \mathcal{H}(X)_F : \hat{\varrho}(f, 1_X) < {}^1\!/\!_2\varepsilon\}$. By the Effros Theorem, the set Vz is open in Gx for every $z \in Gx$. Let $\delta > 0$ be a Lebesgue number for the open cover $\{Vz : z \in \overline{O}\}$ of $\overline{O}$. Now let P be an open neighborhood of x such that $\operatorname{diam} P < \delta$ and $P \subseteq O$. Pick an arbitrary element $a \in P$. There exists $z \in \overline{O}$ such that $x, a \in Vz$. Pick $f, g \in V$ such that $f(z) = x$ and $g(z) = a$. Then $h = g \circ f^{-1}$, restricts to the identity on F, sends x onto a and $\hat{\varrho}(h, 1_X) < \varepsilon$. □

In the light of Corollary 1.3 and Example 3.8, the following question is quite natural.

Question 4.2. Let X be a locally compact CDH-space. Is X SCDH?

References

[1] F. D. Ancel, *An alternative proof and applications of a theorem of E. G. Effros*, Michigan Math. J. **34** (1987), 39–55.
[2] R. D. Anderson, D. W. Curtis, and J. van Mill, *A fake topological Hilbert space*, Trans. Amer. Math. Soc. **272** (1982), 311–321.
[3] R. Bennett, *Countable dense homogeneous spaces*, Fund. Math. **74** (1972), 189–194.
[4] C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, PWN—Polish Scientific Publishers, Warsaw, 1975, Monografie Matematyczne, Tom 58.
[5] L. E. J. Brouwer, *Some remarks on the coherence type η*, Proc. Akad. Amsterdam **15** (1913), 1256–1263.
[6] T. Dobrowolski, *A Polish AR-space with no nontrivial isotopy*, to appear in Bull. Polon. Acad. Sci. Sér. Math. Astronom. Phys.
[7] E. G. Effros, *Transformation groups and C^*-algebras*, Annals of Math. **81** (1965), 38–55.
[8] R. Engelking, *General topology*, Heldermann Verlag, Berlin, second ed., 1989.

[9] I. Farah, M. Hrušák, and C. Martínez Ranero, *A countable dense homogeneous set of reals of size* $\aleph_1$, Fund. Math. **186** (2005), 71–77.
[10] B. Fitzpatrick, Jr. and Zhou Hao-xuan, *Densely homogeneous spaces. II*, Houston J. Math. **14** (1988), 57–68.
[11] B. Fitzpatrick, Jr. and Zhou Hao-xuan, *Some open problems in densely homogeneous spaces*, Open problems in topology (J. van Mill and G. M. Reed, eds.), North-Holland, Amsterdam, 1990, pp. 251–259.
[12] B. Fitzpatrick, Jr. and Zhou Hao-xuan, *Countable dense homogeneity and the Baire property*, Topology Appl. **43** (1992), 1–14.
[13] P. Fletcher and R. A. McCoy, *Conditions under which a connected representable space is locally connected*, Pacific J. Math. **51** (1974), 433–437.
[14] M. Fort, *Homogeneity of infinite products of manifolds with boundary*, Pac. J. Math. **12** (1962), 879–884.
[15] M. Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann. **68** (1910), 145–168.
[16] J. de Groot, *Topological Hilbert space and the drop-out effect*, Rapport ZW 1969-016, Mathematisch Centrum, Amsterdam, 1969.
[17] M. Hrušák and B. Zamora Avilés, *Countable dense homogeneity of definable spaces*, Proc. Amer. Math. Soc. **133** (2005), 3429–3435.
[18] J. Kennedy, *A condition under which 2-homogeneity and representability are the same in continua*, Fund. Math. **121** (1984), 89–98.
[19] K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, 1980.
[20] J. van Mill, *Strong local homogeneity does not imply countable dense homogeneity*, Proc. Amer. Math. Soc. **84** (1982), 143–148.
[21] J. van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Publishing Co., Amsterdam, 2001.
[22] J. van Mill, *A note on the Effros Theorem*, Amer. Math. Monthly. **111** (2004), 801–806.
[23] J. van Mill, *On countable dense and strong local homogeneity*, Bull. Polon. Acad. Sci. Sér. Math. Astronom. Phys. **53** (2005), 401–408.
[24] J. van Mill, *Homogeneous spaces and transitive actions by Polish groups*, 2006, to appear in Israel J. Math.
[25] J. van Mill, *A countable dense homogeneous space with a dense rigid open subspace*, 2007, to appear in Fund. Math.
[26] W. Sierpiński, *Sur une propriété topologique des ensembles dénombrables denses en soi*, Fund. Math. **1** (1920), 11–16.
[27] G. S. Ungar, *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc. **212** (1975), 393–400.
[28] G. S. Ungar, *Countable dense homogeneity and n-homogeneity*, Fund. Math. **99** (1978), 155–160.
[29] S. Watson and P. Simon, *Open subspaces of countable dense homogeneous spaces*, Fund. Math. **141** (1992), 101–108.

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