Linear Algebra: Assignment 5

Exercise 1

Find all Eigenvalues and corresponding Eigenspaces of the following matrices:

[2 + 4 pts]

(a)

$$A = \begin{pmatrix} 2 & 1 \\ 15 & 4 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Exercise 2

(a) Does every $n \times n$ matrix necessarily have at least one (real) Eigenvalue? Explain. [1 pt]

(b) Does every 3×3 matrix necessarily have at least one Eigenvalue? Explain. [1 pt]

(c) Suppose an 8×8 matrix A has three Eigenvalues: λ_1, λ_2 and λ_3. Suppose the dimension of the Eigenspace of λ_1 is 3 and the dimension of the Eigenspace of λ_2 is 4. Is this information sufficient to conclude that A is diagonalizable? Explain. [2 pts]

(d) Suppose an 7×7 matrix A has three Eigenvalues: λ_1, λ_2 and λ_3. Suppose the dimension of the Eigenspace of λ_1 is 2 and the dimension of the Eigenspace of λ_2 is 3. What are the possible values of the dimension of $\text{Eig}(\lambda_3)$? Can we conclude that A is diagonalizable? Explain. [2 pts]

Exercise 3 (*population dynamics*)

In this question we will see a real-world application of the theory. In the Netherlands, some people live in big cities, and some live in small towns. People sometimes move from one to the other: for example, a young person who grows up in a small town might move to the city for university.

Let x_n be the fraction of people living in cities in year n, and let y_n be the fraction of people living in small towns (so that $x_n + y_n = 1$). We can record both of these in a vector:

$$\vec{v}_n = \left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix} \right)$$

Suppose that census data shows that the change in successive years is given by a matrix:

$$\vec{v}_{n+1} = A\vec{v}_n$$

where

$$A = \begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix}.$$

(a) What is the real-world meaning of the bottom left entry of the matrix? [1 pts]

(b) Diagonalize A: that is, find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$ [3 pts]

(c) After some years have passed, it is noticed that $\vec{v_n}$ has stopped changing: $\vec{v}_{2026} = \vec{v}_{2027} = \vec{v}_{2028} = \cdots = \vec{v}$. Calculate this 'steady state' vector $\vec{v}$, and hence give the fraction of people living in big cities in 2026 (and subsequent years). [2 pts]

Exercise 4 Match each linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the eigenvalues of its matrix.

You do not need to justify your answers. [2 pts]

Eigenvalues:

(a) None

(b) -1

(c) 2

(d) 0.5

(e) 1 and -1

(f) 1 and 0

(g) 1

Linear transformation:

(1) Stretching by a factor of 2. (i.e. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$).

(2) Reflection in the line $y = x$.

(3) Projection onto the x-axis (i.e. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$).

(4) A clockwise rotation by 90°.

(5) Shrinking by a factor of 2.

(6) The identity transformation.

(7) A rotation by 180°.