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Integer partitions detect the primes

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We show that integer partitions, the fundamental building blocks in additive number theory, detect prime numbers in an unexpected way. Answering a question of Schneider, we show that the primes are the solutions to special equations in partition functions. For example, an integer $n \geq 2$ is prime if and only if

$$(3n^3 - 13n^2 + 18n - 8)M_1(n) + (12n^2 - 120n + 212)M_2(n) - 960M_3(n) = 0,$$

where the $M_a(n)$ are MacMahon's well-studied partition functions. More generally, for MacMahonesque partition functions $M_{\overrightarrow{a}}(n)$, we prove that there are infinitely many such prime detecting equations with constant coefficients, such as

$$80M_{(1,1,1)}(n) - 12M_{(2,0,1)}(n) + 12M_{(2,1,0)}(n) + \cdots - 12M_{(1,3)}(n) - 39M_{(3,1)}(n) = 0.$$

partitions | primes | quasimodular forms

Significance

Integer partitions arise naturally in additive number theory, algebraic geometry, combinatorics, mathematical physics, and representation theory. We have identified a surprising role for partitions in multiplicative number theory. The prime numbers, the building blocks of multiplicative number theory, are the solutions of infinitely many special "Diophantine equations" in well-studied partition functions. In other words, integer partitions detect the primes in infinitely many natural ways.

1. Introduction and Statement of Results

Diophantine sets S are subsets of $\mathbb{Z}_+^n$ that arise from the positive integer solutions of a Diophantine equation. For example, the set of composite numbers is Diophantine, which is easily seen by considering the polynomial equation

$$C(x_1, x_2, x_3) := x_1 - (x_2 + 1)(x_3 + 1) = 0.$$

Indeed, we have that $S_C := \{x_1 \in \mathbb{Z}_+ \ : \ C(x_1, x_2, x_3) = 0 \text{ with } x_2, x_3 \in \mathbb{Z}_+\}$ is the set of composite numbers. In 1970, Matiyasevich (1) famously proved that a set is Diophantine if and only if it is computably enumerable. Building on work by Davis, Putnam, and Robinson, this result resolved Hilbert's tenth problem in the negative: there is no algorithm that determines whether a generic Diophantine equation has integer solutions.

In contrast to the simple case of the composite numbers, Matiyasevich's work implies that the set of primes is Diophantine, and even more, is polynomially representable. In 1976, Jones, Sato, Wada, and Wiens (2) made this explicit; they produced a degree 25 polynomial in 26 variables whose positive values, as the variables vary over nonnegative integers, is the set of primes.

In analogy, we consider whether the set of primes is partition theoretic, by which we mean solutions to equations in partition functions (for background on partitions see ref. 3). This is a question of Schneider, as an example of his general philosophy and research program where classical number theoretic topics (e.g., multiplicative number theory, prime number distribution, and zeta-functions) are informed by partition theory [for example, see refs. 4 and 5 and his works on arithmetic densities (6) and partition zeta-functions (7)].

We show that the primes are partition theoretic in infinitely many ways. Our first glimpse of this phenomenon arises from MacMahon's (8) q-series:

$$\mathcal{U}_a(q) = \sum_{n \geq 1} M_a(n)\, q^n := \sum_{0 < s_1 < s_2 < \cdots < s_a} \frac{q^{s_1 + s_2 + \cdots + s_a}}{(1 - q^{s_1})^2 (1 - q^{s_2})^2 \cdots (1 - q^{s_a})^2}. \quad [1]$$

For $a \geq 1$, the partition function $M_a(n)$ sums the products of the multiplicities of partitions of n with a different part sizes. To be precise, we have

$$M_a(n) = \sum_{\substack{0 < s_1 < s_2 < \cdots < s_a \\ n = m_1 s_1 + m_2 s_2 + \cdots + m_a s_a}} m_1 m_2 \cdots m_a. \quad [2]$$

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We find linear equations in these functions, with polynomial coefficients, that detect the primes.

Theorem 1. *The following are true:*
(1) For positive integers n, we have

$$(n^2 - 3n + 2)M_1(n) - 8M_2(n) \geq 0,$$

and for $n \geq 2$, this expression vanishes if and only if n is prime.
(2) For positive integers n, we have

$$(3n^3 - 13n^2 + 18n - 8)M_1(n) + (12n^2 - 120n + 212)M_2(n) - 960M_3(n) \geq 0,$$

and for $n \geq 2$, this expression vanishes if and only if n is prime.

Remark 2: The expressions in Theorem 1 vanish trivially when $n = 1$.

Remark 3: By taking suitable linear combinations (with polynomial coefficients) of the two expressions in Theorem 1, it is simple to produce infinitely many further examples. For example, we have

$$(3n^4 - 3n^3 - 17n^2 + 27n - 10)M_1(n) - (240n - 400)M_2(n) - 1920M_3(n).$$

There are more examples of such prime detecting expressions involving the $M_a(n)$ functions.

Theorem 4. *Suppose that $E(n)$ is in Table 1 in the Appendix. For positive integers n, we have $E(n) \geq 0$, and for $n \geq 2$, we have $E(n) = 0$ if and only if n is prime.*

Based on a computer search for further such prime detecting partition equations, we believe that this list generates all such expressions involving MacMahon's $M_a(n)$ partition functions.

Conjecture 1. *Let $\vec{P}(x) := (p_1(x), p_2(x), \ldots, p_a(x)) \in \mathbb{Z}[x]^a$ be a vector of relatively prime integer polynomials. For integers $n \geq 2$, suppose that*

$$E(n) = p_1(n)M_1(n) + p_2(n)M_2(n) + \cdots + p_a(n)M_a(n) \geq 0,$$

and vanishes precisely on the primes. Then $E(n)$ is a $\mathbb{Q}[n]$-linear combination of entries in Table 1 in the Appendix.

Despite this conjecture, we show that Theorems 1 and 4 are glimpses of a rich infinite family of such expressions. To make this precise, we define natural generalizations of MacMahon's $M_a(n)$. For $a \geq 1$ and a vector $\vec{a} = (v_1, v_2, \ldots, v_a) \in \mathbb{N}^a$, we define the MacMahonesque partition function*

$$M_{\vec{a}}(n) := \sum_{\substack{0<s_1<s_2<\cdots<s_a \\ m_1,\ldots,m_a>0 \\ n=m_1s_1+m_2s_2+\cdots+m_as_a}} m_1^{v_1} m_2^{v_2} \cdots m_a^{v_a}, \qquad [3]$$

certain sums of degree $|\vec{a}| := v_1 + \cdots + v_a$ monomials in the part multiplicities.

Remark 5: We note that $M_a(n) \neq M_{(a)}(n)$, when $a > 1$. Instead, we have $M_a(n) = M_{\vec{a}}(n)$, where $\vec{a} = (1, \ldots, 1) \in \mathbb{N}^a$.

For these generalized MacMahon functions, we establish the existence of infinitely many linear equations with integer coefficients whose set of solutions, where $n \geq 2$, is the set of primes.

Theorem 6. *If $d \geq 4$, then the following are true.*
1) There are relatively prime integers $c_{\vec{a}}$ such that for positive integers n we have

$$\sum_{|\vec{a}| \leq d} c_{\vec{a}} M_{\vec{a}}(n) \geq 0,$$

and for $n \geq 2$, this expression vanishes if and only if n is prime.
2) As $d \to \infty$, there are $\gg d^2$ many linearly independent expressions satisfying (1).

Example 1. We offer two examples of Theorem 6. For positive integers n, we find that

$$\begin{aligned}\Psi_1(n) := {} & 63M_{(2,2)}(n) - 12M_{(3,0)}(n) - 39M_{(3,1)}(n) - 12M_{(1,3)}(n) \\ & + 80M_{(1,1,1)}(n) - 12M_{(2,0,1)}(n) + 12M_{(2,1,0)}(n) + 12M_{(3,0,0)}(n) \geq 0,\end{aligned}$$

and

$$\begin{aligned}\Psi_2(n) := {} & 14M_{(1)}(n) - 15M_{(2)}(n) - 2M_{(3)}(n) + 3M_{(4)}(n) + 30M_{(2,0)}(n) - 72M_{(3,0)}(n) \\ & - 36M_{(2,1)}(n) - 6M_{(4,0)}(n) - 12M_{(3,1)}(n) + 72M_{(2,1,0)}(n) + 72M_{(3,0,0)}(n) \geq 0.\end{aligned}$$

For $n \geq 2$ these expressions vanish if and only if n is prime. Finally, we note the nontrivial identity $\Psi_2(n) = \frac{36}{11}\Psi_1(n)$.

*These functions were also considered in Bachmann's master's thesis (see Chapter 5 of ref. 9).

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In addition to the equations covered by Theorem 6, there are many prime detecting equations involving convolutions of the MacMahonesque functions. For example, one can show that

$$\Psi_3(n) := 10M_{(1)}(n) - 17M_{(3)}(n) + 7M_{(5)}(n) + 12\sum_{i+j=n} M_{(1)}(i)\Big(M_{(1)}(j) - 10M_{(3)}(j)\Big) + 96\sum_{i+j+\ell=n} M_{(1)}(i)M_{(1)}(j)M_{(1)}(\ell) \geq 0,$$

and similarly detects primes. However, it turns out that this expression can be reformulated much more simply thanks to the identity $\Psi_3(n) = \Psi_2(n)$.

Such simplifications always exist. It turns out that arbitrary convolutions of MacMahonesque functions are always linear combinations of such functions with constant coefficients. Furthermore, linear combinations of MacMahonesque functions with polynomial coefficients can be reduced similarly. To be precise, we have the following result which is implicit in the proof of Theorem 6.

Theorem 7. *The following are true.*
1) If $\vec{\alpha} \in \mathbb{Z}_+^a$ and $\vec{\beta} \in \mathbb{Z}_+^b$, then there are rational numbers $c_{\vec{a}}$ such that for every n we have

$$\sum_{i+j=n} M_{\vec{\alpha}}(i)M_{\vec{\beta}}(j) = \sum_{|\vec{a}|\leq|\vec{\alpha}|+|\vec{\beta}|+a+b-1} c_{\vec{a}}M_{\vec{a}}(n).$$

2) If $\vec{\alpha} \in \mathbb{Z}_+^a$, then there are rational numbers $c_{\vec{a}}$ such that for every n we have

$$nM_{\vec{\alpha}}(n) = \sum_{|\vec{a}|\leq|\vec{\alpha}|+a+1} c_{\vec{a}}M_{\vec{a}}(n).$$

Remark 1. 1) By Theorem 7 (1) and (2), every convolution product of MacMahonesque functions with polynomial coefficients is a linear combination of such functions with rational coefficients.
2) As the proof will show, Theorem 7 is a partition theoretic reformulation of beautiful results of Bachmann and Kühn (see Theorem 1.3 and 1.7 of ref. 10).

Example 2. The identity $\Psi_3(n) = \Psi_2(n)$ offers examples of linear expressions with constant coefficients obtained from Theorem 7 (1). As a simpler example of Theorem 7 (1), we have

$$\sum_{i+j=n} M_{(1)}(i)\,M_{(1)}(j) = \frac{1}{6}M_{(3)}(n) + 2M_{(1,1)}(n) - \frac{1}{6}M_{(1)}(n).$$

As an example of Theorem 7 (2), we have

$$nM_{(1,1)}(n) = \frac{1}{22}\Big(-21M_{(3,1)}(n) + 72M_{(2,2)}(n) - 9M_{(1,3)}(n) + 24M_{(3,0)}(n) - 24M_{(3,0,0)}(n) - 24M_{(2,1,0)}(n) + 24M_{(2,0,1)}(n) - 72M_{(1,1,1)}(n)\Big).$$

Finally, by iterating Theorem 7 (2), we can reformulate the expression in Theorem 1 (1) as

$$(n^2 - 3n + 2)M_1(n) - 8M_2(n) = \frac{3}{11}\Psi_1(n),$$

a linear expression in MacMahonesque functions with constant coefficients.

To obtain these results, we make use of the theory of multiple q-zeta values, largely as developed by Bachmann and Kühn (9, 10), and the theory of quasimodular forms. In Section 2, we recall facts about quasimodular forms, and we establish conditions (Theorem 11) under which their Fourier expansions detect primes, a result which is of independent interest. In Section 3, we use the fact that MacMahon's q-series $\mathcal{U}_a(q)$ arise naturally as quasimodular forms (see refs. 10–12), combined with Theorem 11, to prove Theorems 1 and 4.

In contrast, the generating series of the generalized MacMahonesque partition functions

$$\mathcal{U}_{\vec{a}}(q) := \sum_{n\geq 1} M_{\vec{a}}(n)q^n$$

are generally not quasimodular. However, they occur naturally in the theory of multiple q-zeta values. Bachmann and Kühn show (10) that the MacMahonesque series generate an extraordinary differential algebra whose arithmetic properties encode a rich web of identities satisfied by the MacMahonesque generating functions. To make use of quasimodularity, for those $\vec{a}$ with odd entries, we construct quasimodular forms $\mathcal{U}_{\vec{a}}^{\mathrm{Sym}}(q)$ by taking traces over the natural action of the symmetric group S_a on $\mathcal{U}_{\vec{a}}(q)$. In particular, we prove the following theorem (Theorem 19).

Theorem 8. *All quasimodular forms are linear combinations of the MacMahonesque $\mathcal{U}_{\vec{a}}^{\mathrm{Sym}}(q)$.*

Theorem 6 then follows by applying Theorem 11. These details are carried out in Section 4.

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2. Quasimodular Forms that Detect Primes

Central to all of our results is a theorem (Theorem 11) that characterizes quasimodular forms in the space consisting of linear combinations of derivatives of Eisenstein series whose coefficients are zero at primes and positive at composite $n \geq 2$.

2.1. Preliminaries About Quasimodular Forms. To state and then prove this result, we require several important preliminary facts. For τ in the upper half of the complex plane, set $q := e^{2\pi i \tau}$. For integers $k \geq 1$, we consider the Bernoulli numbers B_k and the Eisenstein series

$$G_k(\tau) := -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \quad [4]$$

where $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. It is well known (for example, see ref. 13) that G_k is quasimodular if and only if k is even, and modular if $k \geq 4$ and even. Furthermore, the algebra of quasimodular forms, which we denote by $\widetilde{M}$, is generated by G_2, G_4, and G_6. We also denote by $\widetilde{M}_k$ the subspace of $\widetilde{M}$ of forms with weight k, and $\widetilde{M}_{\leq k}$ the space of all quasimodular forms of mixed weight $\leq k$.

We require the important q-differential operator

$$D := \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}. \quad [5]$$

For $k \geq 0$, we observe that $D^k : q^n \mapsto n^k q^n$. Ramanujan famously proved the identities (for example, see ref. 13, section 2.3])[†]

$$DG_2 = -2G_2^2 + \frac{5}{6} G_4, \quad DG_4 = -8G_2 G_4 + \frac{7}{10} G_6, \quad DG_6 = -12 G_2 G_6 + \frac{400}{7} G_4^2.$$

These identities show that D defines a map on the algebra of quasimodular forms, which increases weights by 2. Namely, we have that $D : \widetilde{M}_k \to \widetilde{M}_{k+2}$.

2.2. Some Prime-Detecting Quasimodular Forms. Using elementary properties of D, it is straightforward to produce quasimodular forms whose Fourier coefficients detect the primes.

Lemma 9. *Given nonnegative odd integers k, ℓ with $\ell > k$, consider the quasimodular forms*

$$f_{k,\ell} := \left(D^\ell + 1\right) G_{k+1} - \left(D^k + 1\right) G_{\ell+1}.$$

For $n \geq 2$, the nth Fourier coefficient of $f_{k,\ell}$ vanishes if and only if n is prime. Furthermore, all of the coefficients of $f_{k,\ell}$ are nonnegative.

Remark 10: The assumption in Lemma 9 that k and ℓ are odd is imposed to ensure $f_{k,\ell}$ are quasimodular forms. In particular, the vanishing property is true for any nonnegative integers k and ℓ.

Proof of Lemma 9. By inspection, the coefficient of q^1 vanishes. Therefore, without loss of generality we may assume that $n \geq 2$ is an integer. Thanks to Eq. **4**, the nth coefficient of $f_{k,\ell}$ is given by

$$\sum_{\substack{1 < d < n \\ d|n}} \left[\left(1 + n^\ell\right) d^k - \left(1 + n^k\right) d^\ell \right].$$

This sum is empty and therefore vanishing if n is prime, so we assume now that n is composite. Since $d \leq \frac{n}{2}$ must hold, we then have

$$\left(1 + n^\ell\right) d^k - \left(1 + n^k\right) d^\ell \geq d^k \left(1 + n^\ell - \left(\frac{n}{2}\right)^{\ell-k} - n^k \left(\frac{n}{2}\right)^{\ell-k}\right) \geq d^k \left(1 + n^\ell \left(1 - \frac{1}{2^{\ell-k-1}}\right)\right) \geq 1.$$

Thus, the coefficients of $f_{k,\ell}$ are all nonnegative. □

We now turn to the problem of determining those quasimodular forms whose Fourier expansions detect primes. To this end, we let $\mathcal{E}$ denote the quasimodular Eisenstein space, the subspace of quasimodular forms generated additively by the even weight Eisenstein series and their derivatives. We then let Ω denote the prime-detecting quasimodular forms, where a quasimodular form

$$f = \sum_{n \geq 0} b_n(f) q^n$$

is prime-detecting if for positive n we have $b_n(f) \geq 0$, and for $n \geq 2$ vanish if and only if n is prime.

Based on numerical evidence, we expect that the property of vanishing of the Fourier coefficients at primes is not satisfied by quasimodular forms which are not in the quasimodular Eisenstein space.

Conjecture 2. *We have $\Omega = \mathcal{E} \cap \Omega$.*

2.3. Characterizing $\mathcal{E} \cap \Omega$. We give an explicit description of $\mathcal{E} \cap \Omega$ (Theorem 11). To this end, for even $k \geq 6$, we define the distinguished quasimodular forms of weight k

[†] These identities are often stated for the normalized Eisenstein series $E_k := 1 + \ldots$.

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$$H_k := \sum_{n\geq 0} b_n(H_k)q^n := \begin{cases} \frac{1}{6}\left((D^2 - D + 1)G_2 - G_4\right) & \text{if } k = 6, \\ \frac{1}{24}(-D^2 G_{k-6} + (D^2+1)G_{k-4} - G_{k-2}) & \text{if } k \geq 8. \end{cases} \quad [6]$$

The quasimodular forms H_k give rise to all quasimodular forms which are linear combinations of derivatives of Eisenstein series that are prime-detecting.

Theorem 11. *For all $n \geq 0$ and $k \geq 6$, we have $D^n H_k \in \Omega$. Conversely, if $f \in \mathcal{E} \cap \Omega$, then f is a linear combination of the forms $D^n H_k$ for $n \geq 0$ and $k \geq 6$.*

Before we prove this theorem, we consider two additional results on properties of these quasimodular forms H_k.

Lemma 12. *For $k \geq 6$ even, the following are true:*
1) We have that $b_1(H_k) = 0$.
2) The Fourier coefficients $b_n(H_k)$ for $n \geq 2$ are integers.
3) The Fourier coefficients $b_n(H_k)$, where $n \geq 2$, generate $\mathbb{Z}$ as a ring.

Proof of Lemma 12. Claim (1) is clear by inspection. Therefore, we now assume that $n \geq 2$ is an integer. Noting that $b_4(H_6) = 3$ and $b_6(H_6) = 20$ are relatively prime, we see that claim (3) holds for H_6 assuming that (2) holds. To show claim (2) for H_6, we must prove that

$$(n^2 - n + 1)\sigma_1(n) - \sigma_3(n) \equiv 0 \pmod 6.$$

Evenness is trivial, and so we consider this expression modulo 3. If $n \not\equiv 2 \pmod 3$, then after multiplying by $(n+1)$, the 3-divisibility follows from

$$\left(n^3 + 1\right)\sigma_1(n) - (n+1)\,\sigma_3(n) \equiv 0 \pmod 3,$$

which is an easy consequence of Fermat's little theorem. If $n \equiv 2 \pmod 3$, then there is a prime $p \equiv 2 \pmod 3$ dividing n with odd multiplicity $2m+1$, and then 3-divisibility follows from multiplicativity of divisor sums and $\sigma_{2j+1}\left(p^{2m+1}\right) \equiv 0 \pmod 3$. Thus, both (2) and (3) hold for H_6.

We now consider even $k \geq 8$. To prove claim (2), we must show that

$$-n^2\sigma_{k-7}(n) + \left(n^2+1\right)\sigma_{k-5}(n) - \sigma_{k-3}(n) \equiv 0 \pmod{24}.$$

It is straightforward to see this congruence modulo 3 from the fact that $\sigma_{j+2}(n) \equiv \sigma_j(n) \pmod 3$ for any j, and so we consider the congruence above modulo 8. For any odd prime p, we have from $p^2 \equiv 1 \pmod 8$ that $\sigma_{2j+1}(p^m) \equiv \sigma_1(p^m) \pmod 8$ for any integer $j \geq 0$. Now, writing $n = 2^a m$ for m odd, it follows from the multiplicativity of σ_j that

$$\begin{aligned} &-n^2\sigma_{k-7}(n) + \left(n^2+1\right)\sigma_{k-5}(n) - \sigma_{k-3}(n) \\ &\quad\equiv \left(-4^a m^2\sigma_{k-7}\left(2^a\right) + \left(4^a m^2 + 1\right)\sigma_{k-5}\left(2^a\right) - \sigma_{k-3}\left(2^a\right)\right)\sigma_1(m) \pmod 8 \\ &\quad\equiv 4^a m^2\left(\sigma_{k-5}\left(2^a\right) - \sigma_{k-7}\left(2^a\right)\right)\sigma_1(m) - \left(\sigma_{k-3}\left(2^a\right) - \sigma_{k-5}\left(2^a\right)\right)\sigma_1(m) \pmod 8. \end{aligned} \quad [7]$$

Now, for $j \geq 1$ we observe that

$$\sigma_{2j+1}\left(2^a\right) - \sigma_{2j-1}\left(2^a\right) = \sum_{b=0}^{a}\left(2^{(2j+1)b} - 2^{(2j-1)b}\right) = \sum_{b=0}^{a} 2^{(2j-1)b}\left(4^b - 1\right).$$

It is therefore clear that if $j \geq 2$, $\sigma_{2j+1}\left(2^a\right) - \sigma_{2j-1}\left(2^a\right) \equiv 0 \pmod 8$. If $j = 1$, then from the fact that $\sigma_3\left(2^a\right) - \sigma_1\left(2^a\right)$ is even, we obtain from Eq. 7 that

$$-n^2\sigma_{k-7}(n) + \left(n^2+1\right)\sigma_{k-5}(n) - \sigma_{k-3}(n) \equiv -\left(\sigma_{k-3}\left(2^a\right) - \sigma_{k-5}\left(2^a\right)\right)\sigma_1(m) \pmod 8.$$

But then for $2j + 1 = k - 3$, the condition $k \geq 8$ corresponds to the condition $j \geq 2$, and so we obtain claim (2) for all H_k.

We now prove claim (3). Let g be the largest integer dividing

$$24b_n(H_k) = -n^2\sigma_{k-7}(n) + \left(n^2+1\right)\sigma_{k-5}(n) - \sigma_{k-3}(n)$$

for all $n \geq 2$. We know $24 \mid g$, and therefore it will be enough to show that each of 16, 9, and p for primes $p \geq 5$ do not divide g. Beginning with any $p \geq 3$ prime and $m \geq 1$ coprime to p, it is easy to see that

$$24b_{pm}(H_k) \equiv \sigma_{k-5}(pm) - \sigma_{k-3}(pm) \equiv \sigma_{k-5}(m) - \sigma_{k-3}(m) \pmod p.$$

We then observe that $24b_{2p}(H_k) \not\equiv 0 \pmod p$ for $p \geq 5$ (which then entails p does not divide g) and $24b_{5p}(H_k) \not\equiv 0 \pmod 9$ (which then entails that 3 exactly divides g). Finally, for m odd we see that

$$24b_{4m}(H_k) \equiv \sigma_{k-5}(4)\sigma_{k-5}(m) - \sigma_{k-3}(4)\sigma_{k-3}(m) \pmod{16}.$$

It is then easy to see that this expression is not a multiple of 16 for $m = 1$ in the case $k = 8$ and for $m = 3$ if $k \geq 10$. Therefore, we have shown that $g \mid 24$, so that $g = 24$ as required. □

To prove Theorem 11, we also require the following about the linear independence of the forms $D^n H_\ell$ and the spaces they generate.

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Lemma 13. *The following are true.*
1) The quasimodular forms $D^n H_\ell$ (with $n \geq 0$ and $\ell \geq 6$) are linearly independent.
2) The subspace of elements of the $D^n H_\ell$ of weight $\leq k$ is of dimension $\frac{1}{8}(k-2)(k-4)$.

Proof of Lemma 13. We first prove claim (1). We show the stronger statement that the quasimodular forms $D^n E_\ell$, generating $\mathcal{E}$, are linearly independent. Observe that any nontrivial relations between the quasimodular forms $D^n H_\ell$ would imply a relation between these derivatives of Eisenstein series.

To prove the claim, first of all, observe that one can restrict to quasimodular forms of homogeneous weight. Namely, by the quasimodular transformation behavior, one recovers the weight. For quasimodular forms of homogeneous weight, the statement follows from (14, Proposition 20 (iii))

$$\widetilde{M}_k = \bigoplus_{r=0}^{k/2-1} D^r M_{k-2r} \oplus \mathbb{C} D^{k/2-1} E_2.$$

For claim (2), recall that the operator D increases the weight by 2 and H_ℓ is of weight ℓ. We count the number of linear independent generators of weight $\leq k$, as follows:

$$\sum_{\ell \geq 3} \sum_{2\ell+2n \leq k} 1 \;=\; \frac{1}{8}(k-2)(k-4).$$

□

Proof of Theorem 11. If f is a quasimodular form, then $f \in \mathbf{\Omega}$ if and only if $Df \in \mathbf{\Omega}$. Therefore, it suffices to check that $H_k \in \mathbf{\Omega}$ for $k \geq 6$. We first observe that for $n \geq 2$, we have

$$(n+1) b_n(H_6) = \frac{n+1}{6}\left[\left(n^2 - n + 1\right)\sigma_1(n) - \sigma_3(n)\right] = \frac{1}{6}\left[\left(n^3+1\right)\sigma_1(n) - (n+1)\,\sigma_3(n)\right].$$

This gives the identity $(D+1)H_6 = \frac{1}{6} f_{1,3}$, and so Lemma 9 implies that $H_6 \in \mathbf{\Omega}$.

For $k \geq 8$, we calculate for $n \geq 2$ that

$$24 b_n(H_k) = \sum_{d|n}\left[-n^2 d^{k-7} + \left(n^2+1\right) d^{k-5} - d^{k-3}\right] = \sum_{\substack{1<d<n \\ d|n}} d^{k-7}\left(d^2-1\right)\left(n^2-d^2\right).$$

By inspection, we see that $H_k \in \mathbf{\Omega}$.

For the converse, let $f \in \mathcal{E} \cap \mathbf{\Omega}$ be of weight $\leq k$. As D increases the weight by 2, we can write

$$f \;=\; \sum_{\ell=1}^{k/2} p_{2\ell}(D)\, G_{2\ell}$$

with $p_{2\ell}$ a polynomial of degree at most $\frac{1}{2}k - \ell$. The vanishing of the Fourier coefficients at all primes is equivalent to

$$\sum_{\ell=1}^{k/2} p_{2\ell}(x)(x^{2\ell-1}+1) = 0. \qquad \textbf{[8]}$$

(Initially, this equality only holds for primes x, and then, since a polynomial of bounded degree is uniquely determined by its values at primes, this is an equality of polynomials in the variable x.) Observe the dimension of the space of sequences of polynomials $p_2, p_4, \ldots, p_k$ with the given degree bound equals

$$\sum_{\ell=1}^{k/2}\left(\frac{1}{2}k - \ell + 1\right) \;=\; \frac{1}{8}k(k+2).$$

Note that the left-hand side of Eq. **8** is of degree $\leq k-1$. Hence, condition Eq. **8** gives $\leq k$ linearly independent constraints between the coefficients of the polynomials. For notational convenience, we let

$$p_{2\ell}(x) = \sum_{j=0}^{k/2-\ell} a_{j,2\ell}\, x^j$$

for real coefficients $a_{j,2\ell}$, and by M denote the matrix of linear relations of the $a_{j,2\ell}$ given by Eq. **8**, where the rows are parameterized by the degrees $0, \ldots, k-1$ and the columns by pairs $(j, 2\ell)$, ordered lexicographically.

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We show that, after removing the 0th row of M, the matrix is (up to permuting the columns) in row echelon form. Namely, observe that in the ith row the smallest pair $(j, 2\ell)$ for which the corresponding entry in M is nonzero is given by

$$\begin{cases} (0,2) & i=0, \\ (i+1,0) & i>0 \text{ odd}, \\ (i,1) & i>0 \text{ even}. \end{cases}$$

Hence, the matrix obtained from M by removing the 0th row is of full rank. We conclude the rank of M is at least $k-1$. As Eq. **8** is divisible by $x+1$ one can easily infer that the rank is exactly $k-1$. Therefore, the dimension of the space of polynomials satisfying Eq. **8** is given by

$$\sum_{\ell=1}^{k/2}\left(\frac{1}{2}k-\ell+1\right)-(k-1)=\frac{1}{8}(k-2)(k-4).$$

Accordingly, this is also the dimension of the vector space of elements $D^n H_{2\ell}$ of weight $\leq k$ by Lemma 13 (2). □

3. Quasimodularity of $\mathcal{U}_a(q)$ and the Proofs of Theorems 1 and 4

Here we use Theorem 11 to prove Theorems 1 and 4.

3.1. Proof of Theorem 1. MacMahon's series $\mathcal{U}_a(q)$ have been the focus of much study (for example, see refs. 12 and 15–18). The first glimpses of their beautiful properties are captured by the identities (for example, see refs. 8 and 15)

$$\mathcal{U}_1(q)=\sum_{n\geq 1}\sigma_1(n)q^n \quad \text{and} \quad \mathcal{U}_2(q)=\frac{1}{8}\sum_{n\geq 1}\left((-2n+1)\sigma_1(n)+\sigma_3(n)\right)q^n.$$

To prove claim (1), we note that the sequence we wish to study is the coefficients of the series F given by

$$F:=\left(D^2-3D+2\right)\mathcal{U}_1(q)-8\,\mathcal{U}_2(q)=\sum_{n\geq 1}\left[\left(n^2-n+1\right)\sigma_1(n)-\sigma_3(n)\right]q^n.$$

Noting the identities $F=6H_6$ and $(D+1)\,F=f_{1,3}$, part (1) of Theorem 1 then follows by either Theorem 11 or Lemma 9, respectively.

To prove claim (2), we begin by noting that (8, p. 102)

$$\mathcal{U}_3(q)=\frac{1}{1920}\sum_{n\geq 1}\left[\left(40n^2-100n+37\right)\sigma_1(n)-10(3n-5)\sigma_3(n)+3\sigma_5(n)\right]q^n.$$

The sequence we wish to study forms the coefficients of the series G given by

$$G:=\left(3D^3-13D^2+18D-8\right)\mathcal{U}_1(q)+\left(12D^2-120D+212\right)\mathcal{U}_2(q)-960\,\mathcal{U}_3(q).$$

It is easy to see using previous formulas for $\mathcal{U}_j(q)$ that $G=36H_8$, and so claim (2) follows by Theorem 11.

3.2. Proof of Theorem 4. We recall the result of Andrews and Rose (11, Corollary 4) that $\mathcal{U}_a(q)\in\widetilde{M}_{\leq 2a}$. By Theorem 11, any linear combination of derivatives of various $\mathcal{U}_a(q)$ detects primes if it can be represented by linear combinations of derivatives of the quasimodular forms H_k. Since the spaces of quasimodular forms of (mixed) weight $\leq 2a$ is finite-dimensional, the examples given in the Appendix can be verified by a straightforward linear algebra calculation.[‡] In particular, the examples in Table 1 in the Appendix are exactly $6H_6$, $36H_8$, $90H_{10}$, $90H_{12}$, and $30\left(D^2+1\right)H_{14}+30H_{16}$ respectively.

4. MacMahonesque Series and the Proofs of Theorems 6 and 7

4.1. Certain q-Analogues of Multiple Zeta Values. The proofs in the previous section rely on the fact that the $\mathcal{U}_a$ are quasimodular forms. Similarly, for odd a, we have that $\mathcal{U}_{(a)}(q)$ is quasimodular, as we have

$$\mathcal{U}_{(a)}(q)=\sum_{n\geq 1}M_{(a)}(n)q^n=\frac{B_{a+1}}{2(a+1)}+G_{a+1}(\tau). \qquad [9]$$

The proof of Theorem 6 relies on a refinement of this phenomenon. Although the generating functions

$$\mathcal{U}_{\vec{a}}(q):=\sum_{n=1}^{\infty}M_{\vec{a}}(n)q^n \qquad [10]$$

[‡] In particular, this can be done using a row-echelon basis for the spaces $\widetilde{M}_{\leq a}$ with rational coefficients.

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are not always quasimodular forms, it turns out that they arise naturally in the study of q-analogues of multiple zeta values (see ref. 10, section 6), which often turn out to be related to quasimodular forms.

For completeness, before we proceed with the tools for the proofs of Theorems 6 and 7, we consider the q-series generating functions for the MacMahonesque $M_{\vec{a}}(n)$ that generalize (Eq. **1**).

Lemma 14 (10, Lemma 2.5). *If $\vec{a} \in \mathbb{N}^a$, then we have that*

$$\mathcal{U}_{\vec{a}}(q) = \sum_{n=1}^{\infty} M_{\vec{a}}(n) q^n = \sum_{0<s_1<s_2<\cdots<s_a} \frac{q^{s_1+s_2+\cdots+s_a} P_{v_1}(q^{s_1}) P_{v_2}(q^{s_2}) \cdots P_{v_a}(q^{s_a})}{(1-q^{s_1})^{v_1+1}(1-q^{s_2})^{v_2+1}\cdots(1-q^{s_a})^{v_a+1}},$$

where $P_k(x)$ is the kth Eulerian polynomial, which are defined by

$$\frac{xP_n(x)}{(1-x)^{n+1}} := \sum_{k\geq 1} k^n x^k.$$

Remark 15: In analogy with Eq. **1**, one might think that it makes sense to make use of the more aesthetically pleasing q-series

$$\mathcal{V}_{\vec{a}}(q) = \sum_{n\geq 1} N_{\vec{a}}(n) q^n := \sum_{0<s_1<s_2<\cdots<s_a} \frac{q^{s_1+s_2+\cdots+s_a}}{(1-q^{s_1})^{v_1+1}(1-q^{s_2})^{v_2+1}\cdots(1-q^{s_a})^{v_a+1}}.$$

However, these q-series are the generating functions for the less pleasing partition functions

$$N_{\vec{a}}(n) = \sum_{\substack{0<s_1<s_2<\cdots<s_a \\ n=m_1s_1+m_2s_2+\cdots+m_as_a}} \binom{m_1+v_1-1}{v_1}\binom{m_2+v_2-1}{v_2}\cdots\binom{m_a+v_a-1}{v_a}.$$

To derive this expression, one notes for nonnegative integers v (and $|q|<1$) that

$$\frac{q}{(1-q)^{v+1}} = \sum_{m=1}^{\infty} \binom{m+v-1}{v} q^m.$$

All of our results regarding $M_{\vec{a}}(n)$ could be reformulated in terms of $N_{\vec{a}}(n)$. Finally, we note that the natural definition of $M_{\vec{a}}(n)$ in Eq. **3** necessitates the Eulerian polynomials in Lemma 14. For convenience, we offer the first few polynomials

$$P_0(x)=1,\ \ P_1(x)=1,\ \ P_2(x)=x+1,\ \ P_3(x)=x^2+4x+1,\ \ P_4(x)=x^3+11x^2+11x+1,\ldots.$$

For $\vec{a}=(1,1,\ldots,1)$, we recover $\mathcal{U}_{\vec{a}}(q)=\mathcal{V}_{\vec{a}}(q)$ as $P_1(x)=1$.

Write $\mathcal{Z}_q$ for the vector space generated additively by the MacMahonesque series $\mathcal{U}_{\vec{a}}$. It turns out that $\mathcal{Z}_q$ is strictly larger than the space of quasimodular forms. Furthermore, this space enjoys two important properties; namely, i) it is an algebra, and ii) it is closed under differentiation by $D = q\frac{d}{dq}$. Using the theory of quasi-shuffle algebras, Bachmann and Kühn proved the following theorem (10, Theorems 1.3 and 1.7).

Theorem 16. *The space $\mathcal{Z}_q$ is a differential subalgebra of $\mathbb{Q}[\![q]\!]$ containing $\widetilde{M}$.*

Observe that this theorem directly implies that any product of (in fact, any polynomial in) the MacMahon series is a linear combination of MacMahonesque series $\mathcal{U}_{\vec{a}}$. Similarly, any derivative of a MacMahon series is a linear combination of the MacMahonesque series $\mathcal{U}_{\vec{a}}$. Expressing these observations in terms of the MacMahon functions, we obtain Theorem 7.

The proof of this theorem allows one to compute the product of two MacMahonesque series recursively, as we explain now. Let $A=\{z_1,z_2,\ldots\}$ be the set of letters z_j, one for each positive integer. Furthermore, we let $\mathbb{Q}A$ be the $\mathbb{Q}$-vector space generated by these letters. A word is a noncommutative product of letters, and we write $\mathbb{Q}\langle A\rangle$ for the (noncommutative) polynomial algebra over $\mathbb{Q}$ generated by words with letters in A. On $\mathbb{Q}A$, we define a commutative and associative product $\diamond$ by[§]

$$z_i \diamond z_j := \sum_{m=0}^{i+j} c_{i,j,m}\, z_{m+1}\,, \qquad \textbf{[11]}$$

where we let

$$c_{i,j,i+j} := \frac{i!j!}{(i+j+1)!},$$

[§] Since our $U_{\vec{a}}$ correspond to $(\prod_i v_i!)\,[v_1+1,\ldots,v_a+1]$ in ref. 10, we chose a different normalization here.

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and for $m < i + j$ we let

$$c_{i,j,m} := \left[(-1)^i \binom{i}{m+1} + (-1)^j \binom{j}{m+1}\right] \frac{B_{i+j-m}}{i+j-m}.$$

Then, for $x, y \in A$ and $w, v \in \mathbb{Q}\langle A\rangle$, we recursively define the quasi-shuffle product by $1 * w = w * 1 = w$ and

$$xw * yv := x(w * yv) + y(xw * v) + (x \diamond y)(w * v).$$

Identify $\mathbb{Q}\langle A\rangle$ with the vector space generated by finite sequences of positive integers, i.e., $z_{v_1} \cdots z_{v_a}$ corresponds to $\vec{a} = (v_1, \ldots, v_a)$. Under this identification, the quasi-shuffle product yields, e.g.,

$$(1) * (1,1) = 3(1,1,1) + \frac{1}{6}(3,1) + \frac{1}{6}(1,3) - \frac{1}{3}(1,1).$$

We stress that this is a formal sum of vectors, which cannot be simplified by coordinate-wise addition. Moreover, we extend the generalized MacMahon series linearly, i.e., for formal finite linear combinations of finite sequences $\sum_{\vec{a}} c_{\vec{a}}\, \vec{a}$ we set

$$\mathcal{U}_{\sum_{\vec{a}} c_{\vec{a}} \vec{a}} := \sum\nolimits_{\vec{a}} c_{\vec{a}} \mathcal{U}_{\vec{a}}. \tag{12}$$

Then by Proposition 2.10 of ref. 10, the power series multiplication of the generalized MacMahon series can be computed recursively using the quasi-shuffle product defined above.

Proposition 17: *For all $\vec{a} \in \mathbb{N}^a$ and $\vec{b} \in \mathbb{N}^b$, one has*

$$\mathcal{U}_{\vec{a}}(q)\mathcal{U}_{\vec{b}}(q) = \mathcal{U}_{\vec{a}*\vec{b}}(q).$$

Moreover, assigning weight $|\vec{a}| + \ell(\vec{a})$ to $\vec{a}$, the algebra $\mathcal{Z}_q$ becomes a filtered algebra.

Example 18. As an example of Proposition 17, we have

$$\mathcal{U}_{(1)}(q)\mathcal{U}_{(1,1)}(q) = 3\mathcal{U}_{(1,1,1)}(q) + \frac{1}{6}\mathcal{U}_{(3,1)}(q) + \frac{1}{6}\mathcal{U}_{(1,3)}(q) - \frac{1}{3}\mathcal{U}_{(1,1)}(q) = \mathcal{U}_{(1)*(1,1)}(q),$$

which is equivalent to

$$\mathcal{U}_1(q)\mathcal{U}_2(q) = 3\mathcal{U}_3(q) + \frac{1}{6}\mathcal{U}_{(3,1)}(q) + \frac{1}{6}\mathcal{U}_{(1,3)}(q) - \frac{1}{3}\mathcal{U}_2(q).$$

This example illustrates the fact that the vector space generated by the MacMahon series is not closed under multiplication, but $\mathcal{Z}_q$ is.

4.2. Symmetric Group Action. As we have emphasized above, a generic MacMahonesque q-series $\mathcal{U}_{\vec{a}}(q)$ is not quasimodular. However, their "symmetrized sums" are quasimodular, and provide canonical q-series that additively generate the ring of quasimodular forms.

To make this precise, we recall that the generalized MacMahon series $\mathcal{U}_{\vec{a}}(q)$ admit the natural action of the a-th symmetric group S_a by

$$\sigma\mathcal{U}_{\vec{a}}(q) := \mathcal{U}_{\sigma\vec{a}}(q), \tag{13}$$

where for $\sigma \in S_a$ we let $\sigma\vec{a} := \left(v_{\sigma(1)}, \ldots, v_{\sigma(a)}\right)$. For those $\vec{a}$ consisting of only odd entries, we consider the symmetrized MacMahonesque q-series

$$\mathcal{U}^{\mathrm{Sym}}_{\vec{a}}(q) := \sum_{\sigma \in S_a} \sigma\mathcal{U}_{\vec{a}}(q). \tag{14}$$

Theorem 19. *The following are true.*
1) For all $\vec{a} \in (2\mathbb{Z}_+ - 1)^a$, we have that $\mathcal{U}^{\mathrm{Sym}}_{\vec{a}}(q) \in \widetilde{M}_{\leq |\vec{a}|+a}$.
2) The $\mathcal{U}^{\mathrm{Sym}}_{\vec{a}}(q)$ with $|\vec{a}| + a \leq k$ additively generate $\widetilde{M}_{\leq k}$.
3) The ring of quasimodular forms is additively generated as follows

$$\widetilde{M} = \left\langle 1, \mathcal{U}_{(v_1)}, \mathcal{U}^{\mathrm{Sym}}_{(v_1,v_2)}, \mathcal{U}^{\mathrm{Sym}}_{(v_1,1,\ldots,1)}, \mathcal{U}^{\mathrm{Sym}}_{(v_1,v_2,1,\ldots,1)} \mid v_1, v_2 \text{ positive odds} \right\rangle.$$

Remark 20: Theorem 19 (1) follows from refs. 9 and 10. Parts (2) and (3) of the theorem are the new observations.

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Proof of Theorem 18. We now prove claim (1). Symmetric sums in the context of quasi-shuffle algebras have been studied broadly. In particular, by Theorem 13 of ref. 19 we have

$$\mathcal{U}_{\vec{a}}^{\mathrm{Sym}}(q) = \sum_{\sigma \in S_a} \sigma \mathcal{U}_{\vec{a}}(q) = \sum_{B \in \Pi_a} c(B) \prod_{\beta \in B} \mathcal{U}_{\diamond_\beta}(q), \qquad [15]$$

where Π_a denotes all set partitions of the set $\{1, \ldots, a\}$, $\diamond_\beta := \diamond_{i \in \beta}(v_i)$ denotes the $\diamond$-product of the z_{v_i} (written as a finite sequence), and

$$c(B) = (-1)^{a-|B|} \prod_{\beta \in B} (|\beta| - 1)!$$

is a constant. In particular, this expresses $\sum_{\sigma \in S_a} \sigma \mathcal{U}_{\vec{a}}(q)$ as a polynomial in the modular Eisenstein series (up to constant terms) $\mathcal{U}_{(v)}$ for certain $v \in \mathbb{Z}_+$. We now show that, in fact, v only takes odd values.

Recall that the entries $v_1, \ldots, v_a$ of $\vec{a}$ are all odd. Now, observe that for odd i, j, we have that $c_{i,j,m} = 0$ if m is odd. Namely, for odd i, j, m the Bernoulli number B_{i+j-m} is zero, unless $i + j - m = 1$. In the latter case $m + 1 > i$ and $m + 1 > j$, and hence $\binom{i}{m+1} = \binom{j}{m+1} = 0$. Hence, in the $\diamond$-product of z_i and z_j, where i and j are odd, we find that the coefficients of z_{m+1} vanish if m is odd. We conclude that $\sum_{\sigma \in S_a} \sigma \mathcal{U}_{\vec{a}}(q)$ can be expressed as a polynomial in $\mathcal{U}_{(v)}$ for certain odd integers v. Now, (1) follows from Eq. **9**.

We now turn to the proof of claims (2) and (3). By Eq. **9**, the symmetrized MacMahonesque q-series generate all the Eisenstein series. If $\vec{a} = (v_1, v_2)$ is of length 2, then by brute force using Eq. **15**, we find that $\mathcal{U}_{\vec{a}}^{\mathrm{Sym}}(q)$ is—up to an Eisenstein series—equal to the product of the Eisenstein series $\mathcal{U}_{(v_1)}(q)$ and $\mathcal{U}_{(v_2)}(q)$. It follows that $\mathcal{U}_{\vec{a}}^{\mathrm{Sym}}(q)$ with $|\vec{a}| + a \leq k$ additively generate the algebra of modular forms $M_{\leq k}$, as it is well known that the space of modular forms is generated by polynomials in Eisenstein series of degree at most 2 (for example, see ref. 20, section 5). Finally, using Eq. **15** again, observe that for $\vec{a} = (v, 1, 1, \ldots, 1)$ and $\vec{a} = (v_1, v_2, 1, 1, \ldots, 1)$ one obtains arbitrary powers of the quasimodular $\mathcal{U}_1$ times Eisenstein series and products of Eisenstein series respectively, modulo quasimodular forms of lower depth. Here, we recall every quasimodular form can be written as a polynomial in the quasimodular Eisenstein series $\mathcal{U}_1$ (which equals G_2 up to a constant), and the depth is defined as the degree in $\mathcal{U}_1$. Claims (2) and (3) now follow by induction on the depth. □

4.3. Proof of Theorem 6. For $d = 4$, we have example $\Psi_1(n)$ in the Introduction. Multiplying by n^{d-4} and using Theorem 7 implies the existence for all $d \geq 4$, thereby confirming claim (1).

We turn to claim (2). By Lemma 19 (2) the subspace of all $\mathcal{U}_{\vec{a}}(q)$ of weight at most k generates $\widetilde{M}_k$. By Lemma 13 there are $\gg k^2$ linearly independent quasimodular forms of weight at most k for which the nth Fourier coefficient ($n \geq 2$) is nonnegative and vanish if and only if n is prime. We also note that the coefficient of q^1 in these forms also vanish. This proves claim (2).

4.4. Proof of Theorem 7. The first part of the statement follows from Proposition 17. In particular, the bound on $\vec{a}$ in the sum in the statement of the theorem, follows from estimating $\ell(\vec{a}) \geq 1$ and the fact that $\mathcal{Z}_q$ is a filtered algebra. The second part of the statement follows from the fact that the operator D preserves $\mathcal{Z}_q$ and increases the weight by 2 (see Theorem 16 and Theorem 1.7 of ref. 10).

Appendix

Table 1. Prime detecting expressions for Theorem 4

Prime detecting expressions	Quasimodular form
$(n^2 - 3n + 2)M_1(n) - 8M_2(n)$	$6H_6$
$(3n^3 - 13n^2 + 18n - 8)M_1(n) + (12n^2 - 120n + 212)M_2(n) - 960M_3(n)$	$36H_8$
$(25n^4 - 171n^3 + 423n^2 - 447n + 170)M_1(n) + (300n^3 - 3554n^2 + 12900n - 14990)M_2(n) +$ $(2400n^2 - 60480n + 214080)M_3(n) - 725760M_4(n)$	$90H_{10}$
$(126n^5 - 1303n^4 + 5073n^3 - 9323n^2 + 8097n - 2670)M_1(n) +$ $(3024n^4 - 48900n^3 + 288014n^2 - 737100n + 695490)M_2(n) +$ $(60480n^3 - 1510080n^2 + 10644480n - 23496480)M_3(n) +$ $(725760n^2 - 36288000n + 218453760)M_4(n) - 580608000M_5(n)$	$90H_{12}$
$(300n^8 - 1542n^7 - 33049n^6 + 377959n^5 - 1651959n^4 + 3726801n^3 - 4575760n^2 + 2903750n - 746500)M_1(n) +$ $(12000n^7 - 91008n^6 - 2799900n^5 + 50637162n^4 - 351366300n^3 + 1239098170n^2 - 2210467000n + 1585493500)M_2(n) +$ $(432000n^6 - 3548160n^5 - 236343840n^4 + 5133219840n^3 - 42370071840n^2 + 161101416000n - 236150560800)M_3(n) +$ $(12096000n^5 - 72817920n^4 - 17599680000n^3 + 396192142080n^2 - 3123876672000n + 8555162112000)M_4(n) +$ $(193536000n^4 - 1056513024000n^2 + 21310248960000n - 112944125952000)M_5(n) +$ $(-46495088640000n + 604436152320000)M_6(n) - 1115882127360000M_7(n)$	$30(D^2+1)H_{14} + 30H_{16}$

See Eq. **6** for the definition of the H_k.

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