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Quasi-Jacobi forms, Appell–Lerch functions, and false theta functions as q-brackets of functions on partitions

Kathrin Bringmann[1] · Jonas Kaszian[1] · Jan-Willem van Ittersum[1,2]

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Abstract
We study certain algebras of theta-like functions on partitions, for which the corresponding generating functions give rise to theta functions, quasi-Jacobi forms, Appell–Lerch sums, and false theta functions.

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1 Introduction and statement of results

Let $\mathscr{P}$ be the set of all partitions of integers and for $\lambda \in \mathscr{P}$ denote by $|\lambda|$ the size of the partition λ. For $f : \mathscr{P} \to \mathbb{Q}$, the *$q$-bracket* of f is given by

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathscr{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathscr{P}} q^{|\lambda|}} \in \mathbb{Q}[\![q]\!]. \tag{1.1}$$

For many classes of functions, the q-bracket is a quasimodular form (see, e.g. [2, 7, 28, 29, 32]). The connection to modular forms is natural, since the denominator of the q-bracket equals $q^{\frac{1}{24}}\eta(\tau)^{-1}$ with $\eta(\tau) := q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)$ the *Dedekind eta function* (with $q := e^{2\pi i\tau}$). Notably, in several of these cases these "classes of functions" are algebras, which is surprising since the q-bracket is linear, but not an algebra homomorphism. This fact is crucial in many applications in enumerative geometry [6,

✉ Jan-Willem van Ittersum
j.w.m.vanittersum@uva.nl

Kathrin Bringmann
kbringma@math.uni-koeln.de

Jonas Kaszian
jonaskaszian@gmail.com

1 Division of Mathematics, Department of Mathematics and Computer Science, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany

2 Present Address: Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Postbus 94248, 1090 GE Amsterdam, The Netherlands

7, 9, 11, 15, 25, 26]. Interpreting the q-bracket as the expectation value of some partition invariant f with respect to the normalized probability distribution $\nu \mapsto \frac{q^{|\nu|}}{\sum_{\lambda\in\mathscr{P}} q^{|\lambda|}}$, by this property all the moments of f are modular objects. In short, the theory of q-brackets forms a bridge between enumerative geometry, statistical properties of partitions and number theory.

To be more specific, two examples of such algebras are the algebra of so-called symmetric functions $\mathcal{S}$ and the algebra of shifted symmetric functions Λ^*. These algebras are generated by the *symmetric functions* S_k and *shifted symmetric functions* Q_k, respectively, for $\lambda \in \mathscr{P}$ given by (λ_j is the j-th part of λ and B_k the k-th Bernoulli number)

$$S_k(\lambda) := -\frac{B_k}{2k} + \sum_{j\geq 1} \lambda_j^{k-1} \qquad (k \geq 2 \text{ even}),$$

$$Q_k(\lambda) := -\left(1-2^{1-k}\right)\frac{B_k}{k} + \sum_{j\geq 1}\left(\left(\lambda_j - j + \frac{1}{2}\right)^{k-1} - \left(-j+\frac{1}{2}\right)^{k-1}\right) \qquad (k \geq 2),$$

i.e., $\mathcal{S} = \mathbb{Q}[S_2, S_4, S_6, \ldots]$ and $\Lambda^* = \mathbb{Q}[Q_2, Q_3, Q_4, \ldots]$. These algebras are graded by $\mathrm{wt}(S_k) = k$ and $\mathrm{wt}(Q_k) = k$. For all homogeneous $f \in \mathcal{S}$ or $f \in \Lambda^*$ the q-bracket $\langle f\rangle_q$ is a quasimodular form [2, 28]. In fact, every function $f \in \mathbb{Q}^{\mathscr{P}}$ such that all moments $\langle f^n\rangle_q$ are quasimodular for $\mathrm{SL}_2(\mathbb{Z})$ seems to be an element of either of these two algebras.

Motivated by the study of invariants of flat surfaces, in [7], extensions of the above results for Λ^* were found, where the q-brackets are Eichler integrals of modular forms (see also [4]). In this paper, we aim to get a better understanding of further modularity properties such q-brackets may admit. In particular, we introduce an algebra $\mathcal{A}$ of theta-like functions on partitions and explain how several types of modularity arise if such functions are paired with elements of $\mathcal{S}$ and their generalizations. More concretely, for $N \in \mathbb{N}$, let

$$t_{(N)}(\zeta;\lambda) := 1 + \sum_{m\geq 1}\left(\zeta^m + \zeta^{-m}\right)\delta_{r_m(\lambda)\geq Nm}. \tag{1.2}$$

Here, $\delta_X := 1$ if a statement X is true and 0 otherwise, and $r_m(\lambda)$ denotes the multiplicity of parts of size m in λ. The probability distribution of $r_m(\lambda)$ was determined in [14]. In particular, as $|\lambda| \to \infty$,

$$P(r_m(\lambda) \geq Nm) \sim \begin{cases} 1 & \text{if } m = o\left(|\lambda|^{\frac{1}{4}}\right), \\ 0 & \text{if } |\lambda| = o\left(m^4\right). \end{cases}$$

We let $\mathcal{A}$ be the algebra generated by $\lambda \mapsto t_{(N)}(\zeta;\lambda)$ for $N \in \mathbb{N}$ and $[\mathcal{A}]_{\zeta_\ell=1}$ the algebra generated by

$$\lambda \mapsto t_{(N)}(1;\lambda) = 1 + 2\sum_{m\geq 1}\delta_{r_m(\lambda)\geq Nm}.$$

Our first theorem states that $\mathcal{A}$ gives rise to an algebra of theta-like functions on partitions.

Theorem 1.1 *The algebra $\mathcal{A}$ satisfies the following properties:*

(1) *If $f \in [\mathcal{A}]_{\zeta_\ell=1}$, then $\langle f\rangle_q$ is a modular theta function of some level.*
(2) *If $f \in \mathcal{A}$, then $\langle f\rangle_q$ is a Jacobi theta function.*

Remarks • In particular, by (1), for $f \in [\Phi(\mathcal{A})]_{\zeta_\ell=1}$, the moments $\langle f^n\rangle_q$ are modular forms for all $n \in \mathbb{N}$. This contrasts the situation for Λ^* and $\mathcal{S}$ for which the vector subspace of elements with a modular q-bracket is not a subalgebra (see [27]).
• The algebras $\mathcal{A}$ and $[\mathcal{A}]_{\zeta_\ell=1}$ are graded by assigning the weight $\frac{1}{2}$ to each generator. Then, if the function f in $[\Phi(\mathcal{A})]_{\zeta_\ell=1}$ or in $\mathcal{A}$ is of weight k, then $\langle f\rangle_q$ is of mixed weight k and mixed index, that is, a linear combination of forms of weight at most k and varying index. In Sect. 3, we introduce a closely related bigraded algebra $\mathcal{A}^\circ$ such that for elements $f \in \mathcal{A}^\circ$ of weight k and index M, the corresponding forms are exactly of weight k and of some index M.

Example Writing $t_{(N)}$ for $\lambda \mapsto t_{(N)}(\zeta,\lambda)$ and $t_{(N_j)}$ for $\lambda \mapsto t_{(N_j)}(\zeta_j,\lambda)$, we have

$$\langle t_{(N)}\rangle_q = \sum_{m\in\mathbb{Z}} \zeta^m q^{Nm^2},$$

$$\langle t_{(N_1)}\, t_{(N_2)}\rangle_q = \langle t_{(N_1)}\rangle_q \langle t_{(N_2)}\rangle_q + \frac{1}{2}\sum_{m\in\mathbb{Z}} \left(\zeta_1^m + \zeta_1^{-m}\right)\left(\zeta_2^m + \zeta_2^{-m}\right) \times \left(q^{\max\{N_1,N_2\}m^2} - q^{(N_1+N_2)m^2}\right).$$

Moreover, we consider the algebra $\mathcal{B}$, generated by

$$s(\varrho;\lambda) := \frac{\varrho}{1-\varrho} + \frac{1}{2} + \sum_{m\geq 1}\left(\varrho^m - \varrho^{-m}\right) r_m(\lambda).$$

Note that s is a generating function for the S_k, i.e.,

$$s(e^z;\lambda) = -\frac{1}{z} + 2\sum_{k\geq 2 \text{ even}} S_k(\lambda)\frac{z^{k-1}}{(k-1)!}. \tag{1.3}$$

Elements of $\mathcal{B}$ give rise to quasi-Jacobi forms (see Sect. 2.4). Moreover, also products of elements of $\mathcal{A}$ and $\mathcal{B}$ yield quasi-Jacobi forms.

Theorem 1.2 *If $f \in \mathcal{A}$ and $g \in \mathcal{B}$, then $\langle fg\rangle_q$ is a meromorphic quasi-Jacobi form.*

Remarks • For comparison, if $f \in \Lambda^*$, $g \in \mathcal{S}$, then $\langle f\rangle_q$ and $\langle g\rangle_q$ are quasimodular, but $\langle fg\rangle_q$ is not necessarily a quasimodular form. The latter q-bracket lies in a space of multiple Eisenstein series, also called the space of q-analogues of multiple zeta values (see [1]).

- The algebra $\mathcal{B}$ is graded by assigning weight 1 to each generator. If $f \in \mathcal{A}$ is of weight k and $g \in \mathcal{B}$ is of weight ℓ, then $\langle fg \rangle_q$ is of mixed weight $k+\ell$, and mixed index.

Example Writing s for the function $\lambda \mapsto s(\rho, \lambda)$ and s_j for the function $\lambda \mapsto s(\rho_j, \lambda)$, we have

$$\begin{aligned}
\langle s \rangle_q &= \frac{\varrho}{1-\varrho} + \frac{1}{2} + \sum_{m,r>0} \left(\rho^m - \rho^{-m}\right) q^{mr}, \\
\langle s_1 s_2 \rangle_q &= \langle s_1 \rangle_q \langle s_2 \rangle_q + \sum_{m,r\geq 1} \left(\varrho_1^m - \varrho_1^{-m}\right)\left(\varrho_2^m - \varrho_2^{-m}\right) r q^{mr}, \\
\langle t_{(N)} s \rangle_q &= \langle t_{(N)} \rangle_q \langle s \rangle_q + N \sum_{m\geq 1} \left(\zeta^m + \zeta^{-m}\right)\left(\varrho^m - \varrho^{-m}\right) m q^{Nm^2}.
\end{aligned}$$

Corollary 1.3 *Let f be in the algebra generated by the S_k for $k \geq 2$ even and the functions*

$$\alpha_{(N),\ell} : \quad \lambda \mapsto \delta_{\ell=0} + 2 \sum_{m\geq 1} m^\ell \delta_{r_m(\lambda)\geq Nm}$$

for $N \in \mathbb{N}$ and $\ell \geq 0$ even. Then the q-bracket $\langle f \rangle_q$ is a quasimodular form.

Example For $k \geq 2$ even, $N \in \mathbb{N}$ and $\ell \geq 0$ even, we have

$$\begin{aligned}
\langle S_k \rangle_q &= -\frac{B_k}{2k} + \sum_{m,r\geq 1} m^{k-1} q^{mr}, \qquad \langle \alpha_{(N),\ell} \rangle_q = \sum_{m\in\mathbb{Z}} m^\ell q^{Nm^2}, \\
\langle S_k\, \alpha_{(N),\ell} \rangle_q &= \langle S_k \rangle_q \langle \alpha_{(N),\ell} \rangle_q + N \sum_{m\geq 1} m^{k+\ell} q^{Nm^2}.
\end{aligned}$$

Finally, consider

$$s^*(\varrho, \xi; \lambda) = \frac{1}{1-\varrho} + \frac{\xi}{1-\xi} + \sum_{m\geq 1} \sum_{r=1}^{r_m(\lambda)} \left(\varrho^m \xi^r - \varrho^{-m} \xi^{-r}\right).$$

Similarly, taking the coefficients of s^* around $w = \mathfrak{z} = 0$ with $\varrho = e^{2\pi i \mathfrak{z}}$ and $\xi = e^{2\pi i w}$ generates an algebra $\mathcal{T}$ containing $\mathcal{S}$, but which is not a graded algebra (see Sect. 2.2 or [28] for more details about the algebras $\mathcal{S}$ and $\mathcal{T}$). Quasi-Jacobi forms admit non-holomorphic completions which transform as Jacobi forms. Taking q-brackets with s^* yields forms for which the completion is more involved than for quasi-Jacobi forms; see Sect. 4 for a more precise statement.

Theorem 1.4 *For $N \in \mathbb{N}$, the bracket $\langle t_{(N)} s^* \rangle_q$ can be expressed as a linear combination of a Jacobi theta function, a false theta function, and an Appell–Lerch sum.*

In Sect. 2 we recall the relevant background, such as the connected q-bracket, quasi-Jacobi forms, Appell functions, and false theta functions. The proof of the first two theorems can be found in Sect. 3 and the proof of Theorem 1.4 in Sect. 4.

2 Preliminaries

2.1 Partitions and the connected q-bracket

Each partition $\lambda \in \mathscr{P}$ is represented as $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_j \in \mathbb{N}_0, \lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_j = 0$ for all but finitely many j, or, equivalently, as a finite multiset where the part $m \in \lambda$ has multiplicity $r_m(\lambda)$. We write $|\lambda| = \sum_j \lambda_j = \sum_{m\geq 1} m r_m(\lambda)$ for the size of λ and $\ell(\lambda) := |\{j : \lambda_j \neq 0\}| = \sum_{m\geq 1} r_m(\lambda)$ for the *length* of λ. Following [28], we refine the q-bracket (see (1.1)) by the $\boldsymbol{u}$*-bracket*, which is the isomorphism of vector spaces $\mathbb{Q}^{\mathscr{P}} \to \mathbb{Q}[\![\boldsymbol{u}]\!]$, where $\boldsymbol{u} = (u_1, u_2, \ldots)$, given by

$$f \mapsto \langle f \rangle_{\boldsymbol{u}} := \frac{\sum_{\lambda\in\mathscr{P}} f(\lambda) u_\lambda}{\sum_{\lambda\in\mathscr{P}} u_\lambda} \qquad (u_\lambda = u_{\lambda_1} u_{\lambda_2} \cdots). \tag{2.1}$$

Note that for $f \in \mathbb{Q}^{\mathscr{P}}$, one has $\langle f \rangle_q = [\langle f \rangle_{\boldsymbol{u}}]_{u_j = q^j}$. A partition is *strict* if it has no repeated parts, i.e., if $r_m(\lambda) \leq 1$ for all $m \in \mathbb{N}$. Define the *Möbius function on partitions* by

$$\mu(\lambda) := \begin{cases} (-1)^{\ell(\lambda)} & \text{if } \lambda \text{ is strict,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then we have

$$\langle f \rangle_{\boldsymbol{u}} = \sum_{\lambda\in\mathscr{P}} f(\lambda) u_\lambda \sum_{\lambda\in\mathscr{P}} \mu(\lambda) u_\lambda.$$

If $f(\lambda) = g(r_m(\lambda))$ for some m, then [28, Proposition 3.1.4] explains how to compute its $\boldsymbol{u}$-bracket. Let the *discrete derivative* ∂ of $g : \mathbb{N} \to \mathbb{Q}$ be defined by $\partial g(n) := g(n) - g(n-1)$ for $n \geq 2$ and $\partial g(1) := g(1)$.

Lemma 2.1 *If $f(\lambda) = g(r_m(\lambda))$ with $m \in \mathbb{N}$, then*

$$\langle f \rangle_{\boldsymbol{u}} = \sum_{r\geq 1} \partial g(r) u_m^r.$$

We define the connected q-bracket [7, pp. 55–57], which naturally arises in enumerative geometry from counting connected coverings. From a combinatorial point of view, often the connected q-bracket is easier to compute than the usual q-bracket. For a set X, write $\pi(X)$ for the set of all set partitions of X. For $l \in \mathbb{N}$, let $[l] := \{1, \ldots, l\}$. Write $\Pi(l)$ for the set of all set partitions of $[l]$. For example, we write $\mathbb{1}_l = \{[l]\}$ for the partition consisting of one element. For $A \subset [l]$, we denote $f_A = \prod_{a\in A} f_a$. The *connected q-bracket* is defined as the multilinear map $\langle\cdot\rangle_q : (\mathbb{Q}^{\mathscr{P}})^{\otimes l} \to \mathbb{Q}$ extending the q-bracket such that for $f_1, \ldots, f_l \in \mathbb{Q}^{\mathscr{P}}$

$$\langle f_1 \otimes \cdots \otimes f_l \rangle_q = \sum_{\alpha\in\Pi(l)} \mu(\alpha, \mathbb{1}_l) \prod_{A\in\alpha} \langle f_A \rangle_q.$$

Here, for $\alpha, \beta \in \Pi(l)$ the *Möbius function* is given by

$$\mu(\alpha, \beta) := \prod_{B \in \beta} (-1)^{|\alpha_B|+1} (|\alpha_B| - 1)!,$$

where α_B for $B \subset [l]$ is the partition on B induced by α. We write $\alpha \leq \beta$ if for all $A \in \alpha$ there exists a $B \in \beta$ such that $A \subseteq B$. By invoking the Möbius inversion formula one finds

$$\begin{aligned} \prod_{B \in \beta} \langle \otimes_{b \in B} f_b \rangle_q &= \sum_{\alpha \leq \beta} \mu(\alpha, \beta) \prod_{A \in \alpha} \langle f_A \rangle_q, \\ \prod_{A \in \alpha} \langle f_A \rangle_q &= \sum_{\beta \leq \alpha} \prod_{B \in \beta} \langle \otimes_{b \in B} f_b \rangle_q. \end{aligned} \tag{2.3}$$

Similarly, we let

$$\langle f_1 \otimes \cdots \otimes f_l \rangle_{\boldsymbol{u}} := \sum_{\alpha \in \Pi(l)} \mu(\alpha, \mathbb{1}_l) \prod_{A \in \alpha} \langle f_A \rangle_{\boldsymbol{u}} \tag{2.4}$$

and have a Möbius inversion formula

$$\prod_{A \in \alpha} \langle f_A \rangle_{\boldsymbol{u}} = \sum_{\beta \leq \alpha} \prod_{B \in \beta} \langle \otimes_{b \in B} f_b \rangle_{\boldsymbol{u}}. \tag{2.5}$$

2.2 The algebras $\mathcal{S}$, $\mathcal{T}$, and Λ^*

In the introduction, the algebras Λ^* and $\mathcal{S}$ are defined and it is explained that the q-brackets of their elements are quasimodular. Besides this property which these algebras share, there are many differences. For example, for $f \in \Lambda^*$ of weight k, one has $f(\lambda') = (-1)^k f(\lambda)$, where λ' denotes the conjugate of the partition λ, whereas there is no easy relation by $f(\lambda')$ and $f(\lambda)$ for $f \in \mathcal{S}$.

The Taylor coefficients of s^* give rise to an extension of $\mathcal{S}$. More concretely, let

$$T_{k,\ell}(\lambda) := -\frac{B_{k+\ell}}{2(k+\ell)} \delta_{k=0 \text{ or } \ell=1} + \sum_{m \geq 1} \sum_{r=1}^{r_m(\lambda)} m^k r_m(\lambda)^{\ell-1} \quad (k \geq 0, \ell \geq 1, k+\ell \text{ even}).$$

Then the $T_{k,\ell}$ are the Taylor coefficients of s^*. In particular, note that $T_{k-1,1} = S_k$. The algebra $\mathcal{T}$ generated by the $T_{k,\ell}$ also has the property that the q-bracket of $f \in \mathcal{T}$ is quasimodular. Moreover, it has the interesting property that $\langle \mathcal{T} \rangle_{\boldsymbol{u}}$ is also an algebra with the multiplication of power series in $\mathbb{Q}[\![\boldsymbol{u}]\!]$. In contrast, $\langle \Lambda^* \rangle_{\boldsymbol{u}}$ and $\langle \mathcal{S} \rangle_{\boldsymbol{u}}$ are not algebras. Further variations of these algebras are known for which the elements are quasimodular under a modification of the q-bracket [9, 12, 13], e.g., for the measure $\lambda \mapsto \frac{\mu(\nu) q^{|\nu|}}{\sum_{\lambda \in \mathscr{P}} \mu(\lambda) q^{|\lambda|}}$.

2.3 Jacobi forms

The *Jacobi theta function* is given by

$$\vartheta(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i n\left(z+\frac{1}{2}\right)} q^{\frac{n^2}{2}}. \tag{2.6}$$

It is a Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$, that is, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $m, \ell \in \mathbb{Z}$, we have (which follows from Table V of [23])

$$\begin{aligned} \vartheta(z+m\tau+\ell;\tau) &= (-1)^{m+\ell}\zeta^{-m}q^{-\frac{m^2}{2}}\vartheta(z;\tau), \\ \vartheta\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) &= \nu_\eta^3(\gamma)\sqrt{c\tau+d}\,e^{\frac{\pi i c z^2}{c\tau+d}}\vartheta(z;\tau), \end{aligned} \tag{2.7}$$

where ν_η is the multiplier of η, that is, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta(\gamma)\sqrt{c\tau+d}\;\eta(\tau).$$

Explicitly, ν_η is given by (see [20])

$$\nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{|c|}\right) e^{\frac{\pi i}{12}\left((a+d)c-bd\left(c^2-1\right)-3c\right)} & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right) e^{\frac{\pi i}{12}\left(ac\left(1-d^2\right)+d(b-c+3)-3\right)} & \text{if } c \text{ is even,} \end{cases} \tag{2.8}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. Write

$$\Theta(z;\tau) := \sum_{n\in\mathbb{Z}} \zeta^n q^{\frac{n^2}{2}} = \zeta^{\frac{1}{2}} q^{\frac{1}{8}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \zeta^n q^{\frac{n^2+n}{2}} = \zeta^{\frac{1}{2}} q^{\frac{1}{8}} \vartheta\left(z-\frac{1}{2}+\frac{\tau}{2};\tau\right).$$

Then, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ and $m, \ell \in \mathbb{Z}$, we have [23, p. 2 and Theorem 7.1]

$$\begin{aligned} \Theta(z+m\tau+\ell;\tau) &= \zeta^{-m}q^{-m^2}\Theta(z;\tau), \\ \Theta\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) &= \left(\frac{-2c}{d}\right)\varepsilon_d^{-1}\sqrt{c\tau+d}\;e^{\frac{\pi i c z^2}{c\tau+d}}\Theta(z;\tau), \end{aligned}$$

where, for d odd, we let $\varepsilon_d := 1$ if $d \equiv 1 \pmod 4$ and $\varepsilon_d := i$ if $d \equiv 3 \pmod 4$.

Another Jacobi form that we encounter in this paper is the *Weierstrass $\wp$-function*

$$\wp(z;\tau) := \frac{1}{z^2} + \sum_{\omega\in\mathbb{Z}\tau+\mathbb{Z}\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

We have, for $m, n \in \mathbb{Z}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ [10],

$$\wp(z+m\tau+n;\tau) = \wp(z;\tau), \qquad \wp\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-2} e^{-\frac{2\pi i c z^2}{c\tau+d}} \wp(z;\tau).$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ ($\in \mathrm{SL}_2(\mathbb{Z})$ if $k \in \mathbb{Z}$ and $\in \Gamma_0(4)$ if $k \in \mathbb{Z} + \frac{1}{2}$) define $j(\gamma, \tau)$

$$j(\gamma,\tau) := \begin{cases} \sqrt{c\tau+d} & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-2k} \sqrt{c\tau+d} & \text{if } k \in \mathbb{Z} + \frac{1}{2}. \end{cases} \tag{2.9}$$

The functions obtained as q-brackets in this paper are Jacobi forms, possibly after adding non-holomorphic parts as we discuss in the sections below.

Definition Let $k, M \in \frac{1}{2}\mathbb{Z}$. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ if $k \in \mathbb{Z}$, and of $\Gamma_0(4)$ if $k \in \mathbb{Z} + \frac{1}{2}$. A *polar Jacobi form* of weight k, index M, and character χ on Γ is a function $\phi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ that is holomorphic in τ, meromorphic in z, and that satisfies:

1. For $m, \ell \in \mathbb{Z}$ we have that

$$\phi(z + m\tau + \ell; \tau) = (-1)^{M(\ell+m)} e^{-2\pi i M\left(m^2\tau + 2mz\right)} \phi(z;\tau).$$

2. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have that

$$\phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \chi(d) j(\gamma,\tau)^{2k} e^{\frac{2\pi i c M z^2}{c\tau+d}} \phi(z;\tau).$$

3. All poles z_0 of $z \mapsto \phi(z;\tau)$ satisfy $z_0 \in \mathbb{Q}\tau + \mathbb{Q}$. Moreover, if $z \in \mathbb{Q}\tau + \mathbb{Q}$ is not a pole of $\phi(z;\tau)$, then $\tau \mapsto \phi(z;\tau)$ is a weakly holomorphic modular form (of some level).

Both ϑ and $\wp$ are polar Jacobi forms. It was discovered by Zwegers [33] that meromorphic Jacobi forms for which poles only occur at torsion points $z \in \mathbb{Q}\tau + \mathbb{Q}$ have a modified theta expansion related to mock modular forms. The notion of polar Jacobi forms is closely related to the so-called strictly meromorphic Jacobi forms in [29].

We mention two other examples of Jacobi forms we encounter in this work. First of all, consider Zagier's polar Jacobi form of weight 1 (see Section 3 of [31]) ($q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$, $\xi = e^{2\pi i w}$)

$$\begin{aligned} F(z,w;\tau) &:= \sum_{n\geq 0} \frac{\xi^{-n}}{q^{-n}\zeta - 1} - \sum_{m \geq 0} \frac{\zeta^m \xi}{q^{-m} - \xi} \\ &= \frac{\zeta\xi - 1}{(\zeta-1)(\xi-1)} - \sum_{m,r\geq 1} \left(\zeta^m \xi^r - \zeta^{-m}\xi^{-r}\right) q^{mr} \end{aligned} \tag{2.10}$$

defined for $z_2 < \tau_2$ and $-w_2 < \tau_2$. Here for $w \in \mathbb{C}$, we write $w = w_1 + i w_2,\ w_1, w_2 \in \mathbb{R}$. The following identity was known to Kronecker [18], [17, pp. 309–318]; see also [30, pp. 70–71], [31, Section 3], [22, equation (1.1)]

$$F(z, w; \tau) = \frac{1}{2\pi i} \frac{\vartheta'(0;\tau)\vartheta(z+w;\tau)}{\vartheta(z;\tau)\vartheta(w;\tau)},$$

where $\vartheta'(z;\tau) := \frac{\partial}{\partial z}\vartheta(z;\tau)$. Moreover, by [28, Theorem 4.4.1], we have

$$\langle s^*(\zeta, \xi)\rangle_q = F(z, w; \tau). \tag{2.11}$$

Remark For $n \in \mathbb{N}$, define[1]

$$F_n(z_1, \ldots, z_{2n}; \tau)$$
$$:= (-2)^{-n} \sum_{\boldsymbol{m},\boldsymbol{r}\in\mathbb{Z}^n} \zeta_{2n+1}^{m_n} \zeta_{2n+2}^{r_n} q^{\boldsymbol{m}\cdot\boldsymbol{r}} \prod_{j=1}^{n} \left(\operatorname{sgn}\left(m_j - m_{j+1} - \frac{1}{2}\right) + \operatorname{sgn}\left(r_1 + \ldots + r_j + \frac{1}{2}\right)\right),$$

where we set $m_{n+1} := 0$. This function is closely related to the generating function of q-analogues of multiple zeta values, given by

$$\sum_{\substack{m_1 > \cdots > m_n > 0 \\ r_1, \ldots, r_n > 0}} \zeta_{2n+1}^{m_n} \zeta_{2n+2}^{r_n} q^{\boldsymbol{m}\cdot\boldsymbol{r}}.$$

The appearance of n sign-factors seems to suggest that F_n is not a Jacobi form. However, surprisingly, we have

$$F_n(z_1, \ldots, z_{2n}; \tau) = \prod_{j=1}^{n} F\left(z_1 + z_3 + \ldots + z_{2j+1}, z_{2j} - z_{2j+2}; \tau\right).$$

Secondly, the *Eisenstein–Kronecker series* E_k are polar Jacobi forms of integer weight $k \geq 3$ and index 0, given by (see [30, Chapter III §2])

$$E_k(z;\tau) := \sum_{c,d\in\mathbb{Z}} \frac{1}{(z + c\tau + d)^k}.$$

2.4 Quasi-Jacobi forms

Extending the above definition of E_k by the Eisenstein summation procedure

$$\sum_{c,d\in\mathbb{Z}} = \lim_{C\to\infty} \sum_{c=-C}^{C} \lim_{D\to\infty} \sum_{d=-D}^{D},$$

[1] Note that, in this remark, z_1 and z_2 should not be confused with the real and the imaginary part of z.

we find that the resulting function $E_1(z;\tau)$ and $E_2(z;\tau)$ do not satisfy the transformation properties of a Jacobi form. Note that $E_2(z;\tau) - e_2(\tau) = \wp(z;\tau)$, where $e_2(\tau) := \sum_{c,d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2}$ with the summation taken by the same Eisenstein summation procedure. Recall their completions (see equations (4) and (6) in [30, Chapter VI §2] with $u = 1$, $v = \tau$, $\delta = 1$, and $2\pi i A = \tau - \overline{\tau}$)

$$\widehat{E}_1(z;\tau) := E_1(z;\tau) + 2\pi i \frac{z_2}{\tau_2}, \qquad \widehat{E}_2(z;\tau) = E_2(z;\tau) - \frac{\pi}{\tau_2},$$

transforming as Jacobi forms of weight 1 and 2 and index 0, respectively. More generally:

Definition Let k, $M \in \frac{1}{2}\mathbb{Z}$. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ if $k \in \mathbb{Z}$, and of $\Gamma_0(4)$ if $k \in \mathbb{Z}+\frac{1}{2}$. A *polar almost Jacobi form* of weight k, index M, and character χ on Γ is a function $\phi : \mathbb{C}\times\mathbb{H} \to \mathbb{C}$ satisfying:

1. For $m, \ell \in \mathbb{Z}$ we have that

$$\phi(z + m\tau + \ell;\tau) = (-1)^{M(\ell+m)} e^{-2\pi i M\left(m^2\tau + 2mz\right)} \phi(z;\tau).$$

2. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have that

$$\phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \chi(d) j(\gamma,\tau)^{2k} e^{\frac{2\pi i c M z^2}{c\tau+d}} \phi(z;\tau).$$

3. We have

$$\phi(z;\tau) = \sum_{\ell,j} \phi_{\ell,j}(z;\tau) \frac{z_2^j}{\tau_2^{\ell+j}}$$

with $\phi_{\ell,j} : \mathbb{C}\times\mathbb{H} \to \mathbb{C}$ meromorphic such that all poles z_0 of $z \mapsto \phi_{\ell,j}(z;\tau)$ satisfy $z_0 \in \mathbb{Q}\tau + \mathbb{Q}$. If $z \in \mathbb{Q}\tau+\mathbb{Q}$ is not a pole of $\phi_{\ell,j}(z;\tau)$, then $\tau \mapsto \phi_{\ell,j}(z;\tau)$ is weakly holomorphic.

We say ψ is a *polar quasi-Jacobi form* if $\psi = \phi_{0,0}$ for some polar almost Jacobi form ϕ.

As a first example, quasimodular forms such as e_2 can be seen as quasi-Jacobi forms which do not depend on the elliptic variable z. Quasi-Jacobi forms naturally appear in enumerative geometry and the theory of vertex operator algebras; the definition and first study go back to these works [19, 21, 24]. We recall a key property of quasi-Jacobi forms. Write $J_{k,M}$ and $\widetilde{J}_{k,M}$ for the space of polar Jacobi forms and polar quasi-Jacobi forms, respectively, of weight k and index M.

Proposition 2.2 [29, Proposition 2.18] *For k, $M \in \frac{1}{2}\mathbb{Z}$*

$$\widetilde{J}_{k,M} = J_{k,M}[E_1, E_2].$$

Moreover, from [24, Subsubsection 1.3.3] and by [29, Proposition 2.24] one has

Proposition 2.3 *The maps $\frac{1}{2\pi i}\frac{\partial}{\partial\tau} : \widetilde{J}_{k,M} \to \widetilde{J}_{k+2,M}$ and $\frac{\partial}{\partial z} : \widetilde{J}_{k,M} \to \widetilde{J}_{k+1,M}$ are well-defined derivations.*

As the specialization $z = 0$ of a quasi-Jacobi form yields a quasimodular form, this implies quasi-Jacobi forms admit a Laurent expansion in z for which the coefficients are quasimodular forms.

2.5 Appell functions

Zwegers' *multivariable Appell functions* [34] are given by

$$A_\ell(z, w; \tau) := \zeta^{\frac{\ell}{2}} \sum_{m\in\mathbb{Z}} \frac{(-1)^{\ell m} q^{\frac{\ell m(m+1)}{2}} \xi^m}{1 - \zeta q^m} \quad \left(\zeta = e^{2\pi i z}, \xi := e^{2\pi i w}\right). \tag{2.12}$$

Now recall the completions

$$\widehat{A}_\ell(z, w; \tau) = A_\ell(z, w; \tau) + \frac{i}{2} \sum_{k=0}^{\ell-1} \zeta^k \vartheta\left(w + k\tau + \frac{\ell-1}{2}; \ell\tau\right) R\left(\ell z - w - k\tau - \frac{\ell-1}{2}; \ell\tau\right),$$

where ϑ is the Jacobi theta function defined in (2.7) and

$$R(z; \tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} \left(\operatorname{sgn}(n) - E\left(\left(n + \frac{z_2}{\tau_2}\right)\sqrt{2\tau_2}\right)\right) (-1)^{n-\frac{1}{2}} q^{-\frac{n^2}{2}} e^{-2\pi i n z}, \tag{2.13}$$

where for $x \in \mathbb{R}$ we define $E(x) := 2\int_0^x e^{-\pi t^2} dt$. By [33, Theorem 2.2], for $m_1, m_2, \ell_1, \ell_2 \in \mathbb{Z}$ we have the *elliptic transformation*

$$\widehat{A}_\ell(z + m_1\tau + \ell_1, w + m_2\tau + \ell_2; \tau) = (-1)^{\ell(m_1+\ell_1)} \zeta^{\ell m_1 - m_2} \xi^{-m_1} q^{\frac{\ell m_1^2}{2} - m_1 m_2} \widehat{A}_\ell(z, w; \tau). \tag{2.14}$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have

$$\widehat{A}_\ell\left(\frac{z}{c\tau + d}, \frac{w}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d) e^{\frac{\pi i c}{c\tau+d}(-\ell z^2 + 2zw)} \widehat{A}_\ell(z, w; \tau). \tag{2.15}$$

Among other places, these Appell–Lerch sums occur in the decomposition of meromorphic Jacobi forms as in [8]; see also [5, 16].

2.6 False theta functions

In [3], false theta functions were embedded in a modular framework. For example, let

$$\psi(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} \operatorname{sgn}(n) e^{2\pi i n\left(z+\frac{1}{2}\right)} q^{\frac{n^2}{2}}. \tag{2.16}$$

Then $\psi(z;\tau) = \lim_{t\to\infty} \widehat{\psi}(z;\tau,\tau+it+\varepsilon)$ if $-\frac{1}{2} < \frac{z_2}{\tau_2} < \frac{1}{2}$ and $\varepsilon > 0$ arbitrary, where

$$\widehat{\psi}(z;\tau,\omega) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} E\left(-i\sqrt{\pi i(\omega-\tau)}\left(n+\frac{z_2}{\tau_2}\right)\right) e^{2\pi i n\left(z+\frac{1}{2}\right)} q^{\frac{n^2}{2}}.$$

Note that

$$\widehat{\psi}(z+m\tau+\ell;\tau,\omega) = (-1)^{m+\ell}\zeta^{-m} q^{-\frac{m^2}{2}} \widehat{\psi}(z;\tau,\omega), \tag{2.17}$$

$$\widehat{\psi}\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d},\frac{a\omega+b}{c\omega+d}\right) = \chi_{\tau,\omega}(\gamma)\nu_\eta(\gamma)^3(c\tau+d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau+d}} \widehat{\psi}(z;\tau,\omega), \tag{2.18}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\ell, m \in \mathbb{Z}$. Here,

$$\chi_{\tau,\omega}(\gamma) := \sqrt{\frac{i(\omega-\tau)}{(c\tau+d)(c\omega+d)}} \frac{\sqrt{c\tau+d}\sqrt{c\omega+d}}{\sqrt{i(\omega-\tau)}}.$$

3 Proof of Theorem 1.1 and Theorem 1.2

Set $u_0 := 1$ and for $m \in \mathbb{N}$ let $u_{-m} := u_m^{-1}$. We consider the "theta function"

$$A_N(\zeta,\boldsymbol{u}) := \sum_{m\in\mathbb{Z}} \zeta^m u_m^{Nm} \in \mathbb{Q}\left[\!\left[\boldsymbol{u},\zeta,\zeta^{-1}\right]\!\right] \qquad (N\in\mathbb{N}),$$

which under the substitution $u_m = q^m$ equals an actual theta function of weight $\frac{1}{2}$ and index $\frac{1}{2}$. Moreover, we consider the "theta function associated to a hyperbolic form"

$$\frac{1}{2}\sum_{m,r\in\mathbb{Z}} \left(\operatorname{sgn}\left(m+\frac{1}{2}\right)+\operatorname{sgn}\left(r-\frac{1}{2}\right)\right) \varrho^m \xi^r u_m^r, \tag{3.1}$$

for which the constant term around $w = 0$, where $\xi = e^{2\pi i w}$, equals

$$B(\varrho,\boldsymbol{u}) := \frac{\varrho}{1-\varrho} + \frac{1}{2} + \sum_{\substack{m,r\in\mathbb{Z}\\ mr>0}} \operatorname{sgn}(m)\varrho^m u_m^r.$$

From (2.11) it follows that under the substitution $u_m = q^m$ the latter equals a meromorphic Jacobi form of weight 1 and index 0.

Proposition 3.1 *We have* $\langle t_{(N)}(\zeta,\cdot)\rangle_{\boldsymbol{u}} = A_N(\zeta,\boldsymbol{u})$ *and* $\langle s(\varrho,\cdot)\rangle_{\boldsymbol{u}} = B(\varrho,\boldsymbol{u})$.

Proof Note that $\langle 1\rangle_{\boldsymbol{u}} = 1$. By Lemma 2.1 we directly find that

$$\begin{aligned}\langle t_{(N)}\big(\zeta;\lambda\big)\rangle_{\boldsymbol{u}} &= \langle 1\rangle_{\boldsymbol{u}} + \sum_{m\geq 1}\Big(\zeta^m+\zeta^{-m}\Big)\,\langle \delta_{r_m(\lambda)\geq Nm}\rangle_{\boldsymbol{u}}\\ &= 1+\sum_{m\geq 1}\sum_{r\geq 1}\Big(\zeta^m+\zeta^{-m}\Big)\,\delta_{r=Nm}u_m^r = \sum_{m\in\mathbb{Z}}\zeta^m u_m^{Nm} = A_N\Big(\zeta,\boldsymbol{u}\Big),\\ \langle s(\varrho;\lambda)\rangle_{\boldsymbol{u}} &= \left\langle \frac{\varrho}{1-\varrho}+\frac{1}{2}\right\rangle_{\boldsymbol{u}} + \sum_{m\geq 1}\big(\varrho^m-\varrho^{-m}\big)\,\langle r_m(\lambda)\rangle_{\boldsymbol{u}}\\ &= \frac{\varrho}{1-\varrho}+\frac{1}{2}+\sum_{m\geq 1}\sum_{r\geq 1}\big(\varrho^m-\varrho^{-m}\big)\,u_m^r = B(\varrho,\boldsymbol{u}).\end{aligned}$$

□

A reformulation of [28, Theorem 3.5.4] generalizing Lemma 2.1 is the following. For $g,h:\mathbb{N}\to\mathbb{Q}$, define the *discrete convolution product* of g and h by

$$(g*h)(n) := \sum_{j=1}^{n-1} g(j)h(n-j)$$

and denote the convolution product of functions $g_1,\ldots,g_n$ by

$$\mathop{\ast}_{j=1}^{n} g_j := g_1*\cdots*g_n.$$

Proposition 3.2 *Let* $n\in\mathbb{N}$ *and* $g_1,\ldots,g_n:\mathbb{N}_0\to\mathbb{C}$. *For* $m_1,\ldots,m_n\in\mathbb{N}$, *write* $G_j:\mathscr{P}\to\mathbb{C}$ *for the function* $G_j(\lambda) := g_j(r_{m_j}(\lambda))$. *Then the following hold:*

(1) *We have*

$$\langle G_1\otimes\cdots\otimes G_n\rangle_{\boldsymbol{u}} = \delta_{m_1=\ldots=m_n}\sum_{r\geq 1} g_{[n]}(r)u_m^r,$$

where $m := m_1 = \ldots = m_n$ *and*

$$g_{[n]} = \sum_{\alpha\in\Pi(n)}\mu(\alpha,\mathbb{1}_l)\mathop{\ast}_{A\in\alpha}\partial g_A.$$

(2) *If, moreover,* $g_n(r) = r$ *for all* $r\in\mathbb{N}_0$, *then*

$$\langle G_1\otimes\cdots\otimes G_n\rangle_{\boldsymbol{u}} = \delta_{m_1=\ldots=m_n}\sum_{r\geq 1} r\,g_{[n-1]}(r)u_m^r.$$

We let $\mathcal{A}^\circ$ be the algebra generated by the A_N for all $N \in \mathbb{N}$ and $\mathcal{B}^\circ$ be the algebra generated by the B. These algebras are bigraded by assigning weight $\frac{1}{2}$ and index $\frac{1}{2}$ to A_N and weight 1 and index 0 to B, respectively. Note that for an element $f \in \mathcal{A}^\circ$ or $f \in \mathcal{B}^\circ$ of weight k and index M, the q-series obtained by setting $u_m = q^m$ is a (meromorphic) Jacobi form of weight k and index M.[2]

Theorem 3.3 *The algebras $\mathcal{A}$ and $\mathcal{A}^\circ$ are linearly isomorphic.*

Proof Recall that the $\boldsymbol{u}$-bracket is a linear map. First we show that the image $\mathcal{A}$ under the $\boldsymbol{u}$-bracket is contained in $\mathcal{A}^\circ$. Let $\boldsymbol{N} = (N_1, \dots, N_l) \in \mathbb{N}^l$. As every $\boldsymbol{u}$-bracket can be expressed polynomially in terms of connected $\boldsymbol{u}$-brackets (see (2.5)), it suffices to show that the connected bracket of the $t_{N_j} = t_{(N_j)}$, defined by (1.2), is an element of $\mathcal{A}^\circ$. By Proposition 3.2 (1), we obtain

$$\langle t_{N_1} \otimes t_{N_2} \otimes \cdots \otimes t_{N_l} \rangle_{\boldsymbol{u}} = \delta_{l,1} + \sum_{m,r \geq 1} g_{m,\boldsymbol{N}}(r) u_m^r \prod_j \left(\zeta_j^m + \zeta_j^{-m} \right),$$

where

$$\begin{aligned} g_{m,\boldsymbol{N}}(r) &:= \sum_{\substack{\alpha \in \Pi(\boldsymbol{N}) \\ \alpha = \{A_1, \dots, A_t\}}} \mu(\alpha, \mathbb{1}_l) \sum_{\substack{\ell_1, \dots, \ell_t \\ \ell_1 + \dots + \ell_t = r}} \prod_{j=1}^{t} \partial \left(x \mapsto \prod_{N \in A_j} \delta_{x \geq Nm} \right) (\ell_j) \\ &= \sum_{\substack{\alpha \in \Pi(\boldsymbol{N}) \\ \alpha = \{A_1, \dots, A_t\}}} \mu(\alpha, \mathbb{1}_l) \sum_{\substack{\ell_1, \dots, \ell_t \\ \ell_1 + \dots + \ell_t = r}} \prod_{j=1}^{t} \delta_{\ell_j = (\max A_j) m} = \sum_{\alpha \in \Pi(\boldsymbol{N})} \mu(\alpha, \mathbb{1}_l) \delta_{r = Mm}. \end{aligned}$$

Here we set $M := \sum_{j=1}^t \max\{A_j\}$. Note that $\sum_{\alpha \in \Pi(n)} \mu(\alpha, \mathbb{1}_l) = 0$ for $n \geq 2$. Hence, the connected $\boldsymbol{u}$-bracket is a linear combination of elements of $\mathcal{A}$, as desired:

$$\langle t_{N_1} \otimes t_{N_2} \otimes \cdots \otimes t_{N_l} \rangle_{\boldsymbol{u}} = \frac{1}{2} \sum_{\alpha \in \Pi(\boldsymbol{N})} \mu(\alpha, \mathbb{1}_l) \sum_{m \in \mathbb{Z}} u_m^{Mm} \prod_j \left(\zeta_j^m + \zeta_j^{-m} \right). \quad (3.2)$$

Next, we show that the $\boldsymbol{u}$-bracket restricts to a linear isomorphism between $\mathcal{A}$ and $\mathcal{A}^\circ$. Injectivity is clear since the $\boldsymbol{u}$-bracket is an isomorphism between the vector spaces $\mathbb{Q}^{\mathscr{P}}$ and $\mathbb{Q}[\![\boldsymbol{u}]\!]$, containing $\mathcal{A}$ and $\mathcal{A}^\circ$, respectively. For surjectivity, it suffices to show that the $\boldsymbol{u}$-bracket is surjective on monomials $\prod_{j=1}^d A_{N_j}(\varrho_j, \boldsymbol{u}) = \prod_{j=1}^d \langle t_{N_j}(\varrho_j, \cdot) \rangle_{\boldsymbol{u}}$ (by Proposition 3.1). We prove the stronger statement that for $f, g \in \mathcal{A}$ there exists $h \in \mathcal{A}$ such that

$$\langle f \rangle_{\boldsymbol{u}} \langle g \rangle_{\boldsymbol{u}} = \langle h \rangle_{\boldsymbol{u}}. \quad (3.3)$$

First of all, if $f = s(\varrho_1, \cdot)$ and $g = s(\varrho_2, \cdot)$, then $\langle f \rangle_{\boldsymbol{u}} \langle g \rangle_{\boldsymbol{u}} = \langle fg \rangle_{\boldsymbol{u}} - \langle f \otimes g \rangle_{\boldsymbol{u}}$ and by (3.2) we have that $\langle f \otimes g \rangle_{\boldsymbol{u}} = \langle h \rangle_{\boldsymbol{u}}$ for some h. Finally, by [28, Proposition 3.2.7],

[2] The forms are of matrix index M, where the dimensions of M are given by the number of variables $\zeta_1, \zeta_2, \dots, \zeta_n$. Alternatively, if one restricts to $\zeta_1 = \zeta_2 = \dots = \zeta_n$, then M restricts to the usual index of Jacobi variables in one elliptic variable, as defined in Sect. 2.3.

one can reduce the general case to this case. Hence, (3.3) follows, which implies the surjectivity. □

We are now ready to prove our first two main theorems.

Proof of Theorem 1.1 As the q-bracket of elements in $\mathcal{A}$ are obtained by letting $u_j = q^j$ in (3.2), both statements follow immediately. □

Proof of Theorem 1.2 Recall the Jacobi form $F(z, w; \tau)$, see (2.10). Note this function has a pole if $z = 0$ or $w = 0$. The constant term in the expansion around $w = 0$ equals

$$\operatorname{coeff}_{[w^0]} F(z, w; \tau) = \frac{\zeta}{\zeta - 1} - \frac{1}{2} - \sum_{m,r\geq 1} \left(\zeta^m - \zeta^{-m}\right) q^{mr}$$

and for $n \in \mathbb{N}$, the coefficient of w^{n-1} of $F(z, w; \tau)$ in the expansion around $w = 0$ equals

$$\frac{(n-1)!}{(2\pi i)^{n-1}} \operatorname{coeff}_{[w^{n-1}]} F(z, w; \tau) = \frac{\zeta_1}{\zeta - 1}\delta_{n=1} + \frac{B_n}{n} - \sum_{m,r\geq 1} \left(\zeta^m + (-1)^n \zeta^{-m}\right) r^{n-1} q^{mr}.$$

By Corollary 3.5.6 in [28] the n-fold connected $\boldsymbol{u}$-bracket of s equals

$$\langle s \otimes \cdots \otimes s \rangle_{\boldsymbol{u}} = \left(\frac{\varrho}{1-\varrho} + \frac{1}{2}\right)\delta_{n,1} + \sum_{m,r\geq 1} \prod_{j=1}^{n} \left(\varrho_j^m - \varrho_j^{-m}\right) r^{n-1} u_m^r. \tag{3.4}$$

Note that this is an element of $\mathcal{B}^\circ$ if $n = 1$. Writing $\varrho_j = e^{2\pi i \mathfrak{z}_j}$, we conclude that

$$\langle s \otimes \cdots \otimes s \rangle_q = -\frac{(n-1)!}{2(2\pi i)^{n-1}} \operatorname{coeff}_{[w^{n-1}]} \sum_{\boldsymbol{\nu}\in\{\pm 1\}^n} F(\nu_1\mathfrak{z}_1 + \ldots + \nu_n\mathfrak{z}_n, w; \tau) \prod_j \nu_j.$$

Since the coefficients of a Jacobi form are quasi-Jacobi forms by [24, Subsubsection 1.3.3] and [29, Proposition 2.24] (of which Proposition 2.3 is a special case), we conclude $\langle s \otimes \cdots \otimes s \rangle_q$ is a quasi-Jacobi form. By (3.2) and Proposition 3.2 (2) for $n \in \mathbb{N}$ we have

$$\langle t_{N_1} \otimes \cdots \otimes t_{N_l} \otimes s \otimes \cdots \otimes s \rangle_q = \frac{1}{2} \sum_{\alpha \in \Pi(N)} \mu(\alpha, \mathbb{1}_l) \sum_{m \in \mathbb{Z}} \prod_{j=1}^{l} \left(\zeta_j^m + \zeta_j^{-m} \right) \prod_{j=1}^{n} (\varrho_j^m - \varrho_j^{-m}) (Mm)^n q^{Mm^2},$$

where we take n times the function s and M is as in the proof of Theorem 1.1. □

Remark We have

$$\frac{\partial}{\partial \mathfrak{z}_1} \cdots \frac{\partial}{\partial \mathfrak{z}_n} \langle s \otimes \cdots \otimes s \rangle_q = \frac{1}{2} \left(q \frac{\partial}{\partial q} \right)^{n-1} \sum_{\boldsymbol{\nu} \in \{\pm 1\}^n} E_2(\nu_1 \mathfrak{z}_1 + \ldots + \nu_n \mathfrak{z}_n; \tau).$$

Namely, by (1.3), we find that $\frac{\partial}{\partial \mathfrak{z}_1} \cdots \frac{\partial}{\partial \mathfrak{z}_n} \langle s \otimes \cdots \otimes s \rangle_q$ equals $2^n \cdot 2\pi i$ times the left-hand side of [28, Corollary 3.3.2] after substituting $z_j = 2\pi i \mathfrak{z}_j$. Similarly, we find $E_2(z; \tau) = (2\pi i)^2 P(2\pi i z)$, where the propagator P was defined in [28]. The inversion formula for connected brackets (2.3) allows to replace the sum over $\alpha \in \Pi(n)$ by $A = \{1, \ldots, n\}$. Hence, also the right-hand sides agree up to the same factor and substitution.

4 Proof of Theorem 1.4

In this section, we prove Theorem 1.4, i.e., we show that $\langle t_N \otimes s^* \rangle_q$ is a linear combination of a Jacobi theta function, a false theta function, and an Appell–Lerch function. Set

$$\Theta^*(\zeta; q) := \Theta(z; \tau) = \sum_{n \in \mathbb{Z}} \zeta^n q^{n^2}, \quad T^*(\zeta; q) := T(z; \tau) := \sum_{n \in \mathbb{Z}} \operatorname{sgn}\left(n + \frac{1}{2}\right) \zeta^n q^{n^2},$$

$$f_N(z, w; \tau) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}} \left(\operatorname{sgn}\left(n + \frac{1}{2}\right) + \operatorname{sgn}\left(m - \frac{1}{2}\right) \right) e^{2\pi i (mz + nw)} q^{Nm^2 + mn}. \quad (4.1)$$

Below we express $\langle t_N \otimes s^* \rangle_q$ as a linear combination of these functions. In the next subsections, we obtain the transformation of T as a false theta function and f_N as an Appell–Lerch sum.

4.1 The false theta function *T*

Let $\omega \in \mathbb{H}$. Define

$$\widehat{T}(z; \tau, \omega) = \sum_{n \in \mathbb{Z}} E\left(-i \sqrt{2\pi i (\omega - \tau)} \left(n + \frac{z_2}{2\tau_2} \right) \right) \zeta^n q^{n^2}.$$

Then, for $\varepsilon > 0$ and $0 \le z_2 \le 2\tau_2$, we have

$$T(z; \tau) = \lim_{t \to \infty} \widehat{T}(z; \tau, \tau + it + \varepsilon).$$

The completed function $\widehat{T}$ satisfies the following transformations:

Proposition 4.1

(1) *For $m, \ell \in \mathbb{N}$ one has*

$$\widehat{T}(z+2m\tau+\ell;\tau,\omega)=\zeta^{-m}q^{-\frac{m^2}{2}}\widehat{T}(z;\tau,\omega).$$

(2) *For $\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(4)$ one has*

$$\widehat{T}\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d},\frac{a\omega+b}{c\omega+d}\right)=\left(\frac{-1}{d}\right)\chi_{\tau,\omega}(\gamma)\,j(\gamma,\tau)e^{\frac{\pi icz^2}{2(c\tau+d)}}\widehat{T}\,(z;\tau,\omega)\,,$$

where the automorphy factor $j(\gamma,\tau)$ for $k=\frac{1}{2}$ is given by (2.9).

Proof (1) The claim follows directly from (2.17).
(2) Note that

$$\begin{aligned}T(z;\tau)&=\sum_{n\in\mathbb{Z}+\frac{1}{2}}\operatorname{sgn}(n)\zeta^{n-\frac{1}{2}}q^{\left(n-\frac{1}{2}\right)^2}=\zeta^{-\frac{1}{2}}q^{\frac{1}{4}}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\operatorname{sgn}(n)e^{2\pi in\left(z-\tau-\frac{1}{2}+\frac{1}{2}\right)}q^{n^2}\\&=\zeta^{-\frac{1}{2}}q^{\frac{1}{4}}\psi\left(z-\tau-\frac{1}{2};2\tau\right),\end{aligned}$$

where ψ is the false theta function given in (2.16). Note that,

$$\widehat{T}(z;\tau,\omega)=q^{\frac{1}{4}}\zeta^{-\frac{1}{2}}\widehat{\psi}\left(z-\tau-\frac{1}{2};2\tau,2\omega\right).$$

Hence, for $\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(2)$, we obtain from (2.18)

$$\begin{aligned}&\widehat{T}\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d},\frac{a\omega+b}{c\omega+d}\right)\\&=e^{\frac{\pi i}{2}\frac{a\tau+b}{c\tau+d}-\frac{\pi iz}{c\tau+d}}\widehat{\psi}\left(\frac{z}{c\tau+d}-\frac{a\tau+b}{c\tau+d}-\frac{1}{2};\frac{a(2\tau)+2b}{\frac{c}{2}(2\tau)+d},\frac{a(2\omega)+2b}{\frac{c}{2}(2\omega)+d}\right)\\&=e^{\frac{\pi i}{2}\frac{a\tau+b}{c\tau+d}-\frac{\pi iz}{c\tau+d}}\chi_{2\tau,2\omega}\left(\begin{smallmatrix}a & 2b\\ \frac{c}{2} & d\end{smallmatrix}\right)\nu_\eta\left(\begin{smallmatrix}a & 2b\\ \frac{c}{2} & d\end{smallmatrix}\right)^3(c\tau+d)^{\frac{1}{2}}\\&\qquad\times e^{\frac{\pi ic}{2(c\tau+d)}\left(z-(a\tau+b)-\frac{1}{2}(c\tau+d)\right)^2}\widehat{\psi}\left(z-(a\tau+b)-\frac{1}{2}(c\tau+d);2\tau,2\omega\right).\end{aligned}$$

We now write

$$\widehat{\psi}\left(z-(a\tau+b)-\frac{1}{2}(c\tau+d);2\tau,2\omega\right)$$
$$=\widehat{\psi}\left(z-\tau-\frac{1}{2}+\left(\frac{1-a}{2}-\frac{c}{4}\right)2\tau+\frac{1-d}{2}-b;2\tau,2\omega\right).$$

Assuming $4 \mid c$ and using (2.17) with $\tau \mapsto 2\tau$, $\omega \mapsto 2\omega$, $m = \frac{1-a}{2} - \frac{c}{4}$, $\ell = \frac{1-d}{2} - b$, and $z \mapsto z - \tau - \frac{1}{2}$

$$\widehat{\psi}\left(z-\tau-\frac{1}{2}+\left(\frac{1-a}{2}-\frac{c}{4}\right)2\tau+\frac{1-d}{2}-b;2\tau,2\omega\right)$$
$$=(-1)^{\frac{1-a}{2}-\frac{c}{4}+\frac{1-d}{2}+b}e^{2\pi i\left(\frac{a-1}{2}+\frac{c}{4}\right)\left(z-\tau-\frac{1}{2}\right)-2N\pi i\left(\frac{1-a}{2}-\frac{c}{4}\right)^2\tau}\widehat{\psi}\left(z-\tau-\frac{1}{2};2\tau,2\omega\right).$$

Now, a computations using $ad - bc = 1$ and $4 \mid c$ yields

$$e^{\frac{\pi i}{2}\frac{a\tau+b}{c\tau+d}-\frac{\pi iz}{c\tau+d}+\frac{\pi ic}{2(c\tau+d)}\left(z-(a\tau+b)-\frac{1}{2}(c\tau+d)\right)^2}e^{2\pi i\left(\frac{a-1}{2}+\frac{c}{4}\right)(z-\tau)-2\pi i\left(\frac{1-a}{2}-\frac{c}{4}\right)^2\tau}$$
$$=e^{\frac{\pi i\tau}{2}-\pi iz+\frac{\pi icz^2}{2(c\tau+d)}+2\pi i\left(\frac{ab}{4}+\frac{cd}{16}\right)}.$$

Also, note that

$$\chi_{2\tau,2\omega}\begin{pmatrix}a & 2b\\ \frac{c}{2} & d\end{pmatrix}=\chi_{\tau,\omega}\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

We thus have

$$\widehat{T}\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d},\frac{a\omega+b}{c\omega+d}\right)=(-1)^{\frac{1-d}{2}+b}\chi_{\tau,\omega}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\nu_\eta\begin{pmatrix}a & 2b\\ \frac{c}{2} & d\end{pmatrix}^3(c\tau+d)^{\frac{1}{2}}$$
$$\times e^{\frac{\pi i\tau}{2}-\pi iz+\frac{\pi icz^2}{2(c\tau+d)}+2\pi i\left(\frac{ab}{4}+\frac{cd}{16}\right)}\widehat{\psi}\left(z-\tau-\frac{1}{2};2\tau,2\omega\right)$$
$$=(-1)^{\frac{1-d}{2}+b}\chi_{\tau,\omega}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\nu_\eta\begin{pmatrix}a & 2b\\ \frac{c}{2} & d\end{pmatrix}^3(c\tau+d)^{\frac{1}{2}}e^{\frac{\pi icz^2}{2(c\tau+d)}+2\pi i\left(\frac{ab}{4}+\frac{cd}{16}\right)}\widehat{T}(z;\tau,\omega).$$

Using (2.8), it is not hard to finish the proof of (2). □

4.2 The Appell–Lerch function f_N

Next, we want to find modularity properties of f_N defined in (4.1). In particular, we are interested in the cases $z \in \{0, Nw\}$. Define

$$\widehat{f}_N(z,w;\tau) = f_N(z,w;\tau) + \frac{i}{2}\zeta^{-N}\sum_{k=0}^{2N-1}\xi^k\vartheta\left(z-N\tau+k\tau+N-\frac{1}{2};2N\tau\right)R\left(2Nw-z+N\tau-k\tau-N+\frac{1}{2};2N\tau\right).$$

The completed function $\widehat{f}_N$ satisfies the following transformations.

Proposition 4.2 *For $m_1, m_2, \ell_1, \ell_2 \in \mathbb{Z}$, we have*

$$\widehat{f}_N(z+m_1\tau+\ell_1, w+m_2\tau+\ell_2;\tau) = \zeta^{-m_2}\xi^{2Nm_2-m_1}q^{Nm_2^2-m_1m_2}\widehat{f}_N(z,w;\tau).$$

For $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\widehat{f}_N\left(\frac{z}{c\tau+d}, \frac{w}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)e^{\frac{2\pi ic}{c\tau+d}(-Nw^2+zw)}\widehat{f}_N(z,w;\tau).$$

Proof We write

$$f_N(z,w;\tau) = \left(\sum_{\substack{n\geq 0\\ m\geq 1}} - \sum_{\substack{n\leq -1\\ m\leq 0}}\right) e^{2\pi i(mz+nw)}q^{Nm^2+mn}.$$

Assume that $1 < |w| < |q|^{-1}$, i.e., $0 < -w_2 < \tau_2$. For $m \geq 1$, we have

$$\sum_{n\geq 0} e^{2\pi inw}q^{mn} = \frac{1}{1-e^{2\pi iw}q^m}.$$

For $m \leq 0$, we have

$$\sum_{n\leq -1} e^{2\pi inw}q^{mn} = \sum_{n\geq 1} e^{-2\pi inw}q^{-mn} = \frac{e^{-2\pi iw}q^{-m}}{1-e^{-2\pi iw}q^{-m}} = -\frac{1}{1-e^{2\pi iw}q^m}.$$

Thus we have

$$f_N(z,w;\tau) = \sum_{m\in\mathbb{Z}}\frac{e^{2\pi imz}q^{Nm^2}}{1-e^{2\pi iw}q^m} = \sum_{m\in\mathbb{Z}}\frac{q^{\frac{2Nm(m+1)}{2}}q^{-Nm}\zeta^m}{1-\xi q^m} = \xi^{-N}A_{2N}(w, z-N\tau;\tau).$$

From this the completion of f_N follows directly, i.e.,

$$\widehat{f}_N(z,w;\tau) = \xi^{-N}\widehat{A}_{2N}(w, z-N\tau;\tau).$$

Using (2.14), we obtain $\widehat{A}_{2N}(w, z-N\tau;\tau) = \xi^N\widehat{A}_{2N}(w,z;\tau)$ and thus

$$\widehat{f}_N(z,w;\tau) = \widehat{A}_{2N}(w,z;\tau).$$

Therefore the transformations follow from (2.15) and (2.14). □

The functions $\widehat{f}_N(0,z;\tau)$ and $\widehat{f}_N(Nz,z;\tau)$ transform as Jacobi forms of weight 1 and index $-N$ and 0, respectively:

Corollary 4.3 *For $m,\ell\in\mathbb{Z}$, we have*

$$\begin{aligned}\widehat{f}_N(0,z+m\tau+\ell;\tau)&=\zeta^{2Nm}q^{Nm^2}\widehat{f}_N(0,z;\tau),\\ \widehat{f}_N(N(z+m\tau+\ell),z+m\tau+\ell;\tau)&=\widehat{f}_N(Nz,z;\tau).\end{aligned}$$

For $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned}\widehat{f}_N\left(0,\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)&=(c\tau+d)e^{-\frac{2\pi icNz^2}{c\tau+d}}\widehat{f}_N(0,z;\tau),\\ \widehat{f}_N\left(\frac{Nz}{c\tau+d},\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)&=(c\tau+d)\widehat{f}_N(Nz,z;\tau).\end{aligned}$$

4.3 Proof of Theorem 1.4

Proof of Theorem 1.4 Let $N\in\mathbb{Z}$. Consider functions on $\mathbb{N}$ given by

$$D_{N,m}(r)=\delta_{r\geq Nm},\quad X_\xi(r)=\sum_{j=1}^{r}\xi^j.$$

Now, first we compute using Proposition 3.2

$$\langle t_N\otimes s^*\rangle_q=\sum_{m,r\geq 1}\left(\zeta^m+\zeta^{-m}\right)\left(\varrho^m g_{N,m,\xi}(r)-\varrho^{-m}g_{N,m,\xi^{-1}}(r)\right)q^{mr}$$

with

$$g_{N,m,\xi}(r)=\partial\left(D_{N,m}X_\xi\right)-\partial(D_{N,m})*\partial\left(X_\xi\right)=\delta_{r,Nm}X_\xi(r)+\xi^r\delta_{r>Nm}-\xi^{r-Nm}\delta_{r>Nm}.$$

Hence, we have

$$\begin{aligned}\langle t_N\otimes s^*\rangle_q&=\sum_{m\geq 1}\left(\zeta^m+\zeta^{-m}\right)\varrho^m\left(\left(\xi+\ldots+\xi^{Nm}\right)q^{Nm^2}+\sum_{r>Nm}\left(\xi^r-\xi^{r-Nm}\right)q^{mr}\right)\\&\quad-\sum_{m\geq 1}\left(\zeta^m+\zeta^{-m}\right)\varrho^{-m}\left(\left(\xi^{-1}+\ldots+\xi^{-Nm}\right)q^{Nm^2}+\sum_{r>Nm}\left(\xi^{-r}-\xi^{-r+Nm}\right)q^{mr}\right)\\&=\sum_{m\in\mathbb{Z}}\mathrm{sgn}(m)\left(\zeta^m+\zeta^{-m}\right)\varrho^m\\&\qquad\times\left(\left(\xi^{\mathrm{sgn}(m)}+\ldots+\xi^{Nm}\right)q^{Nm^2}+\sum_{\substack{s\in\mathbb{Z}\\ms>0}}\left(\xi^{s+Nm}-\xi^s\right)q^{m(Nm+s)}\right).\end{aligned}$$

Note that for $m \neq 0$ one has

$$\xi^{\operatorname{sgn}(m)} = \frac{\xi^m - \xi^{-m}}{2} + \operatorname{sgn}(m)\frac{\xi^m + \xi^{-m}}{2}.$$

Hence,

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \operatorname{sgn}(m)\left(\zeta^m + \zeta^{-m}\right)\varrho^m\left(\xi^{\operatorname{sgn}(m)} + \ldots + \xi^{Nm}\right)q^{Nm^2} \\
&= \sum_{m\in\mathbb{Z}} \left(\zeta^m + \zeta^{-m}\right)\varrho^m \frac{1-\xi^{-Nm}}{2(\xi-1)}\left(\left(\xi^{Nm+1}+1\right) + \operatorname{sgn}\left(m+\frac{1}{2}\right)\left(\xi^{Nm+1}-1\right)\right)q^{Nm^2}.
\end{aligned}$$

We conclude

$$\begin{aligned}
\langle t_N \otimes s^*\rangle_q = \frac{1}{2(\xi-1)}&\Big(\Theta^*\left(\zeta\varrho\xi^N;q\right) + \Theta^*\left(\zeta^{-1}\varrho\xi^N;q\right) \\
&- \Theta^*\left(\zeta\varrho\xi^{-N};q\right) - \Theta^*\left(\zeta^{-1}\varrho\xi^{-N};q\right) \\
&+ T^*\left(\zeta\varrho\xi^N;q^N\right) + T^*\left(\zeta^{-1}\varrho\xi^N;q^N\right) + T^*\left(\zeta\varrho\xi^{-N};q^N\right) \\
&+ T^*\left(\zeta^{-1}\varrho\xi^{-N};q^N\right) \\
&- 2T^*\left(\zeta\varrho;q^N\right) - 2T^*\left(\zeta^{-1}\varrho;q^N\right)\Big) + f_N\left(Nz,z;\tau\right) - f_N\left(0,z;\tau\right).
\end{aligned}$$

The transformation properties of T and f_N in Proposition 4.1 and Corollary 4.3 complete the proof. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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