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Partitions, multiple zeta values and the q-bracket

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Abstract
We provide a framework for relating certain q-series defined by sums over partitions to multiple zeta values. In particular, we introduce a space of polynomial functions on partitions for which the associated q-series are q-analogues of multiple zeta values. By explicitly describing the (regularized) multiple zeta values one obtains as $q \to 1$, we extend previous results known in this area. Using this together with the fact that other families of functions on partitions, such as shifted symmetric functions, are elements in our space will then give relations among (q-analogues of) multiple zeta values. Conversely, we will show that relations among multiple zeta values can be 'lifted' to the world of functions on partitions, which provides new examples of functions for which the associated q-series are quasimodular.

Keywords Functions on partitions · Modular forms · q-Bracket · Multiple zeta values

Mathematics Subject Classification Primary 05A17 · 11M32; Secondary 11F11

1 Introduction

The purpose of this note is to introduce a framework that can be seen as a bridge between the theory of functions on partitions and (q-analogues of) multiple zeta values. Multiple zeta values (see (1.5)) are real numbers appearing in various areas of mathematics and theoretical physics. These real numbers satisfy numerous relations, such as the so-called double shuffle relations [20]. For these numbers, there exist var-

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ious different q-analogue models, which are q-series degenerating to multiple zeta values as $q \to 1$. For most of these q-analogues, there exist counterparts for the double shuffle relations, and their algebraic setups are well-understood [6, 11, 13, 27, 32]. In this note, we will generalize this setup even further. We will show that there is a natural analogue of the double shuffle relations on all functions on partitions and then introduce a class of functions that we call *partition analogues of multiple zeta values*. These functions can be seen as the counterpart of q-analogues of multiple zeta values after applying the so-called q-bracket, introduced by Bloch and Okounkov in [8]. The space of partition analogues of multiple zeta values contains various classical types of functions on partitions, such as the shifted symmetric functions, which then, by using the results in this work, provide new tools to obtain relations among (q-analogues of) multiple zeta values.

Denote by $\mathscr{P}$ the set of all partitions of integers. To a function $f : \mathscr{P} \to \mathbb{Q}$, we associate (i) a degree, (ii) a limit $\mathrm{Z}^{\deg}(f)$ and (iii) a power series $\langle f \rangle_q \in \mathbb{Q}[\![q]\!]$, in such a way that asymptotically

$$(1-q)^{\deg(f)} \langle f \rangle_q = \mathrm{Z}^{\deg}(f) + O(1-q)$$

for real q.

To start with the latter, the *q-bracket* of f is defined as

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathscr{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda \in \mathscr{P}} q^{|\lambda|}} \in \mathbb{Q}[\![q]\!], \tag{1.1}$$

where $|\lambda|$ denotes the integer that λ is a partition of. In case $f(\lambda)$ has at most polynomial growth in $|\lambda|$, its q-bracket is holomorphic for $|q| < 1$. Moreover, we can associate a *degree* to f by

$$\deg(f) = \inf_{a \in \mathbb{R}} \left\{ \lim_{q \to 1} (1-q)^a \langle f \rangle_q \text{ converges} \right\}. \tag{1.2}$$

We, then, define $\mathrm{Z}^{\deg}(f) \in \mathbb{R} \cup \{\pm\infty\}$ to be the value of the corresponding limit[1] $\lim_{q \to 1} (1-q)^{\deg(f)} \langle f \rangle_q$ whenever it exists (as it does for all functions in our work). For example, for $f(\lambda) = |\lambda|$ we have

$$\langle f \rangle_q = \sum_{n=1}^{\infty} \sigma(n)\, q^n, \quad \deg(f) = 2, \quad \mathrm{Z}^{\deg}(f) = \zeta(2),$$

where $\sigma(n) = \sum_{d|n} d$ denotes the divisor sum of n and $\zeta(k) = \sum_{m \geq 1} \frac{1}{m^k}$ a Riemann zeta value.

Such limits $\mathrm{Z}^{\deg}(f)$ occur as the volumes of certain moduli spaces, e.g., in the case of the stratum of one-cylinder square-tiled surfaces [16] or in the case of flat surfaces [14]. In both cases, there are associated functions f in Λ^*, the space of

[1] Here and in the rest of the work we understand $q \to 1$ as the limit where q is real and $0 < q < 1$.

shifted symmetric functions. That is, $\Lambda^* = \mathbb{Q}[Q_2, Q_3, \ldots]$, where $Q_k : \mathscr{P} \to \mathbb{Q}$ is given by

$$Q_k(\lambda) := \beta_k + \frac{1}{(k-1)!} \sum_{i=1}^{\infty} \left(\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i + \tfrac{1}{2}\right)^{k-1} \right), \tag{1.3}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\beta_k = \left(\frac{1}{2^{k-1}} - 1\right)\frac{B_k}{k!}$ with B_k the k-th Bernoulli number. By the Bloch–Okounkov theorem [8] (going back to work of Dijkgraaf and Kaneko–Zagier [15, 21]) the q-brackets $\langle f \rangle_q$ for $f \in \Lambda^*$ are quasimodular forms. The space of quasimodular forms $\widetilde{M} = \mathbb{Q}[G_k \mid k = 2, 4, 6, \ldots]$ is generated by the Eisenstein series G_k for all even $k \geq 2$

$$G_k(q) := -\frac{B_k}{2k!} + \frac{1}{(k-1)!} \sum_{m,r \geq 1} m^{k-1} q^{mr}, \tag{1.4}$$

which are holomorphic functions for $|q| < 1$ (or equivalently, for τ in the complex upper half plane, with $q = e^{2\pi i \tau}$). In fact, $\widetilde{M} = \mathbb{Q}[G_2, G_4, G_6]$. As $\mathrm{Z}^{\deg}(G_k) = \zeta(k)$ and every modular form can be written as the sum of an Eisenstein series and a cusp form F, for which $\mathrm{Z}^{\deg}(F) = 0$, it follows that for $f \in \Lambda^*$ the corresponding limits $\mathrm{Z}^{\deg}(f)$ are single zeta values. Besides the shifted symmetric functions there are various other functions on partitions which give rise to quasimodular forms (see [30]). All of these have limits which lie in $\mathbb{Q}[\zeta(2)]$. We will introduce a space $\mathbb{P} \subset \mathbb{Q}^{\mathscr{P}}$ of partition analogues of multiple zeta values, whose elements always have *multiple zeta values* as their limit. Multiple zeta values are defined for $r \geq 1$ and $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ by

$$\zeta(k_1, \ldots, k_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}. \tag{1.5}$$

A natural subspace of $\mathbb{P}$ is the space $\mathbb{M} = \{f \in \mathbb{P} \mid \langle f \rangle_q \in \widetilde{M}\} \subset \mathbb{P}$, containing Λ^*, whose elements have a quasimodular q-bracket and a limit in $\mathbb{Q}[\zeta(2)]$. In particular, the space $\mathbb{P}$ completes the following diagram:

$$\begin{array}{ccccc}
\mathbb{Q}^{\mathscr{P}} & \xrightarrow{\langle\,\rangle_q} & \mathbb{Q}[\![q]\!] & \xrightarrow{\text{“}\lim_{q\to 1}\text{”}} & \mathbb{R} \\
\cup & & \cup & & \cup \\
\mathbb{P} & \longrightarrow & \mathcal{Z}_q & \longrightarrow & \mathcal{Z} \\
\cup & & \cup & & \cup \\
\Lambda^* \subset \mathbb{M} & \longrightarrow & \widetilde{M} & \longrightarrow & \mathbb{Q}[\zeta(2)].
\end{array} \tag{1.6}$$

Here, $\mathcal{Z}$ denotes the $\mathbb{Q}$-vector space generated by multiple zeta values, $\mathcal{Z}_q$ denotes the space of q-analogues of multiple zeta values, studied by many authors (see e.g., [6,

12, 32] for an overview of different types of q-analogues; in this work, we define $\mathcal{Z}_q$ in (3.3), following [6]), and the map $\lim_{q\to 1}$ is between quotation marks to indicate that it concerns infinitely many maps $f \mapsto \lim_{q\to 1}(1-q)^a f$, which are well-defined/ill-defined/regularized depending on f and the value of a.

Polynomial functions on partitions

Let $p \in \mathbb{Q}[x, y]$. In [28], the second author studied functions $T_p : \mathscr{P} \to \mathbb{Q}$ (not to be confused with Hecke operators) of the form

$$T_p(\lambda) = \sum_{m=1}^{\infty} \sum_{r=1}^{r_m(\lambda)} p(m, r),$$

where $r_m(\lambda)$ denotes the number of times m occurs as a part in the partition λ. Similar to the Bloch–Okounkov theorem, for every $f \in \mathbb{Q}[T_p \mid p \in \mathbb{Q}[x, y], \deg(p) \text{ is odd}]$, the q-bracket $\langle f \rangle_q$ is a quasimodular form. Motivated by this construction, we define the space of *partition analogues of multiple zeta values* $\mathbb{P}$ as the following space which can be thought of as a space of polynomial functions on partitions.

Definition 1.1 Let $\mathbb{P}$ be the image of

$$\Psi : \bigoplus_{n\geq 0} \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] \to \mathbb{Q}^{\mathscr{P}},$$

where Ψ maps the polynomial $p(x_1, \ldots, x_n, y_1, \ldots, y_n)$ to

$$\lambda \mapsto \sum_{m_1>\ldots>m_n>0} \sum_{r_1=1}^{r_{m_1}(\lambda)} \cdots \sum_{r_n=1}^{r_{m_n}(\lambda)} p(m_1, \ldots, m_n, r_1, \ldots, r_n).$$

Moreover, define $\mathbb{M} = \{f \in \mathbb{P} \mid \langle f \rangle_q \in \widetilde{M}\}$.

We show that this space $\mathbb{P}$ completes the diagram (1.6). In particular, we are able to compute the *degree* and *limit* of elements of $\mathbb{P}$:

Theorem 1.2 *Given $r \geq 1$ and $d_i, l_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, r$, let $f = \Psi(\prod_i x_i^{d_i} y_i^{l_i})$. Then,*

$$\deg(f) = \max_{j\in\{0,\ldots,r\}} \left\{ \sum_{i\leq j} (d_i + 1) + \sum_{i>j} (l_i + 1) \right\}.$$

Moreover, if the maximum is attained for a unique value of j, then $\mathrm{Z}^{\deg}(f) \in \mathcal{Z}_{\leq \deg(f)}$, where $\mathcal{Z}_{\leq k}$ denotes the $\mathbb{Q}$-vector space of multiple zeta values $\zeta(k_1, \ldots, k_r)$ with $k_1 + \ldots + k_r \leq k$.

We will see that the regularized limit of polynomial functions on partitions always gives regularized multiple zeta values. These limits will be given by *bi-multiple zeta values* $\zeta\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \in \mathbb{R}[T]$, which are defined for $k_1,\dots,k_r \geq 1$, $d_1,\dots,d_r \geq 0$ in Definition 4.22, and which generalize (harmonic regularized) multiple zeta values in the sense that for $k_1 \geq 2, k_2,\dots,k_r \geq 1$

$$\zeta\begin{pmatrix} k_1,\dots,k_r \\ 0,\dots,0 \end{pmatrix} = \zeta(k_1,\dots,k_r)\,.$$

The other special case of $d_1,\dots,d_{r-1} \geq 0, d_r \geq 1$ and $k_1 = \cdots = k_r = 1$ is given by

$$\zeta\begin{pmatrix} 1,\dots,1 \\ d_1,\dots,d_r \end{pmatrix} = \xi(d_1,\dots,d_r)\,,$$

where we define the *conjugated zeta values* ξ as follows:

Definition 1.3 For $d_1,\dots,d_{r-1} \geq 0, d_r \geq 1$, define the *conjugated multiple zeta value* by

$$\xi(d_1,\dots,d_r) := \sum_{0<m_1<\ldots<m_r} \frac{1}{m_1\cdots m_r}\,\Omega\Big[\prod_{i=1}^{r}\Big(\frac{1}{m_i}+\ldots+\frac{1}{m_r}\Big)^{d_i}\Big],$$

where $\Omega : \mathbb{Q}[m_1^{-1},\dots,m_r^{-1}] \to \mathbb{Q}[m_1^{-1},\dots,m_r^{-1}]$ is the linear mapping

$$\Omega\Big[\frac{1}{m_1^{l_1}\cdots m_r^{l_r}}\Big] := \frac{l_1!\cdots l_r!}{m_1^{l_1}\cdots m_r^{l_r}}\,. \tag{1.7}$$

By definition it is clear that the conjugated zeta values can be written as linear combinations of multiple zeta values and as a result of Theorem 4.23 their product can be expressed by the index-shuffle product formula, e.g., $\xi(d_1)\xi(d_2) = \xi(d_1,d_2) + \xi(d_2,d_1)$ for $d_1,d_2 \geq 1$. In general, the bi-multiple zeta values will be given by sums of products of multiple zeta values and their conjugated analogues ξ, e.g., we have (for $k_1 \geq 2, d_m \geq 1$)

$$\zeta\begin{pmatrix} 1,\dots,1,k_1,\dots,k_r \\ d_1,\dots,d_m,0,\dots,0 \end{pmatrix} = \xi(d_1,\dots,d_m)\,\zeta(k_1,\dots,k_r)\,.$$

The product of the bi-multiple zeta values can be expressed by a generalization of the usual harmonic product, e.g. we have

$$\zeta\binom{k_1}{d_1}\zeta\binom{k_2}{d_2} = \zeta\binom{k_1,k_2}{d_1,d_2} + \zeta\binom{k_2,k_1}{d_2,d_1} + \zeta\binom{k_1+k_2}{d_1+d_2}\,.$$

Further, we will see in Theorem 4.23 that the bi-multiple zeta values are invariant under a certain involution ι_{s}, which will appear naturally when considering functions

on partitions. In particular, these give a realization of the so-called formal double Eisenstein space introduced in [7].

In Sect. 4.2, we will introduce an algebraic setup for the elements in $\mathbb{P}$, which can be seen as a generalization of the classical algebraic setup for multiple zeta values and their double shuffle relations (cf., [17, 20]). Starting with a linear combination of multiple zeta values which evaluates to an element in $\mathbb{Q}[\zeta(2)]$, we then show, with diagram (1.6) in mind, how to 'lift' these to obtain new families of functions on partitions with quasimodular q-bracket (see Proposition 4.14 and Example 5.2). In Sect. 5, we will show that not only the analogue of the double shuffle relations give relations among (q-analogues of) multiple zeta values, but also how families of polynomial functions on partitions with quasimodular q-bracket can be used to obtain relations. For example, we will see that the shifted symmetric functions imply a special case of the Ohno–Zagier relations and that the arm-leg moments of Zagier ([30]) imply the sum formula for multiple zeta values.

2 Functions on partitions

2.1 Multiplication, conjugation, brackets and derivations

Denote by $\mathscr{P}$ the set of all partitions of integers. We make use of the following equivalent definitions of partitions:

(i) Finite non-increasing sequences of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where we write $\ell(\lambda)$ for the *length* ℓ of the partition λ and $|\lambda| = \lambda_1 + \ldots + \lambda_\ell$ for the *size*.

(ii) Infinite sequences $(\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\lambda_j = 0$ for all but finitely many j;

(iii) (*Stanley's multi-rectangular coordinates*) Two sequences $\mathbf{r}$ and $\mathbf{m}$ of non-negative integers, of the same length d, and of which $\mathbf{m}$ is strictly decreasing. These sequences correspond to the partition

$$\mathbf{r} \times \mathbf{m} = (\underbrace{m_1, \ldots, m_1}_{r_1}, \ldots, \underbrace{m_d, \ldots, m_d}_{r_d})$$

in the first definition. This representation is unique if the elements of $\mathbf{r}$ and $\mathbf{m}$ are positive. Often, given $\lambda \in \mathscr{P}$, we write $\mathbf{r}(\lambda) = (r_1, \ldots, r_d)$ and $\mathbf{m}(\lambda) = (m_1, \ldots, m_d)$ for such sequences. The integer d is called the *depth* of the partition.

(iv) Multisets of integers, in which the integer m has

$$r_m(\lambda) = \#\{j \mid \lambda_j = m\}$$

appearances.

Example 2.1 The smallest partition is the empty partition (of the integer 0), written as $()$, $(0, 0, \ldots)$, $() \times ()$, $\emptyset$ respectively. The Stanley coordinates of the partition $\lambda = (6, 4, 4, 3, 2, 2, 2, 1)$ can be read of by the decomposition $\lambda = (1, 2, 1, 3, 1) \times (6, 4, 3, 2, 1)$.

There are two elementary operations on the space of functions $f, g \in \mathbb{Q}^{\mathscr{P}}$:

(i) *pointwise multiplication*, i.e., the multiplication $(f \odot g)(\lambda) = f(\lambda)\, g(\lambda)$ induced by the multiplication on $\mathbb{Q}$,
(ii) *conjugation*, i.e., $\omega(f)(\lambda) = f(\overline{\lambda})$, where $\overline{\lambda}$ denotes the transpose of the partition λ.

We adapt both operations to make them equivariant with respect to the q-bracket (1.1). To do so, we introduce the $\underline{u}$-bracket [28, Definition 3.2.1].

Definition 2.2 The vector space isomorphism $\langle\ \rangle_{\underline{u}} : \mathbb{Q}^{\mathscr{P}} \to \mathbb{Q}[\![u_1, u_2, \ldots]\!]$ is given by

$$\langle f \rangle_{\underline{u}} := \frac{\sum_{\lambda \in \mathscr{P}} f(\lambda)\, u_\lambda}{\sum_{\lambda \in \mathscr{P}} u_\lambda} \qquad (u_\lambda = u_{\lambda_1}\, u_{\lambda_2} \cdots,\ u_0 = 1).$$

For $f \in \mathbb{Q}^{\mathscr{P}}$ we call $\langle f \rangle_{\underline{u}}$ the $\underline{u}$-*bracket* of f. For all $\lambda \in \mathscr{P}$ we write $a_\lambda(f)$ to denote the coefficient of u_λ in $\langle f \rangle_{\underline{u}}$, i.e., $\langle f \rangle_{\underline{u}} = \sum_{\lambda \in \mathscr{P}} a_\lambda(f)\, u_\lambda$.

Note that the $\underline{u}$-bracket reduces to the q-bracket (1.1) by specializing $u_i = q^i$ for all integers i.

The $\underline{u}$-bracket is *not* an algebra homomorphism with respect to the pointwise product on $\mathbb{Q}^{\mathscr{P}}$. Therefore, we introduce the *harmonic product* on $\mathbb{Q}^{\mathscr{P}}$, making the $\underline{u}$-bracket into an algebra homomorphism. Even so, we introduce a conjugation ι making the $\underline{u}$-bracket equivariant.

Definition 2.3 Given $F, G \in \mathbb{Q}[\![u_1, u_2, \ldots]\!]$, we define

(i) the *harmonic product* as the multiplication $F \circledast G = FG$, where FG denotes the standard product of F and G in $\mathbb{Q}[\![u_1, u_2, \ldots]\!]$,
(ii) the *conjugation* of $F = \sum_{\lambda \in \mathscr{P}} a_\lambda u_\lambda$ by $\iota(F) = \sum_{\lambda \in \mathscr{P}} a_\lambda u_{\overline{\lambda}}$, where $\overline{\lambda}$ denotes the transpose of the partition λ,
(iii) the *shuffle product* as the multiplication $F \circledcirc G = \iota(\iota(F) \circledast \iota(G))$,
(iv) the *derivative* of $F = \sum_{\lambda \in \mathscr{P}} a_\lambda u_\lambda$ by $DF = \sum_{\lambda \in \mathscr{P}} a_\lambda |\lambda|\, u_\lambda$.

We extend these definitions to $\mathbb{Q}^{\mathscr{P}}$ by the isomorphism given by the $\underline{u}$-bracket, i.e., for $f, g \in \mathbb{Q}^{\mathscr{P}}$ we define

(i) the *harmonic product* by $\langle f \circledast g \rangle_{\underline{u}} = \langle f \rangle_{\underline{u}} \langle g \rangle_{\underline{u}}$,
(ii) the *conjugation* by $\langle \iota(f) \rangle_{\underline{u}} = \iota \langle f \rangle_{\underline{u}}$,
(iii) the *shuffle product* as the multiplication $\langle f \circledcirc g \rangle_{\underline{u}} = \langle f \rangle_{\underline{u}} \circledcirc \langle g \rangle_{\underline{u}}$,
(iv) the *derivative* of f by $\langle Df \rangle_{\underline{u}} = D \langle f \rangle_{\underline{u}}$.

Remark 2.4 In [28] the harmonic product was called the *induced product*, as it is induced from the product on $\mathbb{Q}[\![u_1, u_2, \ldots]\!]$. In the context of the present work, the name *harmonic product* is more appropriate, as will be indicated in Example 5.1.

Proposition 2.5 *For all* $f, g \in \mathbb{Q}^{\mathscr{P}}$

$$\langle \iota(f) \rangle_q = \langle f \rangle_q, \quad \langle f \circledast g \rangle_q = \langle f \rangle_q \langle g \rangle_q = \langle f \circledcirc g \rangle_q \quad \text{and} \quad q \frac{\partial}{\partial q} \langle f \rangle_q = \langle Df \rangle_q.$$

Proof The first equality follows directly by noting that a partition λ and its conjugate $\overline{\lambda}$ are of the same size, so that substituting q^i for u_i has the same effect on u_λ and $u_{\overline{\lambda}}$. By definition of the shuffle product, the second identity follows from the first. The last equality follows directly by noting that $\langle f\rangle_{\underline{u}}|_{u_i=q^i} = \langle f\rangle_q$. □

The harmonic product, conjugation, shuffle product and derivative can be given by explicit formulas on the level of functions in $\mathbb{Q}^{\mathscr{P}}$. For this, recall that a *strict partition* is a partition for which all (non-zero) parts are distinct. Let the Möbius function $\mu : \mathscr{P} \to \{\pm 1\}$ be given by

$$\mu(\lambda) = \begin{cases} (-1)^{\ell(\lambda)} & \lambda \text{ is strict} \\ 0 & \text{else.} \end{cases}$$

Denote the convolution product of $f, g \in \mathbb{Q}^{\mathscr{P}}$ by $\star$ (we reserve the symbol $*$ for the harmonic product), i.e.,

$$(f\star g)(\lambda) = \sum_{\alpha\cup\beta=\lambda} f(\alpha)\, g(\beta),$$

where in the summation we take the union of partitions considered as multisets. Write $1 : \mathscr{P} \to \mathbb{Q}$ for the inverse of the Möbius function under convolution, i.e., $1(\lambda) = 1$. Then, by a direct computation, for all $f, g \in \mathbb{Q}^{\mathscr{P}}$, we have (see also [28, Proposition 3.2.3])

$$\begin{aligned} f \circledast g &= f\star g\star\mu \\ \iota(f) &= \omega(f\star\mu)\star 1 \\ f \circledcirc g &= \omega(\omega(f\star\mu)\star\omega(g\star\mu))\star 1, \end{aligned} \tag{2.1}$$

where we recall $\omega(f)(\lambda) = f(\overline{\lambda})$. Also, by [28, Proposition 5.1.1] for all $f \in \mathbb{Q}^{\mathscr{P}}$ we have

$$Df(\lambda) = f(\lambda)|\lambda| - (f \circledast |\cdot|)(\lambda).$$

Two subspaces of $\mathbb{Q}^{\mathscr{P}}$ are particularly well-behaved with respect to these operations.

Definition 2.6 We define

$$\begin{aligned} \mathscr{H} &:= \{f \in \mathbb{Q}^{\mathscr{P}} \mid f(\lambda) = f(\rho) \text{ if } \mathbf{r}(\lambda) = \mathbf{r}(\rho) \text{ for all } \lambda, \rho \in \mathscr{P}\} \\ \mathscr{J} &:= \{f \in \mathbb{Q}^{\mathscr{P}} \mid (f\star\mu)(\lambda) = (f\star\mu)(\rho) \text{ if } \mathbf{m}(\lambda) = \mathbf{m}(\rho) \text{ for all } \lambda, \rho \in \mathscr{P}\}, \end{aligned}$$

where $\mathbf{r}(\lambda)$ and $\mathbf{m}(\lambda)$ are the Stanley coordinates for λ.

For example, the Möbius function μ is in $\mathscr{H}$, and for any $m \geq 1$ the function $\lambda \mapsto r_m(\lambda)$ is an element of $\mathscr{J}$.

Lemma 2.7 *The space $\mathscr{H}$ is closed under the harmonic product $\circledast$ and $\mathscr{J}$ is closed under the shuffle product ш. In fact, the spaces are conjugate, i.e., $\iota\mathscr{H} = \mathscr{J}$.*

Proof Suppose $\lambda = \mathbf{r} \times \mathbf{m}$ and $\rho = \mathbf{r} \times \mathbf{s}$ for some sequences $\mathbf{r}, \mathbf{m}, \mathbf{s}$. Given $\alpha \subset \lambda$ (where we consider α and λ to be multisets), there is some sequence $\mathbf{a}$ such that $\alpha = \mathbf{a} \times \mathbf{m}$. Then, $\mathbf{a} \times \mathbf{s} \subset \rho$. This gives a bijection between subsets of λ and of ρ. Moreover, if $f \in \mathscr{H}$, then $f(\mathbf{a} \times \mathbf{m}) = f(\mathbf{a} \times \mathbf{s})$. Hence, by the expression in (2.1), we conclude that $\mathscr{H}$ is closed under the harmonic product.

Next, we show that $\mathscr{H}$ and $\mathscr{J}$ are conjugate under ι. By the same argument as above, if $f \in \mathscr{H}$, then also $f\star\mu$. Note that $a_\lambda(f) = (f\star\mu)(\lambda)$, where $a_\lambda(f)$ is defined by Definition 2.2. Hence, if $f \in \mathscr{H}$ or $f \in \mathscr{J}$, then $a_\lambda(f)$ and $a_\rho(f)$ agree whenever $\mathbf{r}(\lambda) = \mathbf{r}(\rho)$ or $\mathbf{m}(\lambda) = \mathbf{m}(\rho)$ respectively. The statement now follows from the definition of ι using the fact that the conjugate of $(r_1, \ldots, r_d) \times (m_1, \ldots, m_d)$ is given by $(m_1, m_2 - m_1, \ldots, m_d - m_{d-1}) \times (r_1 + \ldots + r_d, \ldots, r_1)$. □

Remark 2.8 A more explicit definition of the space $\mathscr{J}$ is as follows. Given $\lambda \in \mathscr{P}$ of depth d, write $\lambda = \mathbf{r} \times \mathbf{m}$ and for $I \subset [d] := \{1, \ldots, d\}$, let $\mathbf{m}_I$ be the strict partition with parts m_i for $i \in I$. Then $f \in \mathscr{J}$ precisely if $f(\lambda)$ is determined by the value of f on all strict partitions contained in λ in the following way:

$$f(\lambda) = \sum_{I \cup J = [d]} (-1)^{|J|} \prod_{i \in I} r_i \prod_{j \in J} (r_j - 1)\, f(\mathbf{m}_I).$$

2.2 Degree and limits

Recall from (1.1) and (1.2) the definition of the q-bracket, degree $\deg(f)$ and limit $\mathrm{Z}^{\deg}(f)$ of a function f on partitions.

Example 2.9 Let us consider several examples of degree limits of interesting functions on partitions:

(i) Let $d(\lambda)$ equal the number of different parts of λ, i.e., d is the depth of λ. As

$$\sum_{\lambda \in \mathscr{P}} d(\lambda)\, q^{|\lambda|} = \frac{q}{1-q}\Big(\sum_{\lambda \in \mathscr{P}} q^{|\lambda|}\Big),$$

we have $\langle d \rangle_q = \frac{q}{1-q}$. Hence, $\deg(d) = 1$ and $\mathrm{Z}^{\deg}(d) = 1$.

(ii) Next, consider $f(\lambda) = \ell(\lambda)$. Then $\langle f \rangle_q = \sum_{m \geq 1} \frac{q^m}{1-q^m}$ (see, e.g., [28, Proposition 3.1.4]). Hence, for all $\epsilon \geq 0$ one has

$$\lim_{q \to 1} (1-q)^{1+\epsilon} \langle f \rangle_q = \lim_{q \to 1} (1-q)^{\epsilon} \sum_{m \geq 1} \frac{(1-q)\, q^m}{1-q^m}.$$

Note that $\lim_{q \to 1} \frac{(1-q)\, q^m}{1-q^m} = \frac{1}{m}$, so that $\sum_{m \geq 1} \frac{(1-q)\, q^m}{1-q^m}$ diverges as $q \to 1$ at logarithmic rate (see, also, [25]). Hence, $\lim_{q \to 1} (1-q)^{1+\epsilon} \langle f \rangle_q$ converges to 0 for $\epsilon > 0$ and diverges to ∞ for $\epsilon = 0$. In other words, the degree of f is 1 and $\mathrm{Z}^{\deg}(f) = \infty$.

(iii) Thirdly, let $f(\lambda)$ be equal to the number of even parts in λ minus the number of odd parts. That is, $f(\lambda) = \sum_m (-1)^m r_m(\lambda)$. Then, (see again [28, Proposition 3.1.4])

$$(1-q)\langle f\rangle_q = \sum_{m\geq 1}(-1)^m \frac{(1-q)q^m}{1-q^m} \xrightarrow[q\to 1]{} \sum_{m\geq 1}\frac{(-1)^m}{m} = \log(2).$$

Hence, in this example, the corresponding degree limit, conjecturally, is not a multiple zeta value.

(iv) The degree may be any real number $x \in \mathbb{R}$. Namely, let

$$f(\lambda) = \sum_{m\in \mathbf{m}(\lambda)} (-1)^m \binom{x}{m} = \sum_{m\geq 1}(-1)^m \binom{x}{m} \delta_{r_m(\lambda)\geq 1}.$$

Then (also by [28, Proposition 3.1.4])

$$\langle f\rangle_q = \sum_{m\geq 0}(-1)^m \binom{x}{m} q^m = (1-q)^x.$$

(v) It may happen that $\deg(f) = -\infty$, as is the case for the Möbius function:

$$\langle \mu\rangle_q = \Big(\sum_{\lambda\in\mathscr{P}} q^{|\lambda|}\Big)^{-2} = \prod_{m\geq 1}(1-q^m)^2.$$

(vi) If f is such that $g(\tau) = \langle f\rangle_q$ is a cusp form of weight k (with $q = e^{2\pi i\tau}$), the modular transformation $g(\tau) = \tau^{-k} g\big(-\frac{1}{\tau}\big)$ implies that

$$\lim_{q\to 1}(1-q)^k\langle f\rangle_q = \lim_{\tau\to 0}\frac{(1-e^{2\pi i\tau})^k}{\tau^k} g\Big(-\frac{1}{\tau}\Big) = \lim_{q\to 0}(-2\pi i)^k\langle f\rangle_q = 0.$$

This illustrates the fact that if $\langle f\rangle_q$ is a quasimodular form of weight k, the degree of f is k and the limit can be computed using the quasimodular transformation (in fact, for quasimodular forms one can recover the full asymptotic expansion as $q \to 1$ using *growth polynomials*; see [14, Section 9]).

Remark 2.10 Given $F = \sum_{n\geq 0} a_n q^n \in \mathbb{Q}[\![q]\!]$ there are infinitely many functions $f : \mathscr{P} \to \mathbb{Q}$ for which $\langle f\rangle_q = F$ (the only condition on f is that $\sum_{|\lambda|=n} f(\lambda) = \sum_{m\geq 0} a_m\, p(n-m)$ with $p(i)$ the number of partitions of i). Hence, one could define the degree and the degree limit of F as the degree and degree limit of f for one such f. However, in the generality of this section, one does not discover a lot of structure in the values $\mathrm{Z}^{\deg}(F)$. In the next section, we see how this situation alters when one restricts to a certain polynomial subspace of $\mathbb{Q}^{\mathscr{P}}$.

3 Partition analogues of multiple zeta values

3.1 Polynomial functions on partitions

The spaces of polynomials and modular forms are graded algebras with the property that after fixing the degree, or weight and a congruence subgroup respectively, the corresponding vector spaces are finite dimensional. Similarly, the algebra $\mathbb{P}$ admits a weight filtration such that the vector subspace of elements of a fixed weight is finite dimensional.

This weight filtration is most naturally introduced after giving an equivalent definition for $\mathbb{P}$. That is, let Φ be the composition of the $\underline{u}$-bracket (see Definition 2.2), and the mapping Ψ from Definition 1.1. Then, Φ denotes the linear map

$$\Phi : \bigoplus_{n\geq 0} \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] \to \mathbb{Q}[\![u_1, u_2, \ldots]\!]$$

uniquely determined by

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \sum_{\substack{m_1>\ldots>m_n>0 \\ r_1,\ldots,r_n\geq 1}} g(m_1, \ldots, m_n, r_1, \ldots, r_n)\, u_{m_1}^{r_1} \cdots u_{m_n}^{r_n}$$

(see Proposition 3.9). Now, $\mathrm{Im}\Phi$ is an algebra with respect to the natural product on $\mathbb{Q}[\![u_1, u_2, \ldots]\!]$, and Φ becomes an algebra homomorphism if one defines a corresponding product on the domain. Even more, this domain admits a weight and a depth filtration, determined by assigning to $g(x_1, \ldots, x_n, y_1, \ldots, y_n)$ weight $\deg g + n$ and depth n. The main advantage of Φ over Ψ is that the natural product on the codomain of Φ (in contrast to the natural product on the codomain of Ψ) behaves well with respect to the q-bracket (see Proposition 2.5). This yields the following natural definition for the weight and depth filtration on $\mathbb{P}$.

Definition 3.1 We call a function $f \in \mathbb{Q}^{\mathscr{P}}$ a polynomial function on partitions of *weight* $\leq k$ and *depth* $\leq p$ if $\langle f \rangle_{\underline{u}} = \Phi(g)$ for some g of weight $\leq k$ and depth $\leq p$ (where we recall that $g(x_1, \ldots, x_n, y_1, \ldots, y_n)$ has weight $\leq \deg g + n$ and depth $\leq n$).

Note that the vector space of polynomial functions on partitions of bounded weight is finite dimensional. Recall, we denote the $\mathbb{Q}$-vector space of all polynomial functions on partitions by

$$\mathbb{P} = \langle f \in \mathbb{Q}^{\mathscr{P}} \mid f \text{ is a polynomial function} \rangle_{\mathbb{Q}}$$

and write for the subspace of all polynomial functions of weight $\leq k$

$$\mathbb{P}_{\leq k} := \langle f \in \mathbb{P} \mid f \text{ is of weight} \leq k \rangle_{\mathbb{Q}}\,.$$

In the next section, we provide a basis for $\mathbb{P}$ (or, in fact, several), which one can easily find because of the following two results. Note that this contrasts the situation

for multiple zeta values, as well as for q-analogues of multiple zeta values, where so far it has not been possible to prove that a certain generating set forms a basis.

Proposition 3.2 *The linear map Φ is injective.*

Proof Suppose $\Phi(g) = 0$ and write $g = (g_0, g_1, \ldots)$ with $g_n \in \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Suppose there exists a minimal n such that $g_n \not\equiv 0$. Then, there are integers $m_1 > \ldots > m_n > 0$ and $r_1, \ldots, r_n \geq 1$ such that

$$g_n(m_1, \ldots, m_n, r_1, \ldots, r_n) \neq 0.$$

Hence, the coefficient of $u_{m_1}^{r_1} \cdots u_{m_n}^{r_n}$ in $\Phi(g)$ is non-zero, contradicting our assumption. □

Corollary 3.3 *Let $f \in \mathbb{Q}^{\mathscr{P}}$. Then $f \in \mathbb{P}$ precisely if there exist a $p_0 \in \mathbb{Q}$ and for $n \geq 1$ polynomials $p_n \in y_1 \ldots y_n \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $p_n \equiv 0$ for all but finitely many n, such that for any partition λ we have*

$$f(\lambda) = p_0 + \sum_{n \geq 1} \sum_{m_1 > \cdots > m_n > 0} p_n(m_1, \ldots, m_n, r_{m_1}(\lambda), \ldots, r_{m_n}(\lambda)). \tag{3.1}$$

Moreover, the function f

(i) is of weight $\leq k$ *if* $\deg(p_n) \leq k$ *for all* $n \geq 0$.
(ii) is of depth $\leq r$ *if* $p_n \equiv 0$ *for* $n > r$,
(iii) uniquely determines the polynomials p_n.

Proof Here, we use that $\sum_{r=1}^{R} r^d$ is a polynomial which is of degree $d+1$ in R and divisible by R. For (iii) we use the previous proposition. □

Remark 3.4 The function

$$f(\lambda) = \sum_{m_1 > m_2 > 0} m_1 m_2 \, r_{m_1}(\lambda) \tag{3.2}$$

is a polynomial function on partitions, although $x_1 x_2 y_1 \notin y_1 y_2 \mathbb{Q}[x_1, x_2, y_1, y_2]$. Namely, observe that

$$f(\lambda) = \sum_{m_1 > 0} \tfrac{1}{2} m_1^2 (m_1 - 1) \, r_{m_1}(\lambda).$$

Observe that f is of weight ≤ 4 and depth ≤ 1; a fact one could not easily read of from the expression (3.2).

More generally, one could relax the condition $p_n \in y_1 \cdots y_n \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by $p_n \in y_1 \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, at the cost of breaking the uniqueness of the representation (3.1)—as we will see in Corollary 3.8—and making it harder to read of the degree and depth.

Remark 3.5 There exist natural extensions $\mathbb{P}(N)$ of $\mathbb{P}$ to higher levels, i.e., such that $\mathbb{M}(N)$ is the subspace of $\mathbb{P}(N)$ for which the q-brackets are quasimodular forms of level N and for which the corresponding limits as q tends to an Nth root of unity are multiple polylogarithms at roots of unity (sometimes called *colored multiple zeta values*). For example, the alternating multiple zeta value $\log 2$ can be obtained in this way: see Example 2.9(iii). More concretely, one would take

$$\bigoplus_{n\geq 0} \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n,\zeta^{x_1},\ldots,\zeta^{x_n},\zeta^{y_1},\ldots,\zeta^{y_n} \mid \zeta^N = 1]$$

as the domain in the definition of Φ. We do not pursue working out all details here but instead refer to [29], where this has been worked out in a similar setting.

3.2 Bases for the space of polynomial functions on partitions

We define a basis of $\mathbb{P}$, or rather different bases. These bases depend on a polynomial sequence in the following way.

Definition 3.6 Let $\mathcal{F} = \{f_k\}_{k=0}^{\infty}$ be a polynomial sequence (i.e., $f_k \in \mathbb{Q}[x]$ is of degree k) such that $f_0 = 1$ and $f_k(0) = 0$ for $k \geq 1$. For $r \geq 0$, $k_1 \geq 1, k_2, \ldots, k_r \geq 0$ and $d_1, \ldots, d_r \geq 0$, we define the map

$$P_{\mathcal{F}}\begin{pmatrix} k_1,\ldots,k_r \\ d_1,\ldots,d_r \end{pmatrix} : \mathscr{P} \to \mathbb{Q}$$

by

$$P_{\mathcal{F}}\begin{pmatrix} k_1,\ldots,k_r \\ d_1,\ldots,d_r \end{pmatrix}(\lambda) := \sum_{m_1>\cdots>m_r>0} \prod_{j=1}^{r} m_j^{d_j} f_{k_j}(r_{m_j}(\lambda))$$

and by $P_{\mathcal{F}}(\lambda) = 1$ if $r = 0$. We write

$$H_{\mathcal{F}}(k_1,\ldots,k_r) = P_{\mathcal{F}}\begin{pmatrix} k_1,\ldots,k_r \\ 0,\ldots,0 \end{pmatrix}, \quad J_{\mathcal{F}}(d_1,\ldots,d_r) = P_{\mathcal{F}}\begin{pmatrix} 1,\ldots,1 \\ d_1,\ldots,d_r \end{pmatrix}.$$

Note that $P_{\mathcal{F}}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} \in \mathbb{P}$, $H_{\mathcal{F}}(k_1,\ldots,k_r) \in \mathbb{P} \cap \mathscr{H}$ and $J_{\mathcal{F}}(d_1,\ldots,d_r) \in \mathbb{P} \cap \mathscr{J}$, where $\mathscr{H}$ and $\mathscr{J}$ were defined in Definition 2.6.

Definition 3.7 A polynomial sequence $\mathcal{F} = \{f_k\}_{k=0}^{\infty}$ with $f_0 = 1$ and $f_k(0) = 0$ for $k \geq 1$ is called *well-normalized* if the leading coefficient of f_k equals $\frac{1}{k!}$. In Table 1, we define four families of polynomials of which the last three are well-normalized.

Corollary 3.8 (of Proposition 3.2) *For all polynomial sequences $\mathcal{F}$, a basis for $\mathbb{P}_{\leq k}$ is given by the functions $P_{\mathcal{F}}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r}$ for all $r \geq 0$, $k_1,\ldots,k_r \geq 1$, and $d_1,\ldots,d_r \geq 0$ such that $k_1+\ldots+k_r+d_1+\ldots+d_r \leq k$.*

Table 1 Overview of different models of partition analogues

$\mathcal{F}$	Name	Definition of f_k for $k \geq 1$
m	Monomial model	$f_k(x) = x^k$
s	Bernoulli–Seki model[1]	$f_k(x) - f_k(x-1) = \frac{1}{(k-1)!}x^{k-1}$ and $f_k(0) = 0$
b	Binomial model	$f_k(x) = \binom{x}{k}$
b^+	Shifted binomial model	$f_k(x) = \binom{x+1}{k} - \delta_{k,1}$

[1] The corresponding polynomials are often referred to as Faulhaber polynomials, but here we use the name which we believe is historically more correct

3.3 Multiplication, conjugations and brackets of partition analogues

In Sect. 2.1 we introduced the $\underline{u}$-bracket, two natural conjugation operations, and three natural product operations on the space of all functions on partitions. We will now explain how the space of *polynomial* functions on partitions behaves under these operations.

First of all, the $\underline{u}$-bracket of elements of $\mathbb{P}$ is given in Proposition 3.9 below. As a corollary, the q-bracket of such an element can be seen as a q-analogue of multiple zeta values as introduced in [2] and further discussed in [6]. In [6] the authors denote by $\mathcal{Z}_q$ the space spanned by all q-series of the form

$$\sum_{m_1>\cdots>m_r>0} \prod_{j=1}^{r} \frac{Q_{k_j}(q^{m_j})}{(1-q^{m_j})^{k_j}}, \tag{3.3}$$

for $k_1, \ldots, k_r \geq 1$ and polynomials $Q_{k_j}(X) \in \mathbb{Q}[X]$ with $\deg Q_{k_j} \leq k_j$ and $Q_{k_1}(0) = 0$. There it was shown ([6, Theorem 1]) that $\mathcal{Z}_q$ is spanned by the q-series (3.4), whose corresponding polynomial functions on partition also span our space $\mathbb{P}$ (see Example 3.10(i)). That is, $\langle \mathbb{P} \rangle_q$ equals the space of q-analogues $\mathcal{Z}_q$. In Sect. 4.1, we show that the (regularized) limit $q \to 1$ of elements in $\mathcal{Z}_q$ is always a (regularized) multiple zeta value.

Given a polynomial f, for an integer n, denote by $\partial f(n) = f(n) - f(n-1)$ if $n \geq 1$ and let $\partial f(0) = f(0)$. Then [28, Proposition 3.1.4 and Lemma 3.5.2] yields:

Proposition 3.9 *The $\underline{u}$-bracket of $P_{\mathcal{F}}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r}$ is given by*

$$\left\langle P_{\mathcal{F}}\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} \right\rangle_{\underline{u}} = \sum_{m_1>\cdots>m_r>0} \prod_{j=1}^{r} \left(m_j^{d_j} \sum_{n \geq 0} \partial f_{k_j}(n)\, u_{m_j}^n \right).$$

Recall that $f_0 = 1$, so that $\partial f_0(n) = \delta_{n,0}$. Moreover, for $k \geq 1$, we have $f_k(0) = 0$, so that $\partial f_k(0) = 0$.

Example 3.10 For some particular choices of $\mathcal{F}$, the q-bracket of partition analogues of multiple zeta values gives rise to several models for q-analogues of multiple zeta values:

(i) The Bernoulli–Seki model $\mathcal{F} = \mathrm{s}$ corresponds, up to a factor, to the *bi-brackets* introduced by the first author in [2]

$$\left\langle P_{\mathrm{s}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}\right\rangle_q = \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}} \prod_{j=1}^{r} m_j^{d_j} \frac{n_j^{k_j-1}}{(k_j-1)!} q^{m_j n_j} \quad (k_i \geq 1). \tag{3.4}$$

For $r = 1$, $d_1 = 0$ and even $k_1 \geq 2$ these are, up to the constant term, the Eisenstein series $G_{k_1} = -\frac{B_{k_1}}{2k_1!} + \left\langle P_{\mathrm{s}}\binom{k_1}{0}\right\rangle_q$, defined in (1.4). In the case $r = 1, k_1 > 1, d_1 > 0$ and $k_1 + d_1$ even, they are essentially the d_1-th derivative of $G_{k_1-d_1}$. For example for $k > d > 0$ we have

$$\left\langle P_{\mathrm{s}}\binom{k}{d}\right\rangle_q = \frac{(k-d-1)!}{(k-1)!}\left(q\frac{d}{dq}\right)^d G_{k-d}\,. \tag{3.5}$$

Together with

$$\left\langle P_{\mathrm{s}}\binom{1}{1}\right\rangle_q = \left\langle P_{\mathrm{s}}\binom{2}{0}\right\rangle_q = G_2 + \frac{1}{24}$$

and

$$\left\langle P_{\mathrm{s}}\binom{k}{d}\right\rangle_q = \frac{d!}{(k-1)!}\left\langle P_{\mathrm{s}}\binom{d+1}{k-1}\right\rangle_q$$

we see that all q-brackets of $P_{\mathrm{s}}\binom{k}{d}$ with $k+d$ even are in the ring $\widetilde{M}$ of quasi-modular forms.

(ii) The binomial model $\mathcal{F} = \mathrm{b}$ corresponds to the so-called Schlesinger–Zudilin q-analogues of multiple zeta values [26, 33] given by

$$\langle H_{\mathrm{b}}(k_1,\dots,k_r)\rangle_q = \sum_{m_1>\dots>m_r>0} \prod_{j=1}^{r} \frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}} \qquad (k_1 \geq 1, k_i \geq 0).$$

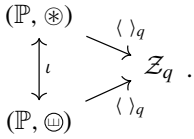
(iii) Shifted binomial polynomials correspond to the q-analogues of multiple zeta values introduced by Bradley [11] and Zhao given by

$$\langle H_{\mathrm{b}^+}(k_1,\dots,k_r)\rangle_q = \sum_{m_1>\dots>m_r>0} \prod_{j=1}^{r} \frac{q^{m_j(k_j-1)}}{(1-q^{m_j})^{k_j}} \qquad (k_1 \geq 2, k_i \geq 1).$$

The three products on the space of all functions on partitions naturally reduce to $\mathbb{P}$. That is, we obtain three commutative $\mathbb{Q}$-algebras $(\mathbb{P}, \circledast)$, $(\mathbb{P}, \circledcirc)$ and $(\mathbb{P}, \odot)$ with a natural involution, as stated in the following result.

Proposition 3.11 *The space $\mathbb{P}$ of polynomial function on partitions*

(i) is closed under the conjugation ι.
(ii) is closed under the harmonic, shuffle and pointwise products $\circledast$, $\circledcirc$ and $\odot$.
(iii) has D (see Definition 2.3) as a derivation on both $(\mathbb{P}, \circledast)$ and $(\mathbb{P}, \circledcirc)$.
(iv) admits the algebra homomorphisms given in the following commutative diagram of differential $\mathbb{Q}$-algebras (with the derivation $q\frac{d}{dq}$ on $\mathcal{Z}_q$)



Proof The first statement follows since the conjugate of $(r_1, \ldots, r_d) \times (m_1, \ldots, m_d)$ is given by $(m_1, m_2 - m_1, \ldots, m_d - m_{d-1}) \times (r_1 + \ldots + r_d, \ldots, r_1)$ and therefore the conjugate of a polynomial in Stanley coordinates is again a polynomial in Stanley coordinates. Statement (ii) follows from Proposition 3.9 together with Corollary 3.8, since the harmonic, as well as the pointwise product, of two $\underline{u}$-brackets of $P_{\mathcal{F}}\binom{\cdots}{\cdots}$ can again be expressed as a linear combination of $\underline{u}$-brackets of $P_{\mathcal{F}}\binom{\cdots}{\cdots}$. We will see this explicitly in Proposition 4.13. Together with (i) this then also shows that the space is closed under $\circledcirc$. Since D is the derivation on $\mathbb{Q}[\![u_1, u_2, \ldots]\!]$, defined on the generators by $Du_n = nu_n$, (iii) follows immediately by the fact that both $\circledast$ and $\circledcirc$ are defined via the $\underline{u}$-bracket. Finally (iv) follows from (i)–(iii) and the definition of $\circledast$ by the $\underline{u}$-bracket after setting $u_n = q^n$. □

Remark 3.12 The space $\mathbb{M} \subset \mathbb{P}$ is also closed under ι and the products $\circledast$, $\circledcirc$. Moreover, since the space of quasimodular forms is closed under $q\frac{d}{dq}$, the space $\mathbb{M}$ is closed under D as well, i.e., $\mathbb{M}$ satisfies almost all properties of Proposition 3.11. But one can check that $\mathbb{M}$ is not closed under $\odot$. In general, it is not clear how $\mathbb{M}$ can be described explicitly as a subspace of $\mathbb{P}$. Since it contains the kernel of the q-bracket, it might be difficult to give an explicit description. In this note, we will give several examples of explicit elements in $\mathbb{M}$, but expect that there are many more (see, e.g., Example 5.2). Another open question is if there are other interesting subalgebras of $\mathbb{P}$, which satisfy (some of) the properties of Proposition 3.11.

Remark 3.13 The space $\mathbb{P}$ is not closed under the conjugation ω. For example, for $\mathcal{F}$ well-normalized $\omega(P_{\mathcal{F}}\binom{1}{0})(\lambda) = \lambda_1$ equals the biggest part of the partition λ. It is a small computation to verify that

$$\langle \lambda_1 \rangle_{\underline{u}} = -\sum_{\lambda \in \mathscr{P}} \mu(\lambda)\, \lambda_\ell\, u_\lambda \notin \langle \mathbb{P} \rangle_{\underline{u}} \qquad (\ell = \ell(\lambda) \text{ is the } length \text{ of } \lambda).$$

The subspace Λ^* of $\mathbb{P}$, defined in the introduction, however, has the remarkable property that it is closed under ω but not under ι.

Remark 3.14 For the Bernoulli–Seki model $\mathcal{F} = \mathrm{s}$ the involution ι is explicitly given by

$$\iota\, P_{\mathrm{s}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r} = \sum_{\substack{a_1,\dots,a_r\geq 1\\ |\underline{a}|=|\underline{d}|+r}} \sum_{\substack{b_1,\dots,b_r\geq 0\\ |\underline{b}|=|\underline{k}|-r}} C^{\underline{a},\underline{k}}_{\underline{b},\underline{d}}\, P_{\mathrm{s}}\binom{a_1,\dots,a_r}{b_1,\dots,b_r}, \tag{3.6}$$

where for $\underline{a},\underline{b},\underline{d},\underline{k}\in\mathbb{Z}^r$ the constant $C^{\underline{a},\underline{k}}_{\underline{b},\underline{d}}$ is given by

$$C^{\underline{a},\underline{k}}_{\underline{b},\underline{d}} = (-1)^{|\underline{b}|}\prod_{j=1}^{r}\tbinom{s_j(\underline{d})-s^j(\underline{a})+j-1}{a_{r-j+1}-1}\tbinom{k_{r-j+1}-1}{s_j(\underline{b})-s^j(\underline{k})+j-1}\frac{(a_j-1)!}{(k_j-1)!}(-1)^{s_j(\underline{k})+s^j(\underline{b})+j}$$

and where for $\underline{\ell}\in\mathbb{Z}^r$ and $j=1,\dots,r$, we write

$$s_j(\underline{\ell}) = \sum_{i=1}^{j}\ell_i, \qquad s^j(\underline{\ell}) = \sum_{i=r-j+2}^{r}\ell_i.$$

Since the q-bracket is ι-invariant, (3.6) gives explicit linear relations among the q-brackets of the Bernoulli–Seki model (3.4). This corresponds to the partition relation in [2, Theorem 2.3]. There it was also conjectured that these families of relations together with the stuffle product are enough to write any q-series (3.4) as a linear combination of those with either $d_1=\cdots=d_r=0$ or $k_1=\cdots=k_r=1$. In our setup here this translates to the following model independent conjecture:

Conjecture 3.15 *We have*

$$\mathcal{Z}_q = \langle\mathbb{P}\rangle_q = \langle\mathbb{P}\cap\mathscr{H}\rangle_q = \langle\mathbb{P}\cap\mathscr{J}\rangle_q\,.$$

Notice that the first equation here is known as explained in the remark at the beginning of Sect. 3.3 and the last equation is clear since $\iota(\mathbb{P}\cap\mathscr{H}) = \mathbb{P}\cap\mathscr{J}$. The functions in $\mathbb{P}\cap\mathscr{H}$ are exactly those polynomial functions on partitions where the polynomials in Definition 1.1 are independent of the variables x_i. This conjecture was also discussed in more detail for the Bernoulli–Seki model in [6] and [13].

3.4 Some remarks about the Möller transform

In contrast to the conjugation, products and derivation above, some natural operations on functions on partitions do not leave the space of polynomial functions invariant. One of the most interesting such operators is the Möller transform $\mathcal{M}:\mathbb{Q}^{\mathscr{P}}\to\mathbb{Q}^{\mathscr{P}}$ in [14]:

Definition 3.16 Given $f:\mathscr{P}\to\mathbb{Q}$, its *Möller transform* $\mathcal{M}f:\mathscr{P}\to\mathbb{Q}$ is the function given on a partition λ of size n by

$$\mathcal{M}f(\lambda) = \frac{1}{n!} \sum_{\mu \in \mathscr{P}(n)} |C_\mu| \, \chi_\lambda(\mu)^2 \, f(\mu),$$

where $|C_\mu|$ denotes the size of the conjugacy class associated to μ in the symmetric group $\mathfrak{S}_n$, and $\chi_\lambda(\mu)$ denotes value of the character of the irreducible representation of $\mathfrak{S}_n$ associated to λ at any element in the conjugacy class associated to μ.

The key property of the Möller transform (which follows directly from the second orthogonality relation for the characters of the symmetric group) is that for all $f : \mathscr{P} \to \mathbb{Q}$ one has

$$\langle \mathcal{M}f \rangle_q = \langle f \rangle_q.$$

For the following subvectorspace of $\mathbb{P}$, the Möller transform is known to preserve the space of polynomial functions:

Proposition 3.17 *Let $f \in \mathbb{P}$. Then $\mathcal{M}(f) \in \mathbb{P}$ if f is in the vector space generated by the constant function* 1 *and*

$$P\begin{pmatrix} 1 \\ k-1 \end{pmatrix} \odot \underbrace{P\begin{pmatrix} 1 \\ 1 \end{pmatrix} \odot \cdots \odot P\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{m}$$

for any $k \geq 2$ and $m \geq 0$.

Remark 3.18 As the statement of this proposition does not depend on the choice of $\mathcal{F}$, we also omit it from the notation.

Proof For all $k \geq 1$, the Möller transform $\mathcal{M}P\begin{pmatrix} 1 \\ k-1 \end{pmatrix}$ equals the hook-length moment T_k, given by

$$T_k(\lambda) := \sum_{\xi \in Y_\lambda} h(\xi)^{k-2},$$

where Y_λ denotes the Young diagram of λ, ξ a cell in this diagram and $h(\xi)$ the hook-length of this cell. Note that for even k this function lies in the space of shifted symmetric functions defined in the introduction, as is shown in [14, Thm. 13.5]. For all $k \geq 2$, the results of Sect. 5.3 imply that $\mathcal{M}P\begin{pmatrix} 1 \\ k-1 \end{pmatrix}$ is a polynomial function on partitions.

Also, any polynomial in $|\lambda|$, that is, any polynomial in $P\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with respect to $\odot$ is invariant under the Möller transform—as follows easily from the first orthogonality relation. In fact, $\mathcal{M}(f \odot P\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \mathcal{M}(f) \odot P\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ from which the result follows. □

Remark 3.19 For $\mathcal{M}P\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, the situation is totally different. In this case

$$\mathcal{M}P\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)(\lambda) = \sum_{\xi \in Y_\lambda} h(\xi)^{-1}.$$

As the q-bracket of a function and its Möller transform agree, we would expect this function to be of weight 1 as well, if it were a polynomial function on partitions. It is a matter of linear algebra to check that $\mathcal{M}P\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is not a polynomial function on partitions of weight ≤ 1.

In fact, the 233-dimensional vector space of polynomial functions on partitions of weight ≤ 6 only has a 10-dimensional subspace for which the Möller transform is again a polynomial function on partitions—this subspace is generated by 1, $P\left(\begin{smallmatrix}1\\k-1\end{smallmatrix}\right)$ for $k = 2, \ldots, 6$, $P\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \odot P\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) \odot P\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $P\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \odot P\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \odot P\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $P\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right) \odot P\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$.

Lemma 3.20 *For $f \in \mathbb{P}_{\leq 6}$ the converse in Proposition 3.17 holds.*

Note that this lemma can be proven by finding the kernel of a matrix containing at least $2 \cdot 233 + 1$ values of all these polynomial functions as well as of their Möller transforms, as we did in Pari/GP. Hence, it seems the converse of Proposition 3.17 holds for all $f \in \mathbb{P}$.

4 From partitions to multiple zeta values

4.1 Weight and degree limits of partition analogues

For functions $f \in \mathbb{P}$, recall $\mathrm{Z}^{\deg}(f)$ and $\deg(f)$ are defined such that

$$(1-q)^{\deg(f)}\langle f \rangle_q = \mathrm{Z}^{\deg}(f) + O(1-q)$$

asymptotically for *real* q. We will see that the degree is a non-negative *integer* bounded above by the weight of f. In this section, we prove Theorem 1.2. That is, we compute this degree and show that $\mathrm{Z}^{\deg}(f)$ is a multiple zeta value, which justifies calling the elements of $\mathbb{P}$ partition analogues of multiple zeta values. From now on, we omit the *well-normalized* family $\mathcal{F}$ from the notation and write P instead of $P_{\mathcal{F}}$, unless the results depend on the choice of $\mathcal{F}$.

To get started we discuss two examples. First of all, we consider the degree of derivatives of polynomial functions on partitions.

Lemma 4.1 *Given $f \in \mathbb{P}$, we have*

$$\deg(Df) = \deg f + 1, \qquad \mathrm{Z}^{\deg}(Df) = \deg f \cdot \mathrm{Z}^{\deg}(f).$$

Proof. This follows directly from L'Hôpital's rule, as for all $k \neq 0$ one has

$$\lim_{q\to 1}(1-q)^k\langle f\rangle_q = \lim_{q\to 1}\frac{1}{k}(1-q)^{k+1}q\frac{\partial}{\partial q}\langle f\rangle_q = \lim_{q\to 1}\frac{1}{k}(1-q)^{k+1}\langle Df\rangle_q\,.$$

□

As another example, we consider polynomial functions on partitions of depth ≤ 1, corresponding to *single* zeta values. To calculate degree limits explicitly we define for $k \geq 1$ the (slightly modified) Eulerian polynomials E_k by

$$\frac{E_k(X)}{(1-X)^k} := \sum_{d\geq 1}\frac{d^{k-1}}{(k-1)!}X^d. \tag{4.1}$$

Also, write $[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$.

Lemma 4.2 *Let $k \geq 1, d \geq 0$. Then,*

$$\deg P\binom{k}{d} = \max(k, d+1), \quad \mathrm{Z}^{\deg} P\binom{k}{d} = \begin{cases}\zeta(k-d) & k \geq d+1\\ \frac{d!}{(k-1)!}\zeta(d-k+2) & k \leq d+1.\end{cases}$$

Proof Consider the limit $q \to 1$ of the function $(1-q)^a \sum_{m,n}\frac{m^{k-1}}{(k-1)!}n^d q^{mn}$, which is the only term contributing to the degree limit for any well-normalized family. For $a = k$, this equals

$$\sum_{n\geq 1} n^d\,\frac{E_k(q^n)}{[n]_q^k}. \tag{4.2}$$

By Proposition 4.5 below the terms in the sum (4.2) are bounded by $(k+1)n^{d-k}$ and therefore the sum converges absolutely if $k - d > 1$. As $\lim_{q\to 1}\frac{E_k(q^n)}{[n]_q^k} = \frac{1}{n^k}$, in that case it follows that

$$\lim_{q\to 1}(1-q)^k\,P\binom{k}{d} = \zeta(k-d).$$

Conversely, one obtains the case $k - d < 1$ by the same argument interchanging the roles of k and d. Finally, in case $k - d = 1$, a more careful analysis as in [25], shows the limit diverges—in accordance with the divergence of $\zeta(1)$. Also, after letting $a = k + \epsilon$ and $k - \epsilon$ for some $\epsilon > 0$, the limit converges to 0 and diverges to $+\infty$ respectively. □

Remark 4.3 Note that Lemma 4.1 and Lemma 4.2 are in agreement with the relation $DP_{\mathrm{s}}\binom{k}{d} = kP_{\mathrm{s}}\binom{k+1}{d+1}$.

In the previous lemma, the proof of absolute convergence of the sum (4.2) was postponed. Note that by applying the techniques for asymptotics of sums of the form $\sum_n f(nt)$ in [31], as was previously observed in [34, Lemma 1], one obtains the following lemma (where the Eulerian polynomials E_k are defined by (4.1)).

Lemma 4.4 *For $k, n \geq 1$*

$$\frac{E_k(q^n)}{[n]_q^k} = \frac{1}{n^k} + O(1-q), \qquad (q \to 1^-).$$

We will give an upper bound for the left-hand side for all $q \in [0, 1]$ in the following proposition. The special cases $k = 1, 2$ were obtained before in [5, Lemma 6.6(ii)].

Proposition 4.5 *For $k, n \geq 1$ and $q \in [0, 1]$ we have*

$$\frac{E_k(q^n)}{[n]_q^k} \leq \frac{k+1}{n^k}.$$

Proof. By continuity, we may assume $q \in (0, 1)$. By definition, we have

$$\frac{E_k(q^n)}{[n]_q^k} = \frac{E_k(q^n)}{(1-q^n)^k}(1-q)^k = \left(\sum_{d\geq 1} \frac{d^{k-1}}{(k-1)!} q^{dn}\right)(1-q)^k.$$

Thus, it suffices to show

$$\left(\sum_{d\geq 1} \frac{d^{k-1}}{(k-1)!} q^{dn}\right)(1-q)^k \leq \frac{k+1}{n^k} \qquad (k, n \geq 1, q \in (0, 1)).$$

We have

$$\begin{aligned}\sum_{d\geq 1} d^{k-1} q^{dn} &= \left(\log \frac{1}{q}\right) \sum_{d\geq 1} d^{k-1} \int_{dn}^{\infty} q^u \, du \\ &= \left(\log \frac{1}{q}\right) \int_n^{\infty} \left(\sum_{1\leq d\leq u/n} d^{k-1}\right) q^u \, du,\end{aligned}$$

where interchanging the sum and integral is allowed since the sum $\sum_{d\geq 1} d^{k-1} q^{dn}$ converges (absolutely) for $q \in (0, 1)$. For $u \geq n$, we now have

$$\begin{aligned}\sum_{1\leq d\leq u/n} d^{k-1} &= \sum_{1\leq d\leq \lfloor u/n\rfloor -1} d^{k-1} + \left(\frac{u}{n}\right)^{k-1} \\ &\leq \sum_{1\leq d\leq \lfloor u/n\rfloor -1} \int_d^{d+1} y^{k-1} \, dy + \left(\frac{u}{n}\right)^{k-1} \\ &\leq \int_1^{u/n} y^{k-1} \, dy + \left(\frac{u}{n}\right)^{k-1} \\ &\leq \frac{k+1}{k}\left(\frac{u}{n}\right)^k.\end{aligned}$$

We thus have

$$\begin{aligned}\sum_{d\geq 1}\frac{d^{k-1}}{(k-1)!}q^{dn} &\leq \frac{k+1}{k!}\frac{1}{n^k}\left(\log\frac{1}{q}\right)\int_n^\infty q^u\,u^k\,du\\ &= \frac{k+1}{k!}\frac{1}{n^k}\left(\log\frac{1}{q}\right)^{-k}\int_{n\log\frac{1}{q}}^\infty e^{-u}\,u^k\,du\\ &\leq \frac{k+1}{k!}\frac{1}{n^k}\left(\log\frac{1}{q}\right)^{-k}\int_0^\infty e^{-u}\,u^k\,du\\ &= \frac{k+1}{n^k}\left(\log\frac{1}{q}\right)^{-k}.\end{aligned}$$

By using the bound

$$\log\frac{1}{q}=\int_q^1\frac{du}{u}\;\geq\;1-q,$$

we get

$$\left(\sum_{d\geq 1}\frac{d^{k-1}}{(k-1)!}q^{dn}\right)(1-q)^k\;\leq\;\frac{k+1}{n^k}\left(\frac{1-q}{\log\frac{1}{q}}\right)^k\;\leq\;\frac{k+1}{n^k}.$$

□ □

Remark 4.6 The stronger bound

$$\frac{E_k(q^n)}{[n]_q^k}\;\leq\;\frac{1}{n^k}\tag{4.3}$$

fails for k divisible by 4, large n and q close to 1. Namely, setting $e^{-t}=q^n$, one finds that this inequality is equivalent to

$$\sum_{d\geq 1}\frac{d^{k-1}}{(k-1)!}\,e^{-dt}\;<\;\frac{1}{(n(1-e^{-t/n}))^k}.$$

Note, $(n(1-e^{-t/n}))$ is bounded from below by t and converges to t as $n\to\infty$. As $t\downarrow 0$, one would deduce from this inequality that $\zeta(1-k)\;<\;0$, which fails if $4\mid k$. Concretely, $k=4$, $q=e^{-10^{-4}}$ and $n=10^4$ provides a counterexample to (4.3).

To determine the degree of a polynomial function on partitions, we start with two estimates on this degree. Together, these statements are an improvement of the statement that the degree is bounded by the weight. In fact, in most cases, the degree is strictly smaller than the weight, so that the weight limit, defined below, vanishes in most cases.

Definition 4.7 For $f \in \mathbb{P}$ of weight $\leq k$ which is not of weight $\leq k-1$, the *weight limit* $Z(f)$ is given by

$$Z(f) := \lim_{q\to 1}(1-q)^k \langle f\rangle_q.$$

Notice that Z is not a linear map.

Lemma 4.8 *For all $\underline{k}$ and $\underline{d}$ one has*

(i) $\deg P\begin{pmatrix} k_1,\ldots,k_r \\ d_1,\ldots,d_r\end{pmatrix} \leq \sum_i \max(k_i, d_i+1)$;

(ii) $\deg P\begin{pmatrix} k_1,\ldots,k_{i+1},\ldots,k_r \\ d_1,\ldots,d_{i+1}+1,\ldots,d_r\end{pmatrix} \leq \deg P\begin{pmatrix} k_1,\ldots,k_i,\ldots,k_r \\ d_1,\ldots,d_i+1,\ldots,d_r\end{pmatrix}$;

Moreover, if there is no index t such that $k_i = 1$ for all $i \leq t$ and $d_i = 0$ for all $i > t$, one has

$$ZP\begin{pmatrix} k_1,\ldots,k_r \\ d_1,\ldots,d_r\end{pmatrix} = 0.$$

Proof Given $F \in \mathbb{Q}[\![q]\!]$, write $[q^n]F$ for the coefficient of q^n in F. Forgetting the inequalities $m_1 > m_2 > \ldots > m_r$, one obtains

$$[q^n]\left\langle P\begin{pmatrix} k_1,\ldots,k_r \\ d_1,\ldots d_r\end{pmatrix}\right\rangle_q \leq [q^n]\left\langle P\begin{pmatrix} k_1 \\ d_1\end{pmatrix}\right\rangle_q \cdots \left\langle P\begin{pmatrix} k_r \\ d_r\end{pmatrix}\right\rangle_q$$

for all $n \in \mathbb{N}$, from which the first part of the statement follows using Lemma 4.2. The second statement follows by estimating $m_{i+1}^{d_{i+1}+1} \leq m_{i+1}^{d_{i+1}} m_i$. Finally, if such a t does not exist, then the first two properties imply that the degree of the function at hand is strictly less than the weight, so that the weight limit vanishes. □

We now prove Theorem 1.2, which determines the degree of a polynomial function

$$f = P\begin{pmatrix} k_1,\ldots,k_r \\ d_1,\ldots,d_r\end{pmatrix}.$$

That is,

$$\deg(f) = \max_{j\in\{0,\ldots,r\}}\left\{\sum_{i\leq j}(d_i+1) + \sum_{i>j} k_i\right\}. \tag{4.4}$$

Recall that if $r = 1$, by Lemma 4.2 the degree limit diverges if $k_1 = d_1 + 1$. In general, this is the case when $k_a + \ldots + k_b = (d_a+1) + \ldots + (d_b+1)$ for some indices $a \leq b$. If such indices do not exist, the maximum in (4.4) is unique and we show the corresponding degree limit is a sum of multiple zeta values. For this, we first show that multiple zeta values are also well-defined for certain indices with negative integers (cf., [22, Theorem 3]).

Lemma 4.9 *For $r \geq 1$ and $\kappa_1, \ldots, \kappa_r \in \mathbb{Z}$ with $\sum_{i=1}^{j} \kappa_i > j$ for all $j \geq 1$,*

$$\zeta(\kappa_1, \ldots, \kappa_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{\kappa_1} \cdots m_r^{\kappa_r}}$$

is a well-defined real number. Moreover, it is an element of $\mathcal{Z}_{\leq |\kappa|}$.

Proof First, we estimate $m_r^{-\kappa_r}$ by $m_{r-1}^{-\kappa_r+1} m_r^{-1}$ if $\kappa_r \leq 0$ and obtain

$$0 < \zeta(\kappa_1, \ldots, \kappa_r) \leq \zeta(\kappa_1, \kappa_2, \ldots, \kappa_{r-1} + \kappa_r - 1, 1)$$

if $\kappa_r \leq 0$. Iteratively (at most $r-1$ times) performing this procedure by estimating $m_i^{-\kappa_i'}$ by $m_{i-1}^{-\kappa_i'+1} m_i^{-1}$ for the largest index i for which the exponent κ_i' of m_i is non-positive, we find

$$0 < \zeta(\kappa_1, \ldots, \kappa_r) \leq \zeta(\kappa_1 + \kappa_2 + \ldots + \kappa_j - j + 1, \ldots) < \infty$$

for some $j \geq 1$, where by the nature of this iterative process only the first entry may be non-positive. By our assumption $\sum_{i=1}^{j} \kappa_i > j$, we find that the first entry is at least equal to 2, from which we indeed conclude that $\zeta(\kappa_1, \ldots, \kappa_r)$ is a well-defined real number.

By iteratively using that $\sum_{m=a}^{b} m^{-k}$ for $k \leq 0$ is a polynomial in a and b, it is not hard to show that $\zeta(\kappa_1, \ldots, \kappa_r) \in \mathcal{Z}_{\leq |\underline{\kappa}|}$, with $|\underline{\kappa}| = \sum_i \kappa_i$. Observe, here, again we make use of the assumption $\sum_{i=1}^{j} \kappa_j > j$ to ensure all multiple zeta values in this linear combination are convergent. □

Proof of Theorem 1.2 Let $j = t$ be the smallest index for which the maximum in (4.4) is attained. Write $g = P\binom{k_1,\ldots,k_t}{d_1,\ldots,d_t}$ and $h = P\binom{k_{t+1},\ldots,k_r}{d_{t+1},\ldots,d_r}$. We will show that $\mathrm{Z}^{\deg} g$ and $\mathrm{Z}^{\deg} h$ are multiple zeta values in case $j = t$ is the unique index for which the maximum in (4.4) is attained. In that case, we also show that $\mathrm{Z}^{\deg} f = (\mathrm{Z}^{\deg} g)(\mathrm{Z}^{\deg} h)$.

Since the degree limit only depends on the leading term we can choose any well-normalized family $\mathcal{F}$. Observe that if we choose the Bernoulli–Seki model we have

$$\langle h \rangle_q = \sum_{m_{t+1} > \ldots > m_r > 0} \prod_i m_i^{d_i} \frac{E_{k_i}(q^{m_i})}{(1-q^{m_i})^{k_i}}. \tag{4.5}$$

By Lemma 4.4 we get for all i

$$\lim_{q \to 1} m_i^{d_i} (1-q)^{k_i} \frac{E_{k_i}(q^{m_i})}{(1-q^{m_i})^{k_i}} = m_i^{d_i - k_i}.$$

First, assume $j = t$ is the unique index for which the maximum in (4.4) is attained. Then, we know that for all $t' > t$

$$\sum_{i=t+1}^{t'} k_i > \sum_{i=t+1}^{t'} (d_i + 1). \tag{4.6}$$

By Proposition 4.5 we find that the sum in (4.5) is absolutely bounded by a constant (depending on the k_i) times $\zeta(k_{t+1} - d_{t+1}, \ldots, k_r - d_r)$. Using (4.6), this multiple zeta value is well-defined by Lemma 4.9, so (4.5) converges absolutely. By interchanging the sum and limit, we conclude

$$\deg h = \sum_{i=t+1}^{r} k_i \quad \text{and} \quad \mathrm{Z}^{\deg} h = \zeta(k_{t+1} - d_{t+1}, \ldots, k_r - d_r).$$

Hence, $\mathrm{Z}^{\deg} h$ is a linear combination of multiple zeta values of weight $\leq \sum_{i=t+1}^{r} k_i$.

In case the index $j = t$ is not unique, let $\epsilon > 0$. Then, as

$$\lim_{q \to 1} m_i^{d_i} (1-q)^{k_i + \epsilon} \frac{E_{k_i}(q^{m_i})}{(1-q^{m_i})^{k_i}} \to 0$$

if $q \to 1$, we see that $\deg h \leq \sum_{i=t+1}^{r} k_i$ in this case. Moreover, as

$$\lim_{q \to 1} m_i^{d_i} (1-q)^{k_i} \frac{E_{k_i}(q^{m_i})}{(1-q^{m_i})^{k_i}} = m_i^{d_i - k_i} > 0,$$

we conclude that also in this case we have $\deg h = \sum_{i=t+1}^{r} k_i$.

Next, (again working with the Bernoulli–Seki model) observe that

$$\begin{aligned} \langle g \rangle_q &= \sum_{\substack{m_1 > \ldots > m_t > 0 \\ r_1, \ldots, r_t > 0}} \prod_i m_i^{d_i} r_i^{k_i - 1} q^{m_i r_i} \\ &= \sum_{\substack{0 < r_1' < \ldots < r_t' \\ m_1', \ldots, m_t' > 0}} \prod_i (m_i' + \ldots + m_t')^{d_i} (r_i' - r_{i-1}')^{k_i - 1} q^{m_i' r_i'}, \end{aligned}$$

where we set $r_0' = 0$. Given a 'monomial' $m_1^{d_1'} \cdots m_t^{d_t'}$, we have

$$\sum_{\substack{0 < r_1 < \ldots < r_t \\ m_1, \ldots, m_t > 0}} \prod_i m_i^{d_i'} q^{m_i r_i} = \sum_{0 < r_1 < \ldots < r_t} \prod_i d_i'! \frac{E_{d_i'+1}(q^{r_i})}{(1-q^{r_i})^{d_i'+1}},$$

which is of degree $\sum_i (d_i' + 1)$ and (in case $d_t' > 0$) with degree limit $\zeta(d_t' + 1, \ldots, d_1' + 1)$ (as follows from above). Hence, by expanding $\prod_i (m_i' + \ldots + m_t')^{d_i}$ in monomials,

we see that

$$\deg g = \sum_i (d_i + 1),$$

provided that the corresponding limit

$$L = \sum_{0<r_1<\ldots<r_t} \prod_i \frac{(r_i - r_{i-1})^{k_i-1}}{r_i} \cdot \Omega\Big[\prod_{i=1}^{t}\Big(\frac{1}{r_i} + \ldots + \frac{1}{r_t}\Big)^{d_i}\Big],$$

converges, where $\Omega : \mathbb{Q}[r_1^{-1}, \ldots, r_t^{-1}] \to \mathbb{Q}[r_1^{-1}, \ldots, r_t^{-1}]$ is the linear mapping (1.7), i.e.,

$$\Omega\Big[\frac{1}{r_1^{l_1}\cdots r_t^{l_t}}\Big] := \frac{l_1!\cdots l_r!}{r_1^{l_1}\cdots r_t^{l_t}} \qquad (l_i \in \mathbb{Z}_{\geq 0}).$$

Note that $\langle g\rangle_q$ is bounded by a constant (only depending on the d_i') times L, which justifies interchanging the sum and limit in this computation (if L converges). We estimate

$$r_i - r_{i-1} \leq r_i \qquad \text{and} \qquad \frac{1}{r_i} + \ldots + \frac{1}{r_t} \leq \frac{t}{r_i}$$

to obtain

$$L \leq \sum_{0<r_1<\ldots<r_t} \prod_i r_i^{k_i-d_i} t^{d_i} d_i! = \Big(\prod_i t^{d_i} d_i!\Big) \cdot \zeta(d_t - k_t, \ldots, d_1 - k_1).$$

Note that if for some s we have that $\sum_{i=s}^{t}(d_i - k_i + 1) \leq 0$, then the sum (4.4) is also maximized for $j = s - 1$, contradicting the definition of t. Hence, Lemma 4.9 implies that L is a well-defined real number. Therefore, $L = \mathrm{Z}^{\deg} g$ and $\mathrm{Z}^{\deg} g$ is a well-defined linear combination of multiple zeta values of weight $\leq \sum_{i=1}^{t}(d_i + 1)$ (the latter one reads of from the definition of L).

Finally, we relate the degree and limits of g and h to the degree and limits of f. Consider the difference $f - g \circledast h$, and observe that $\deg(g \circledast h) = \deg g + \deg h$ (as $\langle g \circledast h\rangle_q = \langle g\rangle_q \langle h\rangle_q$; see Proposition 2.5). First, we assume again that $j = t$ is the unique index for which the maximum in (4.4) is attained. In that case, we claim that $f - g \circledast h$ is of smaller degree than $g \circledast h$. It follows from this claim that the function f has the same degree, as well as degree limit, as $g \circledast h$. Hence, in case $j = t$ is the unique index for which the maximum in (4.4) is attained, it suffices to prove the claim.

To prove the claim, we write this difference $f - g \circledast h$ in terms of the basis elements of $\mathbb{P}$. Then, we will see it contains two types of terms. We show that all these terms are of degree smaller than the degree (4.4) in the statement.

First of all, there are terms in $f - g \circledast h$ where the column with k_t and d_t is on the right of the column with k_{t+1} and d_{t+1} (note that possibly some other column is

stuffed on the columns containing these values). We apply Lemma 4.8(ii) repeatedly, so that:

- The values d_j are replaced by d'_j such that $k_j \leq d'_j + 1$ if $j \leq t$ and $k_t < d'_t + 1$, and with $k_j \geq d'_j + 1$ if $j > t$ and $k_{t+1} > d'_{t+1} + 1$. That this is always possible follows from the construction of t. For the strict inequalities, we use the uniqueness of the maximum defining t.
- Additionally, the values d'_t and d'_{t+1} are replaced by $d'_t - 1$ and $d'_{t+1} + 1$ respectively.

Lemma 4.8(i) now implies that these terms are of lower degree.

Secondly, there are terms of depth $< r$, for which a certain column is given by $\binom{k_t + k_{t+1} - a}{d_t + d_{t+1}}$ for some $a \geq 0$. Then, similarly, Lemma 4.8 also implies that these terms are of lower degree.

Now, the case that $j = t$ is not the unique index for which the maximum in (4.4) is attained remains. By a similar argument as in Lemma 4.8(i), we have

$$\deg f \leq \deg g + \deg h.$$

Namely, for this inequality, we forget all inequalities $m_i > m_j$ if $i \leq t$ and $j > t$.

Let $j = t'$ be the *maximal* index for which the maximum in (4.4) is obtained, and write

$$f' = P\begin{pmatrix} k_1, \ldots, k_{t'-1}, & k_{t'}, & k_{t'+1}, \ldots, k_r \\ 0, \ldots, 0, d_1 + \ldots + d_{t'}, & d_{t'+1}, & \ldots, d_r \end{pmatrix}.$$

Now, we apply Lemma 4.8(ii) for $i = 1, \ldots, t' - 1$, to obtain a lower bound for the degree of f. That is,

$$\deg f = \deg P\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} \geq \deg f' = \sum_{j=1}^{t'} (d_j + 1) + \sum_{j=t'+1}^{r} k_j.$$

Here, the last equality holds as for f' the maximum in (4.4) is uniquely attained at $j = t'$, and we already computed the degree and corresponding limit in that case. Hence, $\deg f$ is bounded from both sides by the value (4.4), which completes the proof in the case that $j = t$ is not the unique index for which the maximum in (4.4) is attained. □

Recall that by Lemma 4.8 we have

$$ZP\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} = 0$$

if there is no index t such that $k_i = 1$ for all $i \leq t$ and $d_i = 0$ for all $i > t$. By going through the proof above for those basis elements in $\mathbb{P}$ for which there does exist such a t, we find that the weight limits of such elements do not vanish:

Corollary 4.10 *For all $d_1, \ldots, d_{t-1} \geq 0, d_t \geq 1, k_{t+1} \geq 2$ and $k_{t+2}, \ldots, k_r \geq 1$, write*

$$f = P\begin{pmatrix} 1, \dots, 1, k_{t+1}, \dots, k_r \\ d_1, \dots, d_t, \ 0, \dots, 0 \end{pmatrix}.$$

We have

$$\deg f = \sum_{i=1}^{t}(d_i + 1) + \sum_{i=t+1}^{r} k_i = \operatorname{wt} f$$

and

$$\mathrm{Z}^{\deg} f = \xi(d_1, \dots, d_t)\, \zeta(k_{t+1}, \dots, k_r) = Zf.$$

4.2 Algebraic setup

We will now introduce the algebraic setup for our space $\mathbb{P}$ by generalizing the classical setup introduced by Hoffman in [17] and used in [20] for the regularization of multiple zeta values. For each model $\mathcal{F} = \{f_k\}_{k=0}^{\infty}$ of well-normalized polynomials we will describe the harmonic product as well as the shuffle product on the level of words in analogy to [17]. For the Bernoulli–Seki model the algebraic setup described here coincides with the one described in [2, Section 3]. Define the set A, also called the set of *letters*, by

$$A := \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \,\middle|\, k \geq 1,\ d \geq 0 \right\}.$$

We will define a product $*_{\mathcal{F}}$ on the space $\mathbb{Q}\langle A\rangle$ of non-commuting polynomials in A and we will call a monic monomial in $\mathbb{Q}\langle A\rangle$ a *word*. This product on the space of words will depend on a product $\diamond_{\mathcal{F}}$ on the space $\mathbb{Q}A$ of letters, which depends on $\mathcal{F}$. Recall that as $\mathcal{F}$ is well-normalized, the f_k are polynomials of degree k with leading coefficient $\frac{1}{k!}$. Then, for $k_1, k_2 \geq 1$ and $1 \leq j \leq k_1 + k_2 - 1$ there exist rational numbers $\alpha_{\mathcal{F}}(k_1, k_2, j) \in \mathbb{Q}$ such that for all $n \geq 1$

$$\sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} \partial f_{k_1}(n_1)\, \partial f_{k_2}(n_2) = \partial f_{k_1+k_2}(n) + \sum_{j=1}^{k_1+k_2-1} \alpha_{\mathcal{F}}(k_1, k_2, j)\, \partial f_j(n), \tag{4.7}$$

where as before $\partial f(n) = f(n) - f(n-1)$ for integers $n \geq 1$.

Example 4.11 (i) If $\mathcal{F} = \mathrm{b}$ is the binomial model, i.e., $f_k(x) = \binom{x}{k}$, we have $\partial f_k(x) = \binom{x-1}{k-1}$ for $k \geq 1$ and since

$$\sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} \binom{n_1 - 1}{k_1 - 1}\binom{n_2 - 1}{k_2 - 1} = \binom{n-1}{k_1 + k_2 - 1}$$

we obtain $\alpha_{\mathrm{b}}(k_1, k_2, j) = 0$.

(ii) For the Bernoulli–Seki model $\mathcal{F} = \mathrm{s}$ we find that $\alpha_{\mathrm{s}}(k_1, k_2, j)$ equals

$$-\left((-1)^{k_1}\binom{k_1+k_2-1-j}{k_2-j}+(-1)^{k_2}\binom{k_1+k_2-1-j}{k_1-j}\right)\frac{B_{k_1+k_2-j}}{(k_1+k_2-j)!}, \tag{4.8}$$

which can be proven by using Bernoulli's/Seki's/Faulhaber's formula for the sum of powers (see, e.g., [28, Lemma 6.1.2(ii)]).

On $\mathbb{Q}A$ we define the product $\diamond_{\mathcal{F}}$ for $k_1, k_2 \geq 1$ and $d_1, d_2 \geq 0$ by

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond_{\mathcal{F}} \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 \\ d_1+d_2 \end{bmatrix} + \sum_{j=1}^{k_1+k_2-1} \alpha_{\mathcal{F}}(k_1, k_2, j) \begin{bmatrix} j \\ d_1+d_2 \end{bmatrix}. \tag{4.9}$$

In the case $\mathcal{F} = \mathrm{b}$ we just write $\diamond = \diamond_{\mathrm{b}}$, which is given by

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 \\ d_1+d_2 \end{bmatrix}.$$

For each well-normalized family of polynomials $\mathcal{F}$ this gives a commutative non-unital $\mathbb{Q}$-algebra $(\mathbb{Q}A, \diamond_{\mathcal{F}})$. We will be interested in $\mathbb{Q}$-linear combinations of words in the letters of A and we define

$$\mathfrak{P} = \mathbb{Q}\langle A \rangle$$

In the following we will use for $k_1, \ldots, k_r \geq 1, d_1, \ldots, d_r \geq 0$ the following notation to write words in $\mathfrak{P}$:

$$\begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix} := \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \ldots \begin{bmatrix} k_1 \\ d_1 \end{bmatrix},$$

where the product on the right is the usual non-commutative product in $\mathbb{Q}\langle A \rangle$.

Definition 4.12 For a well-normalized family of polynomials $\mathcal{F}$ we define the *quasi-shuffle product* $*_{\mathcal{F}}$ on $\mathfrak{P}$ as the $\mathbb{Q}$-bilinear product, which satisfies $1 *_{\mathcal{F}} w = w *_{\mathcal{F}} 1 = w$ for any word $w \in \mathfrak{P}$ and

$$aw *_{\mathcal{F}} bv = a(w *_{\mathcal{F}} bv) + b(aw *_{\mathcal{F}} v) + (a \diamond_{\mathcal{F}} b)(w * v) \tag{4.10}$$

for any letters $a, b \in A$ and words $w, v \in \mathfrak{P}$.

We obtain a commutative $\mathbb{Q}$-algebras $(\mathfrak{P}, *_{\mathcal{F}})$, as shown in [17] (see also [19]). In the case $\mathcal{F} = \mathrm{b}$ we just write $* = *_{\mathrm{b}}$. As an example for the product (4.10) we have

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1, k_2 \\ d_1, d_2 \end{bmatrix} + \begin{bmatrix} k_2, k_1 \\ d_2, d_1 \end{bmatrix} + \begin{bmatrix} k_1+k_2 \\ d_1+d_2 \end{bmatrix}.$$

The algebra $(\mathfrak{P}, *)$ is graded by *weight*, where the weight is defined by

$$\mathrm{wt}\left(\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix}\right) = k_1 + \cdots + k_r + d_1 + \cdots + d_r$$

and it is filtered by *depth*, which is defined by

$$\mathrm{dep}\left(\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix}\right) = r\,.$$

In general the algebra $(\mathfrak{P}, *_{\mathcal{F}})$ is not graded but filtered by weight.

Proposition 4.13 *For any well-normalized family of polynomials $\mathcal{F}$ the maps*

$$\begin{aligned} \Xi_{\mathcal{F}} : (\mathfrak{P}, *_{\mathcal{F}}) &\longrightarrow (\mathbb{P}, \circledast) \\ \begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} &\longmapsto P_{\mathcal{F}}\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (\mathfrak{P}, *) &\longrightarrow (\mathbb{P}, \odot) \\ \begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} &\longmapsto P_{\mathrm{m}}\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix} \end{aligned}$$

are algebra-isomorphisms of filtered algebras. Recall that m *stands for the monomial model (see Table 1).*

Proof Assume $\mathcal{F}$ is a well-normalized family of polynomials. First, notice that whenever one has a sum of the form

$$\sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} m_j^{d_j}\, h(m_j, k_j) \tag{4.11}$$

in a $\mathbb{Q}$-algebra $(A, \times)$ with some function h satisfying

$$h(m, k_1) \times h(m, k_2) = h(m, k_1 + k_2) + \sum_{j=1}^{k_1+k_2-1} \alpha_{\mathcal{F}}(k_1, k_2, j)\, h(m, j)\,, \tag{4.12}$$

then the linear map φ defined by sending the generators $\left[\begin{smallmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{smallmatrix}\right]$ to (4.11) satisfies $\varphi(w) \times \varphi(v) = \varphi(w *_{\mathcal{F}} v)$ for all $w, v \in \mathfrak{P}$. This can be shown by truncating the sum over the m_j by some M and then doing induction on M together with the definition of $*_{\mathcal{F}}$ (see for example [3, Lemma 2.18]). To show the first statement we consider the $\underline{u}$-bracket of a generator of $\mathbb{P}$, which by Proposition 3.9 is given by

$$\left\langle P_{\mathcal{F}}\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix} \right\rangle_{\underline{u}} = \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \Big(m_j^{d_j} \sum_{n>0} \partial f_{k_j}(n)\, u_{m_j}^{n} \Big). \tag{4.13}$$

Since the harmonic product $\circledast$ is defined by the $\underline{u}$-bracket (see Definition 2.3) we see that $\Xi_{\mathcal{F}}$ is an algebra homomorphism, as (4.13) is a sum of the form (4.11) with $h(m,k) = \sum_{n>0} \partial f_k(n)\, u_m^n$ and the property (4.12) is satisfied due to (4.7). With a similar argument, we see that also the second map is an algebra homomorphism since

$$P_{\mathrm{m}}\left(\begin{matrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{matrix};\lambda\right) = \sum_{m_1>\cdots>m_n>0} \prod_{j=1}^{r} m_j^{d_j} r_{m_j}(\lambda)^{k_j}\,,$$

is a sum of the form (4.11) with $h(m,k) = r_m(\lambda)^k$. All maps are also isomorphisms among the elements in $\mathfrak{P}$ and in $\mathbb{P}$ (Corollary 3.8). □

As a direct consequence of describing the product $\circledast$ in terms of a quasi-shuffle product is the following.

Proposition 4.14 *For $k \geq 2$, $d \geq 0$ with $k+d$ even we have $P_{\mathrm{s}}\left(\begin{smallmatrix}k,\dots,k\\ d,\dots,d\end{smallmatrix}\right) \in \mathbb{M}$, i.e., their q-brackets are quasimodular forms (of mixed weight).*

Proof Suppose a $\mathbb{Q}$-algebra R and an algebra homomorphism $\varphi : (\mathfrak{P}, *_{\mathcal{F}}) \to R$ are given. Using [19, (32)] (see also [3, (2.13)]) we have for $a \in A$ the following equation in $R[\![X]\!]$:

$$\exp\left(\sum_{n=1}^{\infty}(-1)^{n-1}\varphi(a^{\diamond_{\mathcal{F}} n})\frac{X^n}{n}\right) = 1 + \sum_{n=1}^{\infty}\varphi(a^n)X^n, \tag{4.14}$$

where $a^{\diamond_{\mathcal{F}} n} = \overbrace{a \diamond_{\mathcal{F}} \cdots \diamond_{\mathcal{F}} a}^{n} \in \mathbb{Q}A$ and $a^n = \overbrace{a \cdots a}^{n} \in \mathfrak{P}$.

Now, let $R = \mathbb{Q}[\![q]\!]$ and let φ be the composition of the algebra homomorphism $\Xi_{\mathcal{F}}$ in Proposition 4.13 with the q-bracket, so that we obtain an algebra homomorphism $\varphi : (\mathfrak{P}, *_{\mathcal{F}}) \to \mathbb{Q}[\![q]\!]$ (see Proposition 3.11). We call a letter $\left[\begin{smallmatrix}k\\ d\end{smallmatrix}\right] \in A$ even if $k \geq 2$, $d \geq 0$ and $k+d$ is even. In the case $\mathcal{F} = \mathrm{s}$ one then checks using (4.8) and (4.9) that for two even letters $a, a' \in A$, the product $a \diamond_{\mathrm{s}} a'$ is again a linear combination of some even letters (the Bernoulli numbers B_k for $k \geq 3$ odd vanish). This inductively gives that for an even letter a the element $a^{\diamond_{\mathcal{F}} n}$ is also a linear combination of even letters. Then (4.14) implies that the q-bracket of $P_{\mathrm{s}}\left(\begin{smallmatrix}k,\dots,k\\ d,\dots,d\end{smallmatrix}\right)$, which is $\varphi(a^n)$ for $a = \left[\begin{smallmatrix}k\\ d\end{smallmatrix}\right]$, can be written as a polynomial in the q-brackets of $P_{\mathrm{s}}\left(\begin{smallmatrix}k'\\ d'\end{smallmatrix}\right)$ with $k'+d'$ even. By (3.5) for $\mathcal{F} = \mathrm{s}$ these are exactly given by derivatives of Eisenstein series and therefore we obtain a quasimodular form (of mixed weight). □

Example 4.15 For example, in the case $k = 2$, $d = 0$ we get in depth two

$$\left\langle P_{\mathrm{s}}\left(\begin{matrix}2,2\\ 0,0\end{matrix}\right)\right\rangle_q = \frac{1}{2}G_2^2 - \frac{1}{2}G_4 + \frac{17}{120}G_2 + \frac{31}{5760}\,,$$

where we used $G_k = -\frac{B_k}{2k!} + \langle P_{\mathrm{s}}\left(\begin{smallmatrix}k\\ 0\end{smallmatrix}\right)\rangle_q$.

The above algebraic setup can be seen as a generalization of the algebra $\mathfrak{H}^1$ defined in [17], as we explain now. Setting $z_k := \left[\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right]$ for $k \geq 1$ we define

$$\mathfrak{H}^1 = \mathbb{Q} + \langle z_{k_1} \dots z_{k_r} \mid r \geq 1, k_1, \dots, k_r \geq 1 \rangle_{\mathbb{Q}} \subset \mathfrak{P}$$

and its subspace of *admissible words*

$$\mathfrak{H}^0 = \mathbb{Q} + \langle z_{k_1} \dots z_{k_r} \in \mathfrak{H}^1 \mid r \geq 1, k_1 \geq 2 \rangle_{\mathbb{Q}} .$$

The quasi-shuffle product on $\mathfrak{P}$ reduces to a product on $\mathfrak{H}^1$ and on $\mathfrak{H}^0$. Moreover, the product $*$ reduces to the classical harmonic product introduced in [17]. One can then define the following $\mathbb{Q}$-linear map[2]

$$\begin{aligned} \zeta : \mathfrak{H}^0 &\longrightarrow \mathcal{Z} \\ z_{k_1} \dots z_{k_r} &\longmapsto \zeta(k_1, \dots, k_r) . \end{aligned} \tag{4.15}$$

By the definition of multiple zeta values (1.5) as nested sum one then checks that this map is an algebra homomorphism from $(\mathfrak{H}^0, *)$ to the algebra of multiple zeta values $\mathcal{Z}$. Since $\mathfrak{H}^1 = \mathfrak{H}^0[z_1]$ ([20, Proposition 1]) one can extend this algebra homomorphism uniquely to an algebra homomorphism $\zeta^* : (\mathfrak{H}^1, *) \to \mathcal{Z}[T]$ which satisfies $\zeta^*(z_1) = T$ and $\zeta^*_{|\mathfrak{H}^0} = \zeta$. This gives the definition of *(harmonic) regularized multiple zeta values* for $k_1, \dots, k_r \geq 1$.

In the following we will generalize this result and the goal is to obtain an algebra homomorphism $(\mathfrak{P}, *) \to \mathcal{Z}[T]$. First, we define the analogue of admissible words in our larger space $\mathfrak{P}$.

Definition 4.16 (i) For $d \geq 0$ set $v_d := \left[\begin{smallmatrix} 1 \\ d \end{smallmatrix}\right]$. As an counterpart of $\mathfrak{H}^1$ and $\mathfrak{H}^0$ we define

$$\begin{aligned} \mathfrak{J}^1 &= \mathbb{Q} + \langle v_{d_1} \dots v_{d_r} \mid r \geq 1, d_1, \dots, d_r \geq 0 \rangle_{\mathbb{Q}} \subset \mathfrak{P} \\ \mathfrak{J}^0 &= \mathbb{Q} + \langle v_{d_1} \dots v_{d_r} \in \mathfrak{J}^1 \mid r \geq 1, d_r \geq 1 \rangle_{\mathbb{Q}} . \end{aligned}$$

(ii) We set $\mathfrak{P}^1 := \mathfrak{J}^1 \mathfrak{H}^1$, which is the space spanned by all words of the form

$$w = \begin{bmatrix} 1, \dots, 1 \\ d_1, \dots, d_m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^j \begin{bmatrix} k_1, \dots, k_r \\ 0, \dots, 0 \end{bmatrix} \tag{4.16}$$

for some $j \geq 0$, $m, r \geq 0$, $d_m \geq 1$ (if $m \geq 1$), and $k_1 \geq 2$ (if $r \geq 1$).

(iii) A word $w \in \mathfrak{P}^1$ of the form (4.16) is called *admissible* if $j = 0$. The subspace of admissible words will be denoted by

$$\mathfrak{P}^0 := \mathfrak{J}^0 \mathfrak{H}^0 = \langle w \in \mathfrak{P}^1 \mid w \text{ is admissible} \rangle_{\mathbb{Q}} .$$

[2] By abuse of notation, we use ζ for the name of the map as well as for the name of the object. From the context, it should always be clear if we are talking about the map or the object.

Notice that this notion of admissibility coincides with the classical notion of admissibility on the subspace $\mathfrak{H}^1$, namely, we have $\mathfrak{H}^0 = \mathfrak{H}^1 \cap \mathfrak{P}^0$. However, in contrast to the classical setup, the spaces $\mathfrak{P}^1$ and $\mathfrak{P}^0$ are not closed under $*$ (or any $*_{\mathcal{F}}$) and therefore they are not subalgebras of $\mathfrak{P}$.

Lemma 4.17 *Let $\mathfrak{N}$ be the subspace of $\mathfrak{P}$ spanned by all words in $\mathfrak{P}\backslash\mathfrak{P}^1$. Then $\mathfrak{N}$ is an ideal in $(\mathfrak{P}, *)$.*

Proof We need to show that for any $w \in \mathfrak{P}$ which is not of the form (4.16) the product $w * v$ is in $\mathfrak{N}$ for any $v \in \mathfrak{P}$. If $w \in \mathfrak{P}$ is not of the form (4.16) then w either contains a letter $\left[\begin{smallmatrix} k \\ d \end{smallmatrix}\right]$ or it contains a letter $\left[\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right]$ which is on the left of a letter $\left[\begin{smallmatrix} 1 \\ d \end{smallmatrix}\right]$ with $k \geq 2$ and $d \geq 1$ in both cases. In the first case, the product of w with any element v will be a sum of words which also contain either a letter $\left[\begin{smallmatrix} k \\ d \end{smallmatrix}\right]$ or a $\left[\begin{smallmatrix} k \\ d \end{smallmatrix}\right] \diamond b$ with some other letter b of v. Since $\diamond$ is the component-wise addition we see that each word therefore also contains a letter with top entry ≥ 2 and bottom entry ≥ 1. In the second case, all words either also have a letter $\left[\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right]$ which is on the left of a letter $\left[\begin{smallmatrix} 1 \\ d \end{smallmatrix}\right]$, or a letter $\left[\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right] \diamond b$ on the left of a letter $\left[\begin{smallmatrix} 1 \\ d \end{smallmatrix}\right] \diamond c$, where b, c are letters of v, etc. In all cases, the words are not of the form (4.16) and are therefore elements in $\mathfrak{N}$. □

As a generalization of $\mathfrak{H}^1 = \mathfrak{H}^0[z_1]$, we will now show that any element in $\mathfrak{P}^1$ can be written as a polynomial with respect to the quasi-shuffle product $*$ with coefficients in $\mathfrak{P}^0$ up to elements in the ideal $\mathfrak{N}$. We write

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{*j} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} * \cdots * \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{j}.$$

Proposition 4.18 *For any word $w \in \mathfrak{P}^1$ of length r there exist unique $w_j \in \mathfrak{P}^0$ such that*

$$w \equiv \sum_{j=0}^{r} w_j * \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{*j} \mod \mathfrak{N}. \tag{4.17}$$

Moreover, if $\mathrm{wt}(w) = k$ then the w_j are linear combinations of words of weight $k - j$.

Proof A word in $w \in \mathfrak{P}^1$ can be written as

$$w = u \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{j} v \qquad (j \geq 0, u \in \mathfrak{J}^0, v \in \mathfrak{H}^0).$$

Write $u = \left[\begin{smallmatrix} 1,\ldots,1 \\ d_1,\ldots,d_m \end{smallmatrix}\right]$ and $v = \left[\begin{smallmatrix} k_1,\ldots,k_r \\ 0,\ldots,0 \end{smallmatrix}\right]$ with $d_m > 0$ and $k_1 > 1$. By definition of the quasi-shuffle product $*$ for $j \geq 1$ we have that $u\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{j-1} v * \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ equals

$$
\begin{aligned}
& j\,w + \sum_{i=1}^{m-1} \begin{bmatrix} 1,\dots,1 \\ d_1,\dots,d_i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1,\dots,1 \\ d_{i+1},\dots,d_m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{j-1} v \\
& \quad + \sum_{i=1}^{r} u \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{j-1} \begin{bmatrix} k_1,\dots,k_i \\ 0,\dots,0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_{i+1},\dots,k_r \\ 0,\dots,0 \end{bmatrix} \\
& \quad + \sum_{i=1}^{m} \begin{bmatrix} 1,\dots,1 \\ d_1,\dots,d_{i-1} \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \diamond \begin{bmatrix} 1 \\ d_i \end{bmatrix} \right) \begin{bmatrix} 1,\dots,1 \\ d_{i+1},\dots,d_m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{j-1} v \\
& \quad + \sum_{i=1}^{j-1} u \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{i-1} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \diamond \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{j-1-i} v \\
& \quad + \sum_{i=1}^{r} u \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{j-1} \begin{bmatrix} k_1,\dots,k_{i-1} \\ 0,\dots,0 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \diamond \begin{bmatrix} k_i \\ 0 \end{bmatrix} \right) \begin{bmatrix} k_{i+1},\dots,k_r \\ 0,\dots,0 \end{bmatrix}.
\end{aligned}
$$

Here every displayed term, except for jw, is either an element in $\mathfrak{N}$ or is of the form $u'\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{j'} v'$, where $u' \in \mathfrak{I}^0$ and $v' \in \mathfrak{H}^0$ and $j' \leq j-1$. By induction on j we therefore see that w can be written as in (4.17).

For the uniqueness assume that $\sum_{j=0}^{r} w_j * \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{*j} \equiv 0 \mod \mathfrak{N}$ for some $w_j \in \mathfrak{P}^0$. Then the term $w_r * \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{*r}$ is the only summand which contains classes of the form (4.16) with $j = r$. Since there are no relations among elements in $\mathfrak{P}^1$ modulo $\mathfrak{N}$ we immediately obtain $w_r = 0$ and therefore $w_j = 0$ for all $j = 1, \dots, r$. □

Example 4.19 For $k \geq 2, d \geq 1$ we have

$$
\begin{bmatrix} 1,1,k \\ d,0,0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1,k \\ d,0 \end{bmatrix}}_{w_1} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} 1,k,1 \\ d,0,0 \end{bmatrix} - \begin{bmatrix} 1,1,k \\ 0,d,0 \end{bmatrix} - \begin{bmatrix} 1,k+1 \\ d,0 \end{bmatrix}}_{w_0} - \underbrace{\begin{bmatrix} 2,k \\ d,0 \end{bmatrix}}_{\in \mathfrak{N}}.
$$

Here $w_0, w_1 \in \mathfrak{P}^0$ and the last term is in $\mathfrak{N}$ due to the 2 in the top left entry.

We end this subsection by defining an analogue of the involution ι on the space $\mathfrak{P}$. Since Proposition 4.13 gives as for any well-normalized family of polynomials $\mathcal{F}$ an isomorphism $\Xi_{\mathcal{F}} : (\mathfrak{P}, *_{\mathcal{F}}) \longrightarrow (\mathbb{P}, \circledast)$ we give the following definition.

Definition 4.20 For any well-normalized family of polynomials $\mathcal{F}$ we define the linear map $\iota_{\mathcal{F}}$ by

$$
\begin{aligned}
\iota_{\mathcal{F}} : \mathfrak{P} &\longrightarrow \mathfrak{P} \\
w &\longmapsto (\Xi_{\mathcal{F}}^{-1} \circ \iota \circ \Xi_{\mathcal{F}})(w).
\end{aligned}
$$

When we choose the Bernoulli–Seki model $\mathcal{F} = \mathrm{s}$, then the involution ι_{s} can be described nicely using generating series. That is, given a depth $r \geq 1$ we write $\mathfrak{A}$ for

the formal power series in $\mathfrak{P}[\![X_1, Y_1, \ldots, X_r, Y_r]\!]$ given by

$$\mathfrak{A}\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} := \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} \begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix} X_1^{k_1-1} \ldots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \ldots \frac{Y_r^{d_r}}{d_r!}. \quad (4.18)$$

Proposition 4.21 *We have*

$$\iota_{\mathrm{s}}\left(\mathfrak{A}\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix}\right) = \mathfrak{A}\begin{pmatrix} Y_1 + \cdots + Y_r, \ldots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \ldots, X_1 - X_2 \end{pmatrix},$$

where the involution ι_{s} on the left is applied coefficient-wise.

Proof This can be proven by using the same change of variables as it was done in [2, Theorem 2.3] by replacing the q-series there with the $\underline{u}$-bracket of P_{s} and then using the explicit description of ι mentioned in the proof of Lemma 2.7. □

4.3 Bi-multiple zeta values

We now combine the results of Sect. 4.1 and 4.2 to define a bi-variant of multiple zeta values. These can be seen as regularized limits of polynomial functions on partitions, since they are defined for arbitrary words $w \in \mathfrak{P}$.

Definition 4.22 We define the linear map

$$\begin{aligned} \zeta : \mathfrak{P} &\longrightarrow \mathcal{Z}[T] \\ w = \begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix} &\longmapsto \zeta(w) = \zeta\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} \end{aligned}$$

as follows:

(i) For $w \in \mathfrak{N}$ we set $\zeta(w) = 0$.

(ii) For $w \in \mathfrak{P}^0$ we write $w = \left[\begin{smallmatrix} 1,\ldots,1, & k_1,\ldots,k_r \\ d_1,\ldots,d_m, & 0,\ldots,0 \end{smallmatrix}\right]$ and set

$$\zeta\begin{pmatrix} 1, \ldots, 1, k_1, \ldots, k_r \\ d_1, \ldots, d_m, 0, \ldots, 0 \end{pmatrix} = \xi(d_1, \ldots, d_m)\, \zeta(k_1, \ldots, k_r),$$

where $k_1 \geq 2$, $d_m \geq 1$ and where ξ is the conjugated multiple zeta value defined in Definition 1.3.

(iii) For $w = \sum_{j=0}^{r} w_j * \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{*j}$ with $w_j \in \mathfrak{P}^0$ we set

$$\zeta(w) = \sum_{j=0}^{r} \zeta(w_j)\, T^j.$$

Notice that due to Proposition 4.18 and part (i) the value $\zeta(w)$ in (iii) is well-defined.

Theorem 4.23 *For any well-normalized family of polynomials $\mathcal{F}$ the map*

$$\zeta : (\mathfrak{P}, *) \longrightarrow \mathcal{Z}[T]$$

is an $\iota_{\mathcal{F}}$-invariant algebra homomorphism.

Proof First, we show that ζ is an algebra homomorphism. Since $\mathfrak{N}$ is an ideal (Lemma 4.17) and $\zeta(w) = 0$ for $w \in \mathfrak{N}$, we can restrict to the case when $w \notin \mathfrak{N}$, i.e., $w \in \mathfrak{P}^1$. Moreover, we can restrict to the elements in $\mathfrak{P}^0$ due to the definition in (iii). If $w \in \mathfrak{P}^0$ we can assume that we can write $w = \left[\begin{smallmatrix} 1,\dots,1,k_1,\dots,k_r \\ d_1,\dots,d_m,0,\dots,0 \end{smallmatrix}\right]$ with $d_m \geq 1$ (or $m = 0$) and $k_1 \geq 2$ (or $r = 0$). Then we have by Corollary 4.10 that

$$\zeta(w) = \mathrm{Z}^{\deg} P_{\mathrm{s}} \begin{pmatrix} 1,\dots,1,k_1,\dots,k_r \\ d_1,\dots,d_m,0,\dots,0 \end{pmatrix} = Z\,\Xi_{\mathrm{s}}(w)\,, \tag{4.19}$$

where the weight limit Z is introduced in Definition 4.7. Now for $k \geq 1$ and $f \in \mathbb{P}$ set

$$Z_k(f) = \lim_{q\to 1} (1-q)^k \langle f \rangle_q\,,$$

i.e., if k is the weight of f then $Z_k(f)$ is exactly the weight limit of f.

Let $w_1, w_2 \in \mathfrak{P}^0$ be given and write $k_1 = \mathrm{wt}(w_1)$, $k_2 = \mathrm{wt}(w_2)$ and $k = k_1 + k_2$. Then, we have by Proposition 4.13 that $\Xi_{\mathrm{s}}(w_1 *_{\mathrm{s}} w_2) = \Xi_{\mathrm{s}}(w_1) \circledast \Xi_{\mathrm{s}}(w_2)$ and $\Xi_{\mathrm{b}}(w_1 * w_2) = \Xi_{\mathrm{b}}(w_1) \circledast \Xi_{\mathrm{b}}(w_2)$. Comparing the definition of $* = *_{\mathrm{b}}$ and $*_{\mathrm{s}}$, we see that they are the same up to lower weight terms. In particular, their image under Z_k is the same. Moreover, since Z_k is defined via the q-bracket, which is an algebra homomorphism with respect to $\circledast$ (Proposition 3.11), we get

$$\begin{aligned} \zeta(w_1)\zeta(w_2) &= Z_{k_1}\,\Xi_{\mathrm{s}}(w_1)\,Z_{k_2}\,\Xi_{\mathrm{s}}(w_2) = Z_k\left(\Xi_{\mathrm{s}}(w_1) \circledast \Xi_{\mathrm{s}}(w_2)\right) \\ &= Z_k\left(\Xi_{\mathrm{s}}(w_1 *_{\mathrm{s}} w_2)\right) = Z_k\left(\Xi_{\mathrm{s}}(w_1 * w_2)\right). \end{aligned}$$

But this also equals $\zeta(w_1 * w_2)$, since $w_1 * w_2$ is a linear combination of words which are either in $\mathfrak{P}^0$ (i.e., they can be written as in (4.19)) or which are in $\mathfrak{N}$. For the latter, the weight limit, and therefore the image under Z_k, vanishes as seen in Lemma 4.8.

It remains to show that ζ is $\iota_{\mathcal{F}}$-invariant for any well-normalized family of polynomials $\mathcal{F}$. Let $w = \sum_{j=0}^{r} w_j * \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{*j}$ with $w_j \in \mathfrak{P}^0$ and define the $\tilde{w}_j \in \mathfrak{P}^0$ by $\iota_{\mathcal{F}}(w) =: \sum_{j=0}^{r} \tilde{w}_j * \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{*j}$. We need to show that $\zeta(w_j) = \zeta(\tilde{w}_j)$ for all $j = 0,\dots,r$. Since the q-bracket is ι invariant we have $\langle \Xi_{\mathcal{F}}(w)\rangle_q = \langle \Xi_{\mathcal{F}}(\iota_{\mathcal{F}}(w))\rangle_q$, i.e., we get, up to q-series which vanish under the weight limit (elements in $\langle \iota_{\mathcal{F}}(\mathfrak{N})\rangle_q$), that

$$\sum_{j=0}^{r} \langle \Xi_{\mathcal{F}}(w_j)\rangle_q \left\langle \Xi_{\mathcal{F}} \begin{bmatrix}1\\0\end{bmatrix} \right\rangle_q^j \equiv \sum_{j=0}^{r} \langle \Xi_{\mathcal{F}}(\tilde{w}_j)\rangle_q \left\langle \Xi_{\mathcal{F}} \begin{bmatrix}1\\0\end{bmatrix} \right\rangle_q^j .$$

For any well-normalized family of polynomials $\mathcal{F}$ we have

$$\left\langle \Xi_{\mathcal{F}}\begin{bmatrix}1\\0\end{bmatrix}\right\rangle_q = \sum_{n>0}\frac{q^n}{1-q^n}.$$

Since[3] $\sum_{n>0}\frac{q^n}{1-q^n} \asymp -\frac{\log(1-q)}{(1-q)}$ (see [25]) and $\langle \Xi_{\mathcal{F}}(w_j)\rangle_q \asymp \frac{1}{(1-q)^{\mathrm{wt}(w_j)}}$ (by Sect. 4.1) as $q \to 1$, we see that for $j \geq 0$ the weight limits of $\langle \Xi_{\mathcal{F}}(w_j)\rangle_q$ and $\langle \Xi_{\mathcal{F}}(\tilde{w}_j)\rangle_q$ are equal. By the discussion above applying ζ to words in $\mathfrak{P}^0$ exactly yields the weight limits, which are also independent of the choice of (well-normalized) $\mathcal{F}$, from which we deduce $\zeta(w_j) = \zeta(\tilde{w}_j)$ for all $j = 1, \ldots, r$. □

5 Induced relations among multiple zeta values

Here we interpret and provide relations among multiple zeta values using our partition analogues.

5.1 Double shuffle relations

In this section, we want to explain why the relations $\langle f \circledcirc g - f \circledast g\rangle_q = 0$ for any $f, g \in \mathbb{Q}^{\mathscr{P}}$ (see Proposition 2.5) can be seen as an analogue of the double shuffle relations for multiple zeta values. For this, we first try to indicate why the product $\circledcirc$ can be seen as the correct analogue of the shuffle product of multiple zeta values. We first recall the classical shuffle product. Setting $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ the spaces $\mathfrak{H}^1$ and $\mathfrak{H}^0$ from the previous section can be identified with the subspaces $\mathbb{Q} + \mathfrak{H}y$ and $\mathbb{Q} + x\mathfrak{H}y$ of $\mathfrak{H}$ via $z_k = x^{k-1}y$. On these spaces, one can then define the shuffle product as the $\mathbb{Q}$-bilinear product, which satisfies $1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1 = w$ for any word $w \in \mathfrak{H}$ and

$$a_1w_1 \sqcup\!\sqcup a_2w_2 = a_1(w_1 \sqcup\!\sqcup a_2w_2) + a_2(a_1w_1 \sqcup\!\sqcup w_2)$$

for any letters $a_1, a_2 \in \{x, y\}$ and words $w_1, w_2 \in \mathfrak{H}$. The spaces $\mathfrak{H}^1$ and $\mathfrak{H}^0$ are closed under this product and a classical result is (see [3, Corollary 2.10]) that the $\mathbb{Q}$-linear map $\zeta : \mathfrak{H}^0 \to \mathcal{Z}$ defined in (4.15) is also an algebra homomorphism with respect to the shuffle product $\sqcup\!\sqcup$. For $w, v \in \mathfrak{H}^0$ this then implies the (finite) double shuffle relations $\zeta(w \sqcup\!\sqcup v - w * v) = 0$.

By the definition of $\circledast$ together with Proposition 4.13 and Theorem 4.23, it becomes clear why $\circledast$ can be seen as the analogue of the stuffle product. On the other hand, the definition of $\circledcirc$ in Definition 2.3 as $F \circledcirc G = \iota(\iota(F) \circledast \iota(G))$ is completely different to the definition of $\sqcup\!\sqcup$ above and it is not obvious why this is the correct analogue. This was recently shown by Brindle in [12], in a slightly different context, by showing that the involution ι (which is called the partition relation) corresponds to a certain duality introduced by Singer in [27]. With this one can then show (see [12, Theorem 3.46])

[3] Here $f(q) \asymp g(q)$ means there exist constants c_1, c_2 with $c_1|g(q)| \leq |f(q)| \leq c_2|g(q)|$ as $q \to 1$.

that the product defined by combining this duality with the stuffle product indeed gives the correct analogue of the shuffle product.

In [4] it will be shown that one not only gets the finite double shuffle relations, but also the extended double shuffle relations, i.e., conjecturally all relations among multiple zeta values.

We will leave out the details here and just indicate this with two examples.

Example 5.1 The harmonic product of z_2 and z_3 is given by $z_2 * z_3 = z_2z_3 + z_3z_2 + z_3$, i.e., $\zeta(2)\zeta(3) = \zeta(2,3)+\zeta(3,2)+\zeta(5)$. The harmonic product $\circledast$ of $P_{\mathrm{s}}\binom{2}{0}$ and $P_{\mathrm{s}}\binom{3}{0}$ is given by

$$P_{\mathrm{s}}\binom{2}{0} \circledast P_{\mathrm{s}}\binom{3}{0} = P_{\mathrm{s}}\binom{2,3}{0,0} + P_{\mathrm{s}}\binom{3,2}{0,0} + P_{\mathrm{s}}\binom{5}{0} - \frac{1}{12}P_{\mathrm{s}}\binom{3}{0},$$

which gives exactly the harmonic product among multiple zeta values after applying the weight 5 limit (defined by Definition 4.7), since the last term vanishes. The shuffle product of z_2 and z_3 is given by $z_2 \sqcup\!\sqcup z_3 = xy \sqcup\!\sqcup xxy = xyxxy + 3xxyxy + 6xxxyy = z_2z_3 + 3z_3z_2 + 6z_4z_1$. In particular, this implies the following shuffle relation for multiple zeta values

$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1). \tag{5.1}$$

The shuffle product $\textcircled{ш}$ of $P_{\mathrm{s}}\binom{2}{0}$ and $P_{\mathrm{s}}\binom{3}{0}$ is given by

$$\begin{aligned} P_{\mathrm{s}}\binom{2}{0} \textcircled{ш} P_{\mathrm{s}}\binom{3}{0} &= \iota\left(\iota P_{\mathrm{s}}\binom{2}{0} \circledast \iota P_{\mathrm{s}}\binom{3}{0}\right) = \iota\left(\frac{1}{2}P_{\mathrm{s}}\binom{1}{1} \circledast P_{\mathrm{s}}\binom{1}{2}\right), \\ &= \frac{1}{2}\iota\left(P_{\mathrm{s}}\binom{1,1}{1,2} + P_{\mathrm{s}}\binom{1,1}{2,1} + P_{\mathrm{s}}\binom{2}{3} - P_{\mathrm{s}}\binom{1}{3}\right) \\ &= P_{\mathrm{s}}\binom{2,3}{0,0} + 3P_{\mathrm{s}}\binom{3,2}{0,0} + 6P_{\mathrm{s}}\binom{4,1}{0,0} + 3P_{\mathrm{s}}\binom{4}{1} - 3P_{\mathrm{s}}\binom{4}{0}. \end{aligned}$$

From the results in Sect. 4 we see that this exactly implies the shuffle product (5.1) by taking the weight 5 limit (observe that the weight 5 limit of $P_{\mathrm{s}}\binom{4}{1}$ and of $P_{\mathrm{s}}\binom{4}{0}$ vanish).

Example 5.2 Recall $\mathrm{Z}^{\deg}(f)$ is a rational multiple of an even power of π for $f \in \mathbb{M}$. Therefore, having diagram (1.6) in mind, given a linear combination of multiple zeta values which is a rational multiple of π^k, one might ask if there exists a 'natural lift' to a polynomial function in $\mathbb{P}$ whose q-bracket is quasimodular. One such family is given by a special case of Proposition 4.14, since $P_{\mathrm{s}}\binom{2m,\dots,2m}{0,\dots,0}$ is quasimodular for $m \geq 1$ with limit $\zeta(2m,\cdots,2m) = \zeta(\{2m\}^n) \in \mathbb{Q}\pi^{2mn}$. Another famous family of multiple zeta values which evaluates to a rational multiple of π^k are given by the 3-1-formula, which states that for any $n \geq 1$ we have

$$\zeta(3,1,\dots,3,1) = \zeta(\{3,1\}^n) = \frac{2\pi^{4n}}{(4n+2)!}.$$

The proof of this fact was first conjectured by Zagier, then proven in [9], and later reduced to the following argument [10] using the shuffle product. One can show that one has the following identity in the algebra $(\mathfrak{H}^0, \sqcup\!\sqcup)$

$$\sum_{j=-n}^{n} (-1)^j z_2^{n-j} \sqcup\!\sqcup z_2^{n+j} = 4^n (z_3 z_1)^n \,. \tag{5.2}$$

The statement then follows after using $\zeta(2,\ldots,2) = \frac{\pi^{2n}}{(2n+1)!}$. We can consider the left-hand side of this expression in our setup and get, by using again Proposition 4.14, that for any $n \geq 1$ the q-bracket of

$$T(n) := \sum_{j=-n}^{n} (-1)^j \, P_{\mathrm{s}}\left(\overbrace{2,\ldots,2}^{n-j} \atop 0,\ldots,0\right) \circledcirc P_{\mathrm{s}}\left(\overbrace{2,\ldots,2}^{n+j} \atop 0,\ldots,0\right) \in \mathbb{M}$$

is quasimodular. Similar to (5.2) this sum evaluates to $4^n P_{\mathrm{s}}\binom{3,1,\ldots,3,1}{0,\ldots,0}$ plus some extra terms with vanishing weight $4n$ limit. For example, we have

$$\begin{aligned}
T(1) =& 4P_{\mathrm{s}}\binom{3,1}{0,0} + 2P_{\mathrm{s}}\binom{3}{1} - 2P_{\mathrm{s}}\binom{3}{0}, \\
T(2) =& 16P_{\mathrm{s}}\binom{3,1,3,1}{0,0,0,0} + 8P_{\mathrm{s}}\binom{3,3,1}{1,0,0} \\
& - 8P_{\mathrm{s}}\binom{3,3,1}{0,1,0} - 8P_{\mathrm{s}}\binom{3,3,1}{0,0,0} + \\
& + 8P_{\mathrm{s}}\binom{3,1,3}{1,0,1} - 8P_{\mathrm{s}}\binom{3,1,3}{1,0,0} - 4P_{\mathrm{s}}\binom{3,3}{0,2} + 4P_{\mathrm{s}}\binom{3,3}{1,1} + \\
& - 4P_{\mathrm{s}}\binom{3,3}{1,0} + 4P_{\mathrm{s}}\binom{3,3}{0,0}.
\end{aligned}$$

5.2 Bloch–Okounkov relations

The exponential isomorphism, introduced in [18] (see also [19]), maps any quasi-shuffle algebra to a shuffle algebra on the same words. We give an analogous definition of an exponential map on our space of polynomial functions on partitions. It turns out that the corresponding basis is convenient for expressing shifted symmetric functions as elements of $\mathbb{P}$ (see, also, [1]).

Definition 5.3 Given $\underline{m} \in \mathbb{Z}^n$, let $\mathrm{Aut}(\underline{m}) = \prod_{j\in\mathbb{Z}} r_j(\underline{m})!$ with $r_j(\underline{m})$ the number of indices i with $m_i = j$. This allows us to define $\exp : \mathbb{P} \to \mathbb{P}$ by

$$\exp P_{\mathcal{F}}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r}; \lambda\Big) = \sum_{m_1\geq\cdots\geq m_r>0} \frac{1}{\mathrm{Aut}(\underline{m})} \prod_{j=1}^{r} m_j^{d_j} f_{k_j}(r_{m_j}(\lambda)).$$

Recall the *Bernoulli polynomial* B_k ($k \geq 0$) is the unique polynomial such that

$$\int_x^{x+1} B_k(u)\,du = x^k.$$

Then, B_k is of degree k with as constant term $B_k(0) = B_k$ the k-th Bernoulli number.

Proposition 5.4 *For all $k \geq 2$ we have*

$$Q_k = \frac{B_k(\frac{1}{2})}{k!} + \sum_{i=0}^{k-2}\sum_{j=0}^{i} \frac{(-1)^{i+j}}{(k-i-1)!}\frac{B_j(\frac{1}{2})}{j!} \exp P_{\mathcal{F}}\begin{pmatrix} 1,\dots,1 \\ \underbrace{0,\dots,0}_{i-j}, k-1-i \end{pmatrix} \in \mathbb{M},$$

where Q_k is a shifted symmetric function defined in (1.3).

Proof Given $m > 0$, abbreviate $\sum_{m'>m} r_{m'}(\lambda)$ by $r_{>m}$. If $\lambda_i = m$, then $r_{>m} < i \leq r_{>m-1}$, hence

$$\begin{aligned} p_k(\lambda) &:= \sum_{i\geq 1}\big((\lambda_i - i + \tfrac{1}{2})^k - (-i+\tfrac{1}{2})^k\big) \\ &= \sum_{m=1}^{\infty}\sum_{j=1}^{r_m(\lambda)}\big((m - r_{>m} - j + \tfrac{1}{2})^k - (-r_{>m} - j + \tfrac{1}{2})^k\big). \end{aligned}$$

Therefore, letting

$$t_b(m,\lambda) := \sum_{j=1}^{r_m(\lambda)} (r_{>m} + j - \tfrac{1}{2})^b,$$

one finds

$$p_k(\lambda) = \sum_{m=1}^{\infty}\sum_{i=0}^{k-1}\binom{k}{i} m^{k-i}(-1)^i\, t_i(m,\lambda).$$

Expanding $t_b(m,\lambda)$ we have

$$t_b(m,\lambda) = \sum_{j=1}^{r_m(\lambda)}\sum_{l=0}^{b}\binom{b}{l}(r_{>m})^l (j-\tfrac{1}{2})^{b-l}.$$

By computing the generating series $\sum_{m\geq 0}\sum_{j=1}^{n}(j-\frac{1}{2})^m \frac{z^m}{m!}$, one finds that

$$\sum_{j=1}^{n}(j-\tfrac{1}{2})^m = \frac{1}{m+1}\sum_{j=0}^{m}(-1)^j\binom{m+1}{j} B_j(\tfrac{1}{2})\, n^{m+1-j},$$

so that

$$t_b(m,\lambda) = \sum_{l=0}^{b}\sum_{j=0}^{b-l}\binom{b}{l}(r_{>m})^l \frac{1}{b-l+1}(-1)^j\binom{b-l+1}{j}B_j(\tfrac{1}{2})r_m(\lambda)^{b-l+1-j}$$
$$= \sum_{l=0}^{b}\sum_{j=0}^{b-l}\frac{b!}{l!(b-l+1-j)!j!}(-1)^j B_j(\tfrac{1}{2})$$
$$\left(\sum_{m_1,\ldots,m_l>m} r_{m_1}(\lambda)\cdots r_{m_l}(\lambda)\right) r_m(\lambda)^{b-l+1-j}.$$

Write $\underline{m} = (m_1,\ldots,m_{b-j+1})$. Then,

$$t_b(m,\lambda) = \sum_{l=0}^{b}\sum_{j=0}^{b-l}\frac{b!}{j!}(-1)^j B_j(\tfrac{1}{2})\left(\sum_{\substack{m_1\geq\ldots\geq m_l>m\\ m_{l+1}=\ldots=m_{b-j+1}=m}}\frac{r_{m_1}(\lambda)\cdots r_{m_{b-j+1}}(\lambda)}{\mathrm{Aut}(\underline{m})}\right)$$
$$= \sum_{j=0}^{b}\frac{b!}{j!}(-1)^j B_j(\tfrac{1}{2})\left(\sum_{m_1\geq\ldots\geq m_{b-j+1}=m}\frac{r_{m_1}(\lambda)\cdots r_{m_{b-j+1}}(\lambda)}{\mathrm{Aut}(\underline{m})}\right), \quad (5.3)$$

where we recall $\mathrm{Aut}(\underline{m}) = \prod_{j\in\mathbb{Z}} r_j(\underline{m})!$. Hence,

$$p_k(\lambda) = \sum_{i=0}^{k-1}\sum_{j=0}^{i}(-1)^{i+j}\frac{k!}{(k-i)!}\frac{B_j(\frac{1}{2})}{j!}\exp P_{\mathcal{F}}\left(\begin{matrix}1,\ldots,1\\ \underbrace{0,\ldots,0}_{i-j},\,k-i\end{matrix}\right).$$

By definition $Q_k(\lambda) = \beta_k + \frac{p_{k-1}(\lambda)}{(k-1)!}$, which finishes the proof. □

Corollary 5.5 *We have*

$$\sum_{k\geq 2}\sum_{i=0}^{k-2}\frac{(-1)^i}{(k-1-i)!}\,\xi(\underbrace{0,\ldots,0}_{i},k-1-i)\,z^k = \exp\left(\sum_{n\geq 2}\zeta(n)\frac{z^n+(-z)^n}{n}\right)-1$$
$$= \sum_{k\geq 2}\sum_{i=0}^{k-2}(-1)^i\,\zeta(k-i,\underbrace{1,\ldots,1}_{i})\,z^k.$$

Proof The first equality follows directly from Proposition 5.4 by taking the weight limit. Note that for all $f\in\mathbb{P}$ we have $Z\exp f = Zf$. Moreover, for computing the weight limit of Q_k we make use of the Bloch–Okounkov theorem, which expresses

the generating series of the Q_k as a certain Jacobi form [8, Theorem 6.1], i.e.,

$$\sum_{k\geq 0}\langle Q_k\rangle_q\, z^{k-1} = \frac{1}{\Theta(q,z)} := \frac{1}{z}\exp\Bigg(2\sum_{\substack{k\geq 2\\ k \text{ even}}} G_k(q)\,\frac{z^k}{k}\Bigg).$$

Recall that the Eisenstein series in this work are normalized such that the weight limit of G_k equals $\zeta(k)$.

For the second equality, we combine Proposition 5.4 with the explicit formula for ι in Remark 3.14 to find that

$$\iota(Q_k) = \frac{B_k(\frac{1}{2})}{k!} + \sum_{i=0}^{k-2}\sum_{j=0}^{i}\frac{(-1)^{i+j}B_j(\frac{1}{2})}{j!}\exp P_{\mathrm{s}}\begin{pmatrix} k-i, 1, \ldots, 1\\ 0, \underbrace{0, \ldots, 0}_{i-j}\end{pmatrix}.$$

Then, similarly, the second equality follows by taking the weight limit. □

Remark 5.6 (i) The second equality in the above formula is a special case of the Ohno–Zagier relations [24, Theorem 1] after setting $s = 1$ and $x = -y$, which goes back to unpublished results of Zagier from 1995.

(ii) By calculating the q-bracket (instead of the weight limit) of both sides of the equality in Proposition 5.4 one obtains Ohno–Zagier relations for q-analogues of multiple zeta values. These might give special cases of the relations proven in [23].

The relation in the corollary above follows from exploiting the facts that $\langle Q_k\rangle_q$ is a quasimodular form, as well as that Q_k is a polynomial function on partitions. The same holds for any shifted symmetric function, i.e., any polynomial in the generators Q_k. We illustrate this by considering Q_4Q_3:

Example 5.7 Recall that by computing the weight limit of Q_3 and Q_4, cf. Corollary 5.5, we obtain $\zeta(3) - \zeta(2,1) = 0$ and $\zeta(4) - \zeta(3,1) + \zeta(2,1,1) = 0$ respectively. If we proceed in the same way computing the weight limit of Q_3Q_4, we find (the trivial statement) that the shuffle product of $\zeta(3) - \zeta(2,1)$ and $\zeta(4) - \zeta(3,1) + \zeta(2,1,1)$ vanishes as well. We can, however, do more.

Instead, consider

$$Q_4Q_3 - Q_4 \circledast Q_3.$$

The q-bracket of this function vanishes, as can be seen from the fact that $Q_3(\lambda) = -\omega(Q_3)(\lambda)$. The degree is equal to 6, whereas the weight is 7. Computing the degree-6 limit we obtain

$$\begin{aligned}-10\zeta(5) + \zeta(2,3) + 3\zeta(3,2) + 16\zeta(4,1) +\\ - 16\zeta(3,1,1) - 3\zeta(2,2,1) - \zeta(2,1,2) + 10\zeta(2,1,1,1) = 0,\end{aligned}$$

a formula which easily follows from the duality relations for multiple zeta values. Hence, we see that not only the weight limits but also the degree limits of polynomial

functions on partitions give rise to interesting (families of) relations between multiple zeta values.

5.3 Zagier's arm-leg moments

In [30, Theorem 8] it is shown that for any even polynomial $g \in \mathbb{Q}[x, y]$ the q-bracket of the function $\mathcal{A}_g : \mathscr{P} \to \mathbb{Q}$ defined by

$$\mathcal{A}_g(\lambda) := \sum_{\xi \in Y_\lambda} g(\underline{a}(\xi), \underline{b}(\xi)) \tag{5.4}$$

has a quasimodular q-bracket. Here the sum is over all cells ξ of the Young diagram of λ and $\underline{a}(\xi) = a(\xi) + \frac{1}{2}$, $\underline{b}(\xi) = b(\xi) + \frac{1}{2}$, where $a(\xi)$ and $b(\xi)$ denote the arm- and leg-lengths of the cell ξ.

We now show that, in fact, $\mathcal{A}_g \in \mathbb{M}$, and that by computing its weight limits we obtain the sum formula for multiple zeta values.

Theorem 5.8 *For all polynomials $g \in \mathbb{Q}[x, y]$, we have $\mathcal{A}_g \in \mathbb{M}$ and the weight of $\mathcal{A}_g$ is $\leq \deg g + 2$. For $g(x, y) = \frac{x^a y^b}{a!b!}$ the difference*

$$\mathcal{A}_g - \sum_{i=0}^{a} (-1)^{a-i} \frac{1}{i!(a+1-i)!} \exp P_{\mathcal{F}} \begin{pmatrix} 1, \dots, 1 \\ i, \underbrace{0, \dots, 0}_{b-1}, a-i+1 \end{pmatrix} \tag{5.5}$$

is of weight $\leq \deg g + 1$. Here, $\exp : \mathbb{P} \to \mathbb{P}$ is defined in Definition 5.3.

Proof Suppose $\xi = (x, y) \in Y_\lambda$. Then, we write $m = \lambda_x$ and $j = 1 - x + r_{\geq m}(\lambda)$, where we denote $r_{\geq \ell}(\lambda) = \sum_{\ell' \geq \ell} r_{\ell'}(\lambda)$. Note that $1 \leq j \leq r_m(\lambda)$. We can rewrite (5.4) as

$$\mathcal{A}_g(\lambda) = \sum_{m \geq y \geq 1} \sum_{j=1}^{r_m(\lambda)} g\big(m - y + \tfrac{1}{2}, r_{\geq y}(\lambda) - r_{\geq m}(\lambda) + j - \tfrac{1}{2}\big).$$

In particular, we see that for any polynomial g we have $\mathcal{A}_g \in \mathbb{P}$.

Analogous to (5.3), for $b \geq 0$ we have

$$\sum_{j=1}^{r_m(\lambda)} (r_{\geq y}(\lambda) - r_{\geq m}(\lambda) + j - \tfrac{1}{2})^b$$
$$= \sum_{j=0}^{b} \frac{b!}{j!} (-1)^j B_j(\tfrac{1}{2}) \left(\sum_{m = m_1 \geq \dots \geq m_{b-j+1} \geq y} \frac{r_{m_1}(\lambda) \cdots r_{m_{b-j+1}}(\lambda)}{\mathrm{Aut}(\underline{m})} \right).$$

Hence, in the special case $g(x, y) = \frac{x^a y^b}{a!b!}$ we get

$$\begin{aligned}\mathcal{A}_g &= \sum_{j=0}^{b} \sum_{m_1 \geq \ldots \geq m_{b-j+1} \geq y \geq 1} \frac{(-1)^j}{a!j!} B_j(\tfrac{1}{2})(m_1 - y + \tfrac{1}{2})^a \frac{r_{m_1} \cdots r_{m_{b-j+1}}}{\mathrm{Aut}(\underline{m})} \\ &= \sum_{j=0}^{b} \sum_{m_1 \geq \ldots \geq m_{b-j+1} \geq 1} \frac{(-1)^j}{a!j!} B_j(\tfrac{1}{2})(\tilde{\mathrm{s}}_{a+1}(m_1) - \tilde{\mathrm{s}}_{a+1}(m_1 - m_{b-j+1})) \\ &\quad \frac{r_{m_1} \cdots r_{m_{b-j+1}}}{\mathrm{Aut}(\underline{m})}, \end{aligned} \tag{5.6}$$

where $\tilde{\mathrm{s}}_{a+1}(n) = \sum_{i=1}^{n} (i - \frac{1}{2})^a$. As $\tilde{\mathrm{s}}_{a+1}(n)$ is a polynomial in n of degree $a+1$ and with leading coefficient $\frac{1}{a+1}$, we have that

$$\tilde{\mathrm{s}}_{a+1}(u) - \tilde{\mathrm{s}}_{a+1}(u - v) = \frac{1}{a+1} \sum_{i=0}^{a} \binom{a+1}{i} (-1)^{a-i} u^i v^{a-i+1} + \ldots, \tag{5.7}$$

where the ommited terms are of total degree at most a. Substituting (5.7) in (5.6) for $u = m_1$ and $v = m_b$ gives the desired result. □

Corollary 5.9 (Sum formula) *For all $a, b \geq 0$, we have*

$$\begin{aligned}&\sum_{i=0}^{a} \frac{(-1)^{a-i}}{i!(a+1-i)!} \xi(i, \underbrace{0, \ldots, 0}_{b-1}, a - i + 1) = \zeta(a + b + 2) \\ &= \sum_{\substack{k_1 + \ldots + k_{b+1} = a+b+2 \\ k_1 \geq 2,\, k_i \geq 1}} \zeta(k_1, \ldots, k_{b+1}). \end{aligned} \tag{5.8}$$

Proof We compute the weight limit of (5.5). By [30, p. 367] we have

$$\langle \mathcal{A}_g \rangle_q = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} g(i + \tfrac{1}{2}, n - i - \tfrac{1}{2}) \frac{q^n}{1 - q^n}$$

for all $g \in \mathbb{Q}[x, y]$. Now let $g(x, y) = \frac{x^a y^b}{a!b!}$. Then, as

$$\sum_{i=0}^{n-1} g(i + \tfrac{1}{2}, n - i - \tfrac{1}{2}) = \frac{1}{(a+b+1)!} n^{a+b+1} + O(n^{a+b}),$$

we find $Z(\mathcal{A}_g) = \zeta(a + b + 2)$, which proves the first equality.

The second equality follows by applying the involution ι. Namely, the left hand side of (5.8) is the coefficient of X^{a+1} in

$$\zeta\left(\mathfrak{A}\begin{pmatrix}0,0,\ldots,0\\X,0,\ldots,0\end{pmatrix}-\mathfrak{A}\begin{pmatrix}0,0,\ldots,0,0\\X,0,\ldots,0,-X\end{pmatrix}\right),$$

where $\mathfrak{A}$ is defined by (4.18). By Proposition 4.21, together with Theorem 4.23 by which we know that ζ is ι_{s}-invariant, this equals

$$\zeta\left(\mathfrak{A}\begin{pmatrix}X,\ldots,X\\0,0,\ldots,0\end{pmatrix}-\mathfrak{A}\begin{pmatrix}0,X,\ldots,X\\0,\ldots,0\end{pmatrix}\right).$$

The coefficients of X^{a+1} in the latter expression is the sum over all admissible multiple zeta values in weight $a+b+2$. □

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