An equivariant version of Lehmer's Mahler measure problem

Jan-Willem van Ittersum

Utrecht University
Max Planck Institute for Mathematics

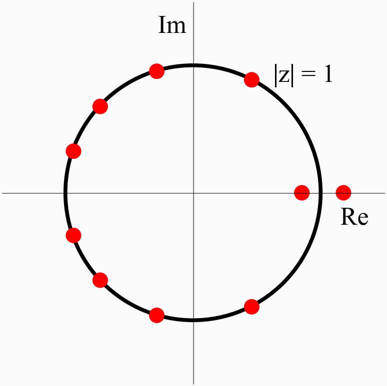
What is Lehmer's conjecture on the Mahler measure?

Definition Given

$$f(x) = a_n \prod_{i=1}^{n} (x - \alpha_i) \in \mathbb{Z}[x] \backslash \{0\},$$

the Mahler measure of f is given by

$$M(f) = |a_n| \prod_{i=1}^{n} \max(|\alpha_i|, 1).$$



The roots of f

What is Lehmer's conjecture on the Mahler measure?

Definition Given

$$f(x) = a_n \prod_{i=1}^{n} (x - \alpha_i) \in \mathbb{Z}[x] \backslash \{0\},$$

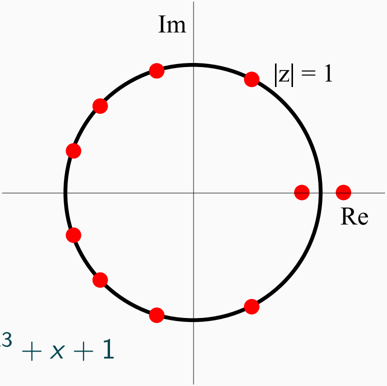
the Mahler measure of f is given by

$$M(f) = |a_n| \prod_{i=1}^{n} \max(|\alpha_i|, 1).$$

Example

$f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$

$M(f) = 1.1762808\ldots$



The roots of f

What is Lehmer's conjecture on the Mahler measure?

Definition Given

$$f(x) = a_n \prod_{i=1}^{n} (x - \alpha_i) \in \mathbb{Z}[x] \backslash \{0\},$$

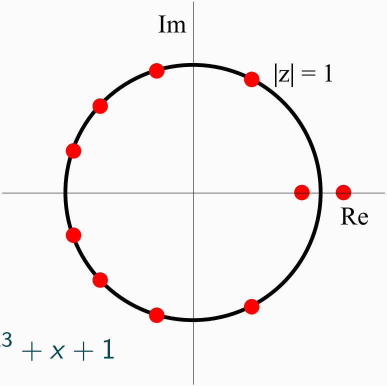
the Mahler measure of f is given by

$$M(f) = |a_n| \prod_{i=1}^{n} \max(|\alpha_i|, 1).$$

Example

$f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$

$M(f) = 1.1762808\ldots$



The roots of f

Conjecture (Lehmer, 1933)

There exists a lower bound $\mu > 1$ such that for all non-zero $f \in \mathbb{Z}[x]$ it holds that

$$M(f) = 1 \quad \text{or} \quad M(f) \geq \mu.$$

An equivariant version of Lehmer's conjecture

For simplicity, restrict to f with coprime coefficients.

An equivariant version of Lehmer's conjecture

For simplicity, restrict to f with coprime coefficients.

Let $\gamma = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PGL}_2(\mathbb{Q})$, the group of rational Möbius transformations, act on $f \in \mathbb{Z}[x]$ of degree k by

$$f^{\gamma}(z) := (cz + d)^k f(\gamma z)$$

taking the representative for γ such that f^{γ} has coprime coefficients.

An equivariant version of Lehmer's conjecture

For simplicity, restrict to f with coprime coefficients.

Let $\gamma = \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] \in \mathrm{PGL}_2(\mathbb{Q})$, the group of rational Möbius transformations, act on $f \in \mathbb{Z}[x]$ of degree k by

$$f^{\gamma}(z) := (cz + d)^k f(\gamma z)$$

taking the representative for γ such that f^{γ} has coprime coefficients.

Theorem (vI)

For every finite subgroup $G < \mathrm{PGL}_2(\mathbb{Q})$ for which $GS^1 \neq S^1$, there exists a computable $\mu > 1$ such that

$$\prod_{\gamma \in G} M(f^{\gamma}) = 1 \quad \textit{or} \quad \prod_{\gamma \in G} M(f^{\gamma}) \geq \mu^k.$$

An equivariant version of Lehmer's conjecture

For simplicity, restrict to f with coprime coefficients.

Let $\gamma = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PGL}_2(\mathbb{Q})$, the group of rational Möbius transformations, act on $f \in \mathbb{Z}[x]$ of degree k by

$$f^\gamma(z) := (cz + d)^k f(\gamma z)$$

taking the representative for γ such that f^γ has coprime coefficients.

Theorem (vI)

For every finite subgroup $G < \mathrm{PGL}_2(\mathbb{Q})$ for which $GS^1 \neq S^1$, there exists a computable $\mu > 1$ such that

$$\prod_{\gamma \in G} M(f^\gamma) = 1 \quad \text{or} \quad \prod_{\gamma \in G} M(f^\gamma) \geq \mu^k.$$

Example (Zagier) $G = \langle z \mapsto 1 - z \rangle$, then $\mu = \sqrt{\varphi} = \sqrt{\frac{1+\sqrt{5}}{2}}$.