

Exercise 1

(a) Let W be the subspace of \mathbb{R}^4 spanned by

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

Find a basis for the orthogonal complement W^\perp . [2 pts]

(b) Give an example of a subspace H of \mathbb{R}^{2026} such that $\dim(H) = \dim(H^\perp)$. [2 pts]

(c) Prove that there exists no subspace $V \subseteq \mathbb{R}^{2026}$ which satisfies $V = V^\perp$. [3 pts]

Exercise 2

An orthonormal matrix is a matrix whose columns are orthogonal and normal. In this exercise, A is an orthonormal matrix and \vec{x} and \vec{y} are vectors.

(a) Prove that $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$. [3 pts]

(b) Conclude from (a) that $\|A\vec{x}\| = \|\vec{x}\|$. [1 pt]

(c) Also, conclude from (a) that $\vec{x} \perp \vec{y}$ implies $A\vec{x} \perp A\vec{y}$. [1 pt]

Exercise 3

(a) Find the distance between the point $\vec{y} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ and the plane $W \subseteq \mathbb{R}^3$ given by [3 pts]

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

(b) Let L be the line in \mathbb{R}^2 with equation $y = 3x$. Find the *orthogonal projection matrix* P onto the line L , i.e., the 2×2 matrix P so that for all $\vec{x} \in \mathbb{R}^2$ we have [3 pts]

$$P\vec{x} = \text{Proj}_L(\vec{x}).$$

(c) Compute P^{2026} . [2 pts]