APPORTIONMENT WITH WEIGHTED SEATS

Julian Chingoma, Ulle Endriss, Ronald de Haan  
ILLC  
University of Amsterdam  
Amsterdam, The Netherlands

Adrian Haret  
MCMP  
LMU Munich  
Munich, Germany

Jan Maly  
DBAI  
TU Wien  
Vienna, Austria

ABSTRACT

Apportionment is the task of assigning resources to entities with different entitlements in a fair manner, and specifically a manner that is as proportional as possible. The best-known application concerns the assignment of parliamentary seats to political parties based on their share in the popular vote. Here we enrich the standard model of apportionment by associating each seat with a weight that reflects the value of that seat, for example because seats come with different roles, such as chair or treasurer, that have different (objective) values. We define several apportionment methods and natural fairness requirements for this new setting, and study the extent to which our methods satisfy our requirements. Our findings show that full fairness is harder to achieve than in the standard apportionment setting. At the same time, for relaxations of those requirements we can achieve stronger results than in the more general model of weighted fair division, where the values of objects are subjective.

1 Introduction

Allocating resources in a proportional manner to entities with different entitlements, also known as apportionment, is one of the core problems of social choice (Balinski, 2005): in federal systems (e.g., the US), states receive seats in parliament according to their populations, while in proportional representation systems (e.g., the Netherlands), parties receive seats according to their share in the popular vote. Elsewhere the need for apportionment arises in the context of fair allocation (Chakraborty et al., 2021), the presentation of statistics (Balinski and Rachev, 1993) and the handling of bankruptcies (Csóka and Herings, 2018). But, in line with the paradigmatic example of apportionment, for the rest of the paper we stick to the terminology of seats being assigned to parties.

The merits of different apportionment methods are well understood, due to an elegant mathematical theory developed for the political realm (Balinski and Young, 1982; Pukelsheim, 2014). However, existing work on apportionment remains restricted by the assumption—often not met in practice—that all seats are of equal value. In this paper, we put forward an enriched model in which seats may have different weights reflecting their (objective) values.

There are numerous scenarios that fit this richer model: from the distribution of non-liquid assets in bankruptcies to beneficiaries with different entitlements, to the assignment of positions on news websites to editorial domains (such as politics, business, or sports), based on the readership’s relative levels of interest in those domains. A particularly salient illustration is offered by the way special-purpose committees are constituted in the Bundestag, Germany’s national parliament (Bundestag, 2024). These are committees with specific responsibilities (e.g., Budget or Defence), established anew in every political cycle. Usually, which party gets to nominate the head of each committee is the result of a negotiation—but when no consensus can be found, which happened in 8 out of 20 parliamentary sessions since 1949, standard apportionment methods are used. But different committees have different size and influence, so positions end up differing in value: the role of chair of the Budget Committee will be valued more highly than, say, that of chair of the Tourism Committee. As the values of positions will factor into the satisfaction of parties, a standard apportionment method (treating all positions the same) cannot possibly do justice to this scenario.

The central problem in apportionment is finding an assignment of seats to parties that is as proportional as possible, given that perfect proportionality is often not feasible (e.g., for a parliament with 100 seats, there is no perfect apportionment for three parties that each obtained exactly one third of the votes). In our enriched model we associate each
We write Apportionment with Weighted Seats

...
A vote vector \( \mathbf{v} = (v_1, \ldots, v_m) \in [n]^m \), with \( v_1 + \cdots + v_m = n \), specifies how many votes (out of the total number \( n \)) party \( p \in [m] \) garnered. Each seat \( t \in [k] \) is associated with a weight \( w_t \) indicating how valuable \( t \) is. Thus, the environment in which the election takes place can be described by a weight vector \( \mathbf{w} = (w_1, \ldots, w_k) \in \mathbb{N}_{\geq 1}^k \), listing these weights in non-increasing order. Let \( \omega = \sum_{t \in [k]} w_t \) be the total weight. A seat assignment is a vector \( \mathbf{s} = (s_1, \ldots, s_k) \in [m]^k \), where \( s_t = p \) means that party \( p \in [m] \) is assigned seat \( t \in [k] \) with weight \( w_t \). Given a seat assignment \( \mathbf{s} \), we write \( \mathbf{s}(p) = \{ t \mid s_t = p \} \) for the vector of seats, in increasing order of index, assigned to party \( p \) under seat assignment \( \mathbf{s} \). An election instance is a pair \((\mathbf{v}, \mathbf{w})\) of a vote vector \( \mathbf{v} \) and a weight vector \( \mathbf{w} \). We speak of a unit-weight instance in case \( w_t = 1 \) for all \( t, t' \in [k] \).

The core question of apportionment is how to distribute the available seats to parties in a proportional manner. This is typically formalised in terms of a party’s quota. We do the same in our weighted setting, with the important caveat that the quota in this case is construed in terms of the total weight. Hence, the quota of party \( p \) is defined as \( q(p) = \omega \cdot v_p / n \).

To judge whether a party satisfies its quota, we need to reason about the weights it accrued via its seat assignment. This leads us to the notion of representation. Formally, the representation of party \( p \) derived from seat assignment \( \mathbf{s} \) is \( r_{\mathbf{s}}(p) = \sum_{t \in \mathbf{s}(p)} w_t \), i.e., the sum of the weights of the seats assigned to \( p \) according to \( \mathbf{s} \). For a weight vector \( \mathbf{w} \), the set of all possible representation values a party can obtain from occupying up to \( h \in [k] \) seats can be computed as follows:

\[
R(\mathbf{w})_h = \left\{ \sum_{t \in T} w_t \mid T \in \mathcal{P}([k]) \mathrm{ with } |T| \leq h \right\}.
\]

We now turn to the methods used to assign seats to parties. A weighted-seat assignment method (WSAM) \( F \) takes an election instance \((\mathbf{v}, \mathbf{w})\) as input and maps it to a winning seat assignment \( F(\mathbf{v}, \mathbf{w}) \). We focus on two types of WSAMs, which generalise the most prominent methods for standard apportionment (Balinski and Young, 1982).

**Definition 1** (Divisor methods). Given an election instance \((\mathbf{v}, \mathbf{w})\) and a function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), the divisor method for \( f \) works in \( k \) rounds, as follows. In round \( t \in [k] \), seat \( t \) is assigned to the party \( p \) with the highest value for:

\[
\text{ratio}_p = \begin{cases} \frac{v_p}{f(g_p(t), w_t)} & \text{if } f(g_p(t), w_t) \neq 0 \\ \infty & \text{if } f(g_p(t), w_t) = 0, \end{cases}
\]

where \( g_p(t) \) is the sum of the weights of the seats assigned to party \( p \) in earlier rounds. If required, a tie-breaking rule is used to choose between parties with equal ratio.

Intuitively, divisor methods allocate the available seats sequentially, starting with the most valuable seat, based on the ratio between \( v_p \) and \( f(g_p(t), w_t) \). It is, of course, possible to allocate the seats in a different (fixed) order but, to anticipate results to come, starting with the most valuable seat leads to particularly nice axiomatic properties.

Only certain choices for the function \( f \) lead to reasonable divisor methods. In the unweighted apportionment setting, it is common to set \( f(g_p(t), w_t) = g_p(t) \) (Adams), \( g_p(t) + 0.5 \) (Sainte-Laguë) and \( g_p(t) + 1 \) (D’Hondt). Since our focus here is on upper and lower quota, we narrow our attention to Adams, the unique divisor method satisfying upper quota, and D’Hondt, the unique divisor method satisfying lower quota (Balinski and Young, 1975). These two rules can be generalised to our setting as follows.

**Definition 2** (Adams and D’Hondt). Adams is the divisor method defined by \( f(g_p(t), w_t) = g_p(t) \) and D’Hondt is the divisor method defined by \( f(g_p(t), w_t) = g_p(t) + w_t \).

Second on our list, the largest remainder method (LRM) assigns each party their lower quota of seats, as defined below, and then assigns the remaining seats based on the fractional remainder of each party’s quota. As we will see in Theorem 2, this version cannot work in the weighted setting. Instead, we put forward the following procedure.

**Definition 3** (Greedy Method). In each round \( t \in [k] \), the seat \( t \) with weight \( w_t \in \mathbf{w} \) is assigned to the party \( p \) for which the value \( q(p) - g_p(t) \) is maximal, with ties broken arbitrarily whenever they arise.

Though not the most obvious way of generalising LRM to the weighted setting, it can be checked that, without weights, the Greedy method reduces to LRM.

**Example 1.** Consider three parties obtaining votes \( \mathbf{v} = (60, 30, 10) \) and four seats of weights \( \mathbf{w} = (10, 6, 4, 2) \) waiting to be filled. Adams maximises the ratio \( v_p / g_p(t) \). Since \( g_p(t) = 0 \) before a party receives any seats, each party gets a seat after the first three rounds; assume tie-breaking assigns party 1, 2 and 3 seat 1, 2 and 3, respectively, for the partial assignment \( \mathbf{s} = (1, 2, 3, \ldots) \). At round \( t = 4 \), ratio_1 is maximised by party 1, with ratio_1 = 60/10 versus ratio_2 = 30/6 and ratio_3 = 10/4. The final assignment is \( \mathbf{s} = (1, 2, 3, 1) \).
Apportionment with Weighted Seats

We note that the WSAMs defined above take the parties’ previously assigned weights into account when deciding on a seat assignment.

We can now define our first proportionality property.

Throughout, we work with election instances (Balinski and Young, 1982; Pukelsheim, 2014). For our weighted-seat setting it would thus be natural to define the natural equivalents of our rules in their setting.

The axioms that follow are defined as properties of seat assignments, and we say that a WSAM $F$ satisfies property $\mathcal{P}$ if for every election instance $(v, w)$ it is the case that every seat assignment $s \in F(v, w)$ satisfies property $\mathcal{P}$. Throughout, we work with election instances $(v, w)$ with $n$ voters, $m$ parties, and $k$ seats.

3 Lower Quota

In the standard apportionment setting, a perfectly proportional allocation would give each party $p$ the share of seats that corresponds precisely to its vote share. Since there is no guarantee that this share, calculated as $k \cdot v_p/n$, is an integer, the immediate fallback is a lower quota axiom stating that each party $p$ should receive at least $\lfloor k \cdot v_p/n \rfloor$ seats (Balinski and Young, 1982; Pukelsheim, 2014). For our weighted-seat setting it would thus be natural to define the weighted lower quota as $\lfloor \omega \cdot v_p/n \rfloor$. However, the following example shows that such a quota is not guaranteed to be achievable, even in the simplest case of two parties and two seats.

**Example 2.** Consider two parties with $v = (1, 1)$ and $w = (99, 1)$, so $\lfloor v_p/n \rfloor = 50$ for both parties $p \in \{2\}$, and there is no way for both to get at least 50 in representation.

Intuitively, the problem is that there may be no combination of seats that actually gives each party its weighted lower quota. Let us then restrict the lower quota of a party to the values the party can achieve with the number of seats it deserves, i.e., $\ell^o(p) = \lfloor k \cdot v_p/n \rfloor$, for party $p$. We now use this quantity to determine the party’s obtainable lower quota of weights $\ell^o(p)$ as follows:

$$\ell^o(p) = \max \{ w \in R(w)_{\ell^o(p)} \mid w \leq q(p) \} .$$

We can now define our first proportionality property.

**Definition 4** (Obtainable Weighted-seat Lower Quota, WLQ$^o$). A seat assignment $s$ provides WLQ$^o$ if for every party $p$, it is the case that $r_s(p) \geq \ell^o(p)$.

Note that with unit weights WLQ$^o$ is equivalent to the standard lower quota. In the weighted setting, however, computing $\ell^o(p)$ for any party $p$ requires solving a SUBSET SUM problem and can hence not be done in polynomial time unless P = NP. Note, as well, that WLQ$^o$ is similar to the Extended Justified Representation (EJR) axiom in participatory budgeting: in this setting, while also computationally difficult to compute the representation deserved by a group of voters, EJR can always be satisfied (Peters et al., 2021). The same holds, now, for WLQ$^o$—in the case of two parties.

**Proposition 1.** For every election instance with two parties there exists a seat assignment that provides WLQ$^o$.

**Proof.** Consider an election instance with parties 1 and 2. By definition of the obtainable weighted lower quota, there exists a set $T$ of seats such that $\sum_{t \in T} w(t) = \ell^o(1)$. Let $s$ now be a seat assignment such that party 1 is assigned all seats in $T$ and party 2 gets all of the other seats. By definition, $r_s(1) = \ell^o(1)$ holds for party 1. For party 2 we have:

$$r_s(2) = \omega - r_s(1) = \omega - \ell^o(1) \geq \omega - \omega \cdot \frac{v_1}{n} = \omega \cdot \frac{n - v_1}{n} = \omega \cdot \frac{v_2}{n} \geq \ell^o(2).$$

Hence, WLQ$^o$ is satisfied.

Observe that computing this seat assignment takes exponential time in the general case, as we cannot compute the set of seats $T$ in polynomial time (unless P = NP). The upcoming Proposition 3 shows that this is unavoidable. However, in contrast to EJR, WLQ$^o$ is not satisfiable in general with more than two parties.

**Theorem 2.** There are election instances for which there exists no seat assignment that provides WLQ$^o$. 

Apportionment with Weighted Seats

Proof. Consider vote vector \( v = (1, 1, 1) \) for three parties, and weight vector \( w = (3, 2, 1) \). We get \( \ell^ω(p) = 2 \) for each party \( p \in [3] \). But there exists no seat assignment that provides at least a weight of 2 to all three parties. \( \square \)

While WLQ\(^ω\) cannot always be satisfied, one might still ask for a WSAM that delivers an allocation satisfying WLQ\(^ω\) on instances where this is possible. Unfortunately, the following result shows that this requirement is not computationally tractable. The proof of this result involves a reduction from the well-known NP-complete problem PARTITION.

**Proposition 3.** If there exists a polynomial-time algorithm \( α \) that finds a seat assignment \( s \) that provides WLQ\(^ω\) whenever such a seat assignment exists, then \( P = NP \). This holds even when restricted to the case where there are only two parties.

Proof. Assume such an algorithm \( α \) exists. Recall that the PARTITION problem asks, for a given multiset \( X = \{x_1, \ldots, x_k\} \) of \( k \) positive integers, whether there exists a partition of \( X \) into two subsets \( X_1 \) and \( X_2 \) such that \( \sum_{x \in X_1} x = \sum_{x \in X_2} x \). Let \( X \) be such an instance of the PARTITION problem. We create the following election instance \((v, w)\), as follows. Set the weight vector \( w = (x)_{x \in X} \) to be the non-increasing vector of the \( k \) elements in \( X \). Thus, we have \( ω = \sum_{x \in X} x \). Take two parties with \( v = (1, 1) \), so each party \( p \in [2] \) receives half of the total votes. Now let \( s \) be the seat assignment produced by \( α \) on input \((v, w)\). We claim \( X \) is a positive instance of PARTITION if and only if \( r_α(p) = ω/2 \) for every party \( p \in [2] \).

\((⇒)\) Assume that \( X \) is a positive instance of PARTITION. Thus, there exist subsets \( X_1 \) and \( X_2 \) such that \( \sum_{x \in X_1} x = \sum_{x \in X_2} x \). In particular, this means that \( \sum_{x \in X_i} x = ω/2 \), for each \( i \in [2] \). Consider the constructed election instance \((v, w)\). Each party deserves \( k/2 \) seats and has a weighted lower quota of \( ω/2 \). As \( \min(|X_1|, |X_2|) \leq k/2 \), there exists a way to receive weight \( ω/2 \) with \( k/2 \) seats. It follows that both parties have an obtainable lower quota of \( ω/2 \). Finally, WLQ\(^ω\) is satisfiable, as the seat assignment that gives all seats that correspond to an element of \( X_1 \) to party 1 and every seat that corresponds to an element of \( X_2 \) to party 2, satisfies WLQ\(^ω\). It follows that in the seat assignment produced by \( α \) each party must have representation \( ω/2 \).

\((⇐)\) Assume that \( r_α(p) = ω/2 \) for both parties \( p \). Let \( X_1 \) be the set of all elements that correspond to a seat that is allocated to party 1 and let \( X_2 \) be the set of all elements that correspond to a seat that is allocated to party 2. Then we must have \( \sum_{x \in X_1} x = ω/2 = \sum_{x \in X_2} x \), and hence \( X \) is a positive instance of PARTITION.

However, that means we can solve PARTITION in polynomial time by transforming it into the election instance \((v, w)\), running \( α \) on that instance and checking whether \( r_α(p) = ω/2 \) holds for both parties \( p \in [2] \). As PARTITION is NP-complete, this implies that \( P = NP \).

Interestingly, Proposition 3 shows that it is also computationally hard to determine whether a seat assignment satisfying WLQ\(^ω\) exists at all (unless \( P = NP \)). But under extra assumptions, we obtain a more positive result.

**Proposition 4.** For a constant number of parties and weights in \( w \) that are polynomial in the input size, finding a seat assignment \( s \) that provides WLQ\(^ω\) can be done in polynomial time, assuming such a seat assignment exists.

Proof. We now describe a dynamic programming algorithm that finds a seat assignment \( s \) that provides WLQ\(^ω\) whenever one exists. Consider an election instance with \( m \) parties.

The algorithm works as follows. For each \( i \in [k] \) (where \( i \) represents the number of seats assessed thus far), it computes \( W_i \) which is a set of tuples of the form \((W_1, \ldots, W_m)\). Here, \( W_p \) indicates the sum of seat weights of the seats assigned to party \( p \in [m] \).

Each \( W_{i+1} \) can be computed using \( W_i \) and the weight \( w_{i+1} \), by looking at every combination of some tuple in \( W_i \) and some choice of party to assign the weight-\( w_{i+1} \) seat to. Once \( W_k \) is computed, we can check every tuple in \( W_k \) and for each tuple, assess whether it satisfies WLQ\(^ω\) (which can be done in polynomial time). Specifically, this check can be done for each tuple \((W_1, \ldots, W_m)\) by assessing whether \( W_p \geq \ell^ω(p) \) for every party \( p \in [m] \). From the assumption on the weights in \( w \) and the observation that computing \( \ell^ω \) requires solving an instance of SUBSET SUM, we can apply a dynamic programming algorithm for the latter problem to compute \( \ell^ω(p) \) in polynomial time. And finally, there are at most \( ω^{2m} \) such tuples, which is polynomially many in the input size due to the assumptions on the weights in \( w \) and the number of parties being constant. Thus, this algorithm runs in polynomial time. \( \square \)

The proof of Proposition 4 makes use of a dynamic programming algorithm. The assumptions of Proposition 4 may be restrictive, but they fit the scenarios envisioned for our model, which are not likely to feature a number of parties, or weight values, exponential in the input size.
We have, as of yet, made no inroads towards our goal of finding an achievable lower quota property for the weighted setting. To do so, it is helpful to view lower quotas in the unit-weight setting from a different perspective: instead of thinking of the lower quota as the closest value to the quota that can be obtained in practice, we interpret it as guaranteeing that each party \( p \) is at most one seat away from surpassing its quota. To make this interpretation of lower quota work with weighted seats one must specify which seat, amongst those not assigned to it, a party has to additionally receive in order to surpass its quota. We parse this in three ways.

**Definition 5** (WLQ up to one seat, WLQ-1). A seat assignment \( s \) provides WLQ-1 if, for every party \( p \), there exists some seat \( t \in [k] \setminus \{ t' \in s(p) \} \) such that \( r_s(p) + w_t > q(p) \).

**Definition 6** (WLQ up to any seat, WLQ-X). A seat assignment \( s \) provides WLQ-X if, for every party \( p \), either \( r_s(p) > q(p) \) or for every seat \( t \in [k] \setminus \{ t' \in s(p) \} \), it holds that \( r_s(p) + w_t > q(p) \).

**Definition 7** (WLQ up to any seat from a sufficiently represented party, WLQ-X-r). A seat assignment \( s \) provides WLQ-X-r if, for every party \( p \), either \( r_s(p) > q(p) \) or for every seat \( t \in \{ t' \in s(p^*) \mid p^* \in [n] \setminus \{ p \}, r_s(p^*) > q(p^*) \} \), it holds that \( r_s(p) + w_t > q(p) \).

WLQ-1 states that for each party \( p \), there exists a seat it can additionally receive so as to surpass \( q(p) \); WLQ-X states that each party \( p \) would surpass \( q(p) \) if they would receive any of the additional seats. WLQ-X-r can then be seen as a weakening of WLQ-X where not all seats are considered, but only the seats that have been assigned to parties that have exceeded their representation quota. The intuition behind this requirement is that if one party receives more than their quota, then this is justified by the fact that we could give none of their seats to another party without that party exceeding their quota. Observe that all three axioms are equivalent to lower quota if restricted to unit-weight instances.

Let us first clarify the relations between WLQ\(^o\), WLQ-X and WLQ-1. Clearly, WLQ-X implies WLQ-X-r, which in turn implies WLQ-1. It turns out that WLQ\(^o\) is incomparable with WLQ-X-r (so also WLQ-X).

**Example 3.** Consider two parties, votes \( v = (1, 1) \) and weights \( w = (97, 1, 1, 1) \). For each party \( p \in \{ 2 \} \), we have \( q(p) = 50 \) and \( \ell^o(p) = 2 \). See that the seat assignment \( s = (1, 1, 2, 2) \) satisfies WLQ\(^o\) but not WLQ-X-r since party 1 is sufficiently represented and party 2 could also receive seat 2 without surpassing \( q(2) = 50 \).

In the other direction, Proposition 3 and (the upcoming) Proposition 6 give us that WLQ-X does not imply WLQ\(^o\), assuming \( P \neq NP \). Here is an explicit example that shows that WLQ-X does not imply WLQ\(^o\) in general.

**Example 4.** Consider three parties, a vote vector \( v = (3, 2, 1) \) and a weight vector \( w = (3, 2, 1) \), and (so \( \omega = 6 \)). For parties 1 and 2, see that \( q(1) = \ell^o(1) = 3 \) and \( q(2) = \ell^o(2) = 2 \) hold, respectively, while for party 3, we have \( q(3) = 1 \) and \( \ell^o(3) = 0 \). Now, take a seat assignment \( s = (3, 2, 1, 1) \). This clearly does not satisfy WLQ\(^o\) with \( r_s(1) = 1 < \ell^o(1) = 3 \) but note that the addition of any of the two seats that party 1 did not get assigned, would suffice in helping it cross \( q(1) = 3 \), so WLQ-X is satisfied.

We follow up by investigating whether WLQ\(^o\) implies WLQ-1.

**Proposition 5.** WLQ\(^o\) implies WLQ-1.

**Proof.** Consider some party \( p \) with \( \ell^o(p) = [k \cdot r_s/n] \). Now, suppose we iterate through the seats in non-increasing order of weight and assign \( h \) seats to party \( p \) such that \( h \leq \ell^o(p) \) and \( \sum_{t \in [h]} w_t \leq q(p) < \sum_{t \in [h+1]} w_t \). We know that \( \sum_{t \in [h]} w_t \leq q(p) \) and that it is obtainable with at most \( \ell^o(p) \) seats, i.e., \( \sum_{t \in [h]} w_t \in R(w)|_{v(e(p))} \). Since \( \ell^o(p) \) is the maximal value amongst such weights, we get \( \ell^o(p) \geq \sum_{t \in [h]} w_t \). Now assume a seat assignment \( s \) provides WLQ\(^o\), so we have \( r_s(p) \geq \ell^o(p) \) for party \( p \). Assume that \( r_s(p) < q(p) \) and consider the seats assigned to party \( p \) in \( s \). If they are exactly the same \( h \) seats as selected above, then seat \( h+1 \) works so that \( r_s(p) + w_{h+1} = \ell^o(p) + w_{h+1} \geq \sum_{t \in [h]} w_t + w_{h+1} > q(p) \). Otherwise, party \( p \) did not receive one of those \( h \) seats. Now, since we have \( w_t \geq w_{h+1} \) for all those seats \( t \in [h] \), then \( r_s(p) + w_t > q(p) \) holds for any seat \( t \in [h] \). Thus, WLQ-1 is satisfied.

We now question whether these axioms are easier to satisfy. First, for the two-party cases, not only can WLQ-X always be provided, but it is possible to do so efficiently.

**Proposition 6.** For two parties, a seat assignment providing WLQ-X always exists and can be found in polynomial time.

**Proof.** Recall that the weight vector \( w = (w_1, \ldots, w_k) \) is non-increasing (so \( w_k \) is minimal). We devise a method to find a seat assignment \( s \) that provides WLQ-X: let \( t \in [k] \) be the minimal value such that \( \sum_{i=1}^k w_i > q(1) \), and assign seats \( t + 1 \) to \( k \) to party 1 (so \( r_s(1) \leq q(1) \)). Then assign the remaining seats to party 2 to obtain seat assignment \( s \). Observe that amongst the seats that party 1 was not assigned, seat \( t \) has the lowest weight \( w_t \). Moreover, we
Apportionment with Weighted Seats

have \( r_s(1) + w_t > q(1) \). Hence, WLQ-X is satisfied with respect to party 1. Now, let us assess for party 2. Since \( r_s(1) \leq q(1) \) holds by the choice of \( t \) and we know \( q(1) + q(2) = r_s(1) + r_s(2) = \omega \) (as \( s \) is complete), it must be the case that \( r_s(2) \geq q(2) \). Hence, WLQ-X is also satisfied with respect to party 2.

In other words, for two parties, we find that WLQ-X is easier to satisfy than WLQ\(^o\). Unfortunately, this does not extend to the case of more than two parties.

**Proposition 7.** For election instances with three or more parties, a seat assignment providing WLQ-X may not exist.

**Proof.** Consider three parties, votes \( v = (1, 1, 1) \), weights \( w = (63, 30, 3, 1, 1, 1) \), and thus \( \omega = 99 \). For each party \( p \in [3] \), we have \( q(p) = 33 \). To achieve WLQ-X, the seats of weight 63 and 30 must be assigned to different parties, say parties 1 and 2. Now, if party 3 is not assigned all of the remaining four seats, then it cannot reach 33 in representation with the addition of any one of these four seats. But, if party 3 is assigned all four of these seats, then party 2 cannot reach 33 in representation by receiving any of the weight-1 seats. Thus, there is no way to provide WLQ-X.

Observe that, in this example, \( q(p) = \ell^o(p) \) for all parties \( p \). Hence, we cannot even guarantee that each party is only seat away from their obtainable lower quota of weights.

While determining how difficult it is to provide WLQ-X is left for future work, a minor adjustment to the dynamic programming algorithm of Proposition 4 delivers the following.

**Proposition 8.** Given a constant number of parties and the weights in \( w \) being polynomial in the input size, finding a seat assignment \( s \) that provides WLQ-X can be done in polynomial time, assuming such a seat assignment exists.

**Proof.** Consider the dynamic programming algorithm from Proposition 4. If we alter the algorithm to also keep track of the smallest seat weight \( t_x \) that is assigned to each party \( p \) (alongside the sum of seat weights \( W_h \) assigned to party \( p \)) within the tuples in \( W_i \), then we can use the modified, final set of tuples \( W_h \) to check in polynomial time if WLQ-X is satisfied.

Off the back of the mostly negative results regarding WLQ\(^o\) and WLQ-X, we take aim at the weaker requirement of WLQ-X-r, and find a positive result using the Greedy method.

**Theorem 9.** The Greedy method satisfies WLQ-X-r.

**Proof.** Assume that WLQ-X-r is violated by some seat assignment \( s \) returned by the Greedy method. Let party \( p_x \) be the party that witnesses it, i.e., it holds that \( r_s(p_x) < r_s(p_x) + w_t \leq q(p_x) \) for some seat \( t \in \{ t' \in s(p') \mid p' \in [n] \setminus \{ p \}, r_s(p') > q(p') \} \). As party \( p_x \) has less than \( q(p_x) \) in representation, there must be a party \( p_y \) where \( r_s(p_y) > q(p_y) \). Let \( h \) be the round after which party \( p_y \) has more than \( q(p_y) \) in representation (so party \( p_y \) was assigned seat \( h \)). By choice of round \( h \), we have \( g_{p_y}(h) + w_t > q(p_y) \) and hence, it holds that \( w_t > q(p_y) - g_{p_y}(h) \). So we have that \( w_t > q(p_y) - g_{p_y}(h) \). Since party \( p_x \) was not assigned seat \( h \), we know that \( w_t > q(p_y) - g_{p_y}(h) \). Thus, it then follows that \( q(p_x) < g_{p_x}(h) + w_t \). So this seat \( h \) is enough for party \( p_x \) to reach their quota with the same holding for all seats assigned to party \( p_y \) in prior rounds (as seats are assigned in non-increasing order).

Recall that in standard apportionment, LRM is known to satisfy LQ (Balinski and Young, 1982). Theorem 9 further justifies the Greedy method as a weighted proxy of LRM. Recall, also, that in the standard setting D’Hondt satisfies LQ as well; we find that D’Hondt\(_w\), now, satisfies WLQ-X-r.

**Theorem 10.** The D’Hondt\(_w\) method satisfies WLQ-X-r.

**Proof.** For a seat assignment \( s \) returned by D’Hondt\(_w\), for the sake of contradiction, assume that there is a party \( p_x \) such that there exists some \( t \in \{ t' \in s(p') \mid p' \in [n] \setminus \{ p \}, r_s(p') > q(p') \} \) such that \( r_s(p_x) < r_s(p_x) + w_t \leq q(p_x) \).

Thus, we know that \( v_{p_x}/(g_{p_x}(k)+w_t) \geq v_{p_x}/q(p_x) \) for some \( t \in \{ t' \in s(p') \mid p' \in [n] \setminus \{ p \}, r_s(p') > q(p') \} \), where \( g_{p_x}(k) \) is the total weight assigned to party \( p_x \) at D’Hondt\(_w\)’s conclusion. This gives us the following:

\[
\frac{v_{p_x}}{g_{p_x}(k) + w_t} \leq \frac{v_{p_x}}{\omega \cdot v_{p_x}/n} = \frac{n}{\omega}
\]

(1)

During D’Hondt\(_w\), there must be some round \( h \) where some party \( p_y \neq p_x \in [m] \) is assigned weight \( w_y \) such that \( n/\omega > v_{p_y}/(g_{p_y}(k)+w_t) \). Assume otherwise, and that for every party \( p \in [m] \setminus \{ p_x \} \), it holds that \( v_{p_y}/g_{p_y}(k) \geq n/\omega \) after D’Hondt\(_w\)’s \( k \) rounds. Then we have that \( \omega \cdot v_{p_y}/n \geq g_{p_y}(k) \) for all \( p \in [m] \setminus \{ p_x \} \). Summing over all parties with
Apportionment with Weighted Seats

ω · v_p/n > g_p_s(k) for party p_s, we get \(\sum_{p\in[m]} \omega \cdot v_p/n = \omega > g_p_s(k) + \sum_{p\in[m]\setminus\{p_s\}} g_p(k)\) which means D’Hondtω did not assign all of the weight, contradicting its definition. So, there must exist some round \(h\) where for some party \(p_y\), we have:

\[
\frac{n}{\omega} > \frac{v_{p_y}}{g_p_s(h) + w_h}
\]

Since weight \(w_h\) was assigned to party \(p_y\) in round \(h\), and not party \(p_x\), then we have that \(v_{p_y}/(g_p_s(h)+w_h) \geq v_{p_x}/(g_p_s(h)+w_h)\) where \(h \in \{t' \in s(p^*) \mid p^* \in \{n\} \setminus \{p\}; r_s(p^*) > q(p^*)\}\). And also considering the fact that \(g_p_s(h) \leq g_p_s(k)\), it follows that:

\[
\frac{v_{p_y}}{g_p_s(h) + w_h} \geq \frac{v_{p_x}}{g_p_s(h) + w_h} \geq \frac{v_{p_x}}{g_p_s(k) + w_h}
\]

Putting equations (1), (2) and (3) together, it follows that \(n/\omega > v_{p_y}/(g_p_s(h)+w_h) \geq n/\omega\). This is a contradiction, so no such party \(p_x\) can exist. Note that we considered a seat weight \(w_h\) assigned to some party \(p_y\) in round \(h\), such that \(p_y\) surpasses its quota. And such a weight \(w_h\) is sufficient in aiding party \(p_x\) in reaching \(q(p_x)\). This holds for all seats assigned to party \(p_y\) before round \(h\) (as such seats \(h^*\) have weight \(w_{h^*} \geq w_h\)), and also those seats assigned to party \(p_y\) after round \(h\) (as such seats \(h^*\) are only assigned to party \(p_y\), and not some party \(p_x\) below its quota \(q(p_x)\) in that round, if the weight \(w_{h^*}\) would lead to party \(p_x\) reaching said quota). □

This improves on a result of Chakraborty et al. (2021) stating that D’Hondt satisfies an axiom called WPROP1 (see Theorem 4.9 in (Chakraborty et al., 2021)) 1 which is weaker than WLQ-X-r. The importance of our findings is strengthened by observing that the “up to any” properties are, in many scenarios, much stronger than the equivalent “up to one” properties, in particular if the values of objects vary a lot. Consider the application of our WSAMs to the allocation of non-liquid assets in a bankruptcy. We may use the approximate monetary value of the assets as a weight. In this case, we might have a few very valuable assets (e.g., a house or other property), and other assets of much lower values (e.g., furniture). In such a case “up to one” properties can become essentially meaningless, while “up to any” properties are still meaningful. This scenario shall be illustrated with the upcoming Example 5.

Cruelly, our stronger result does not only stem from our restricted setting but also from our use of a version of D’Hondt that takes weights into account as the standard, unweighted D’Hondt used by Chakraborty et al. (2021) does not satisfy WLQ-X-r in our setting. Here is an example showing this.

**Example 5** (Seat assignment induced by the standard D’Hondt method fails WLQ-X-r). Consider two parties with votes \(v = (10, 2)\), and a weight vector \(w = (10, 1, 1)\). Then, standard D’Hondt method assigns all three seats to party \(p_1\) as, in the D’Hondt method’s three respective rounds, party \(1\) has the ratios 10, 5, and 2.5 versus party \(2\)’s ratio of 2 in all three rounds. Thus, party \(2\) has representation of 0 from the resulting seat assignment and none of weight-1 seats are enough to add so that party 2 exceeds its quota of \(q(2) = 2\). However, observe that the seat assignment determined by standard D’Hondt provides WLQ-1 while our WSAM D’Hondt returns the seat assignment \(s = (1, 2, 2)\) for this same election instance and this seat assignment \(s\) not only provides WLQ-X-r, but it is a much fairer outcome from an intuitive standpoint. △

This further illustrates the importance of our focus on the weighted apportionment model as we need to use the specific properties of our model to define new voting rules, namely the WSAMs, that can satisfy the stronger “up to any” properties.

Now, regarding the other divisor method, Adams_{ω} fails WLQ-1 (and thus WLQ-X-r), as it is known to violate LQ, which, as mentioned above, is equivalent to WLQ-1 in the unit-weight case (Balinski and Young, 1982).

**Example 6** (Adams_{ω} fails WLQ-1). Consider four parties, \(v = (9, 1, 1, 1)\), and \(w = (1, 1, 1, 1)\). Then, the weighted quota of party \(1\) is \(q(1) = 4 \cdot 9/12 = 3\). However, Adams_{ω} gives each party \(p \in [4]\) an initial ratio of \(ratio_p = \infty\), hence each party receives exactly seat. Consequently, party \(1\) is two seats away from its weighted lower quota. △

### 4 Upper Quota

Parties should get at least as many seats as they deserve (lower quota), but also not more than appropriate: the latter bound is captured by an upper quota (UQ) property. In standard apportionment, UQ states that a party \(p\) amassing \(v_p\) of the \(n\) votes should receive at most \(\lfloor k \cdot v_p/n \rfloor\) of the \(k\) seats (Balinski and Young, 1982). As with lower quota, there is no hope of satisfying the naïve weighted upper quota notion defined by \(\lfloor \omega \cdot v_p/n \rfloor\). We can then define an obtainable

---

1WPROP1 is similar to WLQ-1, but WPROP1 is defined with a weak inequality in the condition on the existence of some seat of sufficient weight.
Apportionment with WeightedSeats

upper quota as for the obtainable weighted lower quota \( \ell^o(p) \). Specifically, we incorporate ‘upper quota of seats’ in defining a weighted upper quota for party \( p \) that is obtainable. So for a party \( p \), we may define its weighted upper quota to be the following where \( u^\#(p) = \lceil k \cdot \frac{v_p}{n} \rceil \):

\[
u_u^*(p) = \max \{ r \in R(w)_{u^\#(p)} \mid r \leq \min \{ w \in R(w)[k] \mid w \geq \lceil \omega \cdot \frac{v_p}{n} \rceil \} \}.
\]

Note that this definition is more involved than, and not exactly symmetric to, that of \( \ell^o(p) \). To see why, observe that we cannot fix \( u_u^*(p) \) to be \( \min \{ r' \in R(w)_{u^\#(p)} : r' \geq \lceil \omega \cdot \frac{v_p}{n} \rceil \} \) as such a value may not exist (as all values in \( R(w)_{u^\#(p)} \) may be less than \( \lceil \omega \cdot \frac{v_p}{n} \rceil \)). Thus, we define \( u_u^*(p) \) to be the largest value in \( R(w)_{u^\#(p)} \) that does not exceed an upper bound of \( \min \{ w \in R(w)[k] \mid w \geq \lceil \omega \cdot \frac{v_p}{n} \rceil \} \) which always exists.

This seems a natural counterpart to the obtainable weighted lower quota \( \ell^o(p) \), but we can show that there exist election instances where the sum of obtainable weighted upper quotas is less than the total weight, and therefore, any reasonable upper quota axiom based on \( u_u^*(p) \) will be violated.

Example 7. Take, five parties with \( v = (40, 15, 15, 15, 15) \) and \( w = (5, 1, 1, 1, 1, 1) \). Then party 1 has an upper quota of seats of 2 seats but and an upper quota of weight of less than 5. All other parties have an upper quota of seats of only 1. Thus, the obtainable upper quotas taking the ‘upper quota of seats’ into account would be 2 for party 1 and 1 for the other parties, which sums to \( 7 < 10 = \omega \). \( \triangle \)

So in contrast to the obtainable lower quota, we do not incorporate an ‘upper quota of seats’ in defining a weighted upper quota for party \( p \) that is obtainable. We thus define the obtainable weighted upper quota as follows:

\[
u_u^o(p) = \min \{ r \in R(w)[k] \mid r \geq q(p) \}.
\]

Now we can define the axiom WUQ\(^o\) based on \( u^o \).

Definition 8 (Obtainable Weighted-seat Upper Quota, WUQ\(^o\)). A seat assignment \( s \) provides WUQ\(^o\) if for every party \( p \) it is the case that \( r_u^s(p) \leq u_u^o(p) \).

Note that for unit-weight election instances WUQ\(^o\) reduces to the standard UQ property. For this definition of the obtainable upper quota, we achieve essentially the same results as for WLQ\(^o\). The following four results can be seen as direct counterparts to those shown for WLQ\(^o\) and they are obtained using similar arguments as for the WLQ\(^o\) results. First, we consider the whether WUQ\(^o\) seat assignments always exist in the two-party case.

Proposition 11. For every election instance with two parties there exists a seat assignment that provides WUQ\(^o\).

Proof. Consider an election instance with parties 1 and 2. By the definition of the obtainable weighted upper quota, there exists a set of seats \( T \) such that \( \sum_{t \in T} w(t) = u^o(1) \). Now, let \( s \) be a seat assignment such that party 1 gets all seats in \( T \), and party 2 all other seats. By definition, \( r_u^s(1) = u^o(1) \) for party 1. Moreover, for party 2, we have:

\[
r_u^s(2) = \omega - r_u^s(1) = \omega - u^o(1) \leq \omega - \omega \cdot \frac{v_1}{n} = \omega \cdot \frac{n - v_1}{n} = \omega \cdot \frac{v_2}{n} \leq u^o(2).
\]

Hence, WUQ\(^o\) is satisfied. \( \square \)

Next, we move beyond two parties and find a similarly negative result as we found WUQ\(^o\).

Proposition 12. There are election instances where no complete seat assignment provides WUQ\(^o\).

Proof. We use the election instance familiar from the proof of Theorem 2, where there are three parties, a weight vector \( w = (3, 2, 1) \), and a vote vector \( v = (1, 1, 1) \). Then each party \( p \in [3] \) has an obtainable weighted upper quota of \( u^o(p) = 2 \). However, providing each party with at most 2 in representation cannot be achieved if we need to assign all three seats. \( \square \)

Proposition 13. If there exists a polynomial-time algorithm \( \alpha \) that finds a seat assignment \( s \) that provides WUQ\(^o\), whenever such a seat assignment exists, then \( P = NP \).

Proof. Consider the reduction from the PARTITION problem in Proposition 3. Observe that in the election instance constructed in this reduction, the obtainable weighted lower quota \( \ell^o(p) \) for each party \( p \in [2] \) is exactly equal to the obtainable weighted upper quota \( u^o(p) \). So the same arguments provided for WLQ\(^o\) work for WUQ\(^o\).

Proposition 14. Given a constant number of parties and the weights in \( w \) being polynomial in the input size, finding a seat assignment \( s \) that provides WUQ\(^o\) can be done in polynomial time, assuming such a seat assignment exists. \( \square \)
Proof. Consider the dynamic programming algorithm from Proposition 4. Observe that the same algorithm can be deployed to find WUQ\(^o\) - providing seat assignments with the following, simple modification: once \(W_k\) is computed, the algorithm checks each tuple \((W_1, \ldots, W_m)\) to determine whether, for every \(p \in [m]\), it holds that \(W_p \leq \ell^o(p)\) instead of whether \(W_p \geq \ell^o(p)\). And we can compute \(u^o(p)\) in polynomial time due to (i) the assumption on the weights in \(w\), and (ii) observing this task is also equivalent to solving the SUBSET SUM problem.

We now move to the up-to-one/any relaxations that allowed us to define satisfiable lower-quota axioms.

**Definition 9** (WUQ up to one seat, WUQ-1). A seat assignment \(s\) provides WUQ-1 if, for every party \(p\), either \(r_s(p) \leq q(p)\) or there exists some seat \(t \in s(p)\) such that \(r_s(p) - w_t < q(p)\).

**Definition 10** (WUQ-X to any seat, WUQ-X). A seat assignment \(s\) provides WUQ-X if, for every party \(p\), either \(r_s(p) \leq q(p)\) or for every seat \(t \in s(p)\), it holds that \(r_s(p) - w_t < q(p)\).

WUQ-X states that, for every party \(p\), disregarding any seat it received would take it below \(q(p)\), while for WUQ-1 there need only exist one such seat assigned to party \(p\) to take it below \(q(p)\). With unit weights, both axioms reduce to UQ. Observe that there is no natural way of defining a counterpart to WLQ-X-\(r\), as the items we remove from each party must be assigned to them.

WUQ-X clearly implies WUQ\(^o\) but how do these two axioms relate to WUQ\(^o\)? Here, we see the first difference between upper and lower quota as WUQ\(^o\) does not only imply WUQ-1, but even WUQ-X. Here we uncover a difference between upper and lower quota: WUQ\(^o\) implies not only WUQ-1, but WUQ-X as well.

**Proposition 15.** WUQ\(^o\) implies WUQ-X.

**Proof.** Assume WUQ-X is violated by a seat assignment \(s\). Then there exists a party \(p\) such that for every seat \(t \in s(p)\), it holds that \(r_s(p) - w_t \geq q(p)\). But then this means that \(r_s(p) - w_t\) is an achievable weight representation value at least as large as \(q(p)\), i.e., \(r_s(p) - w_t \in \{r \in R(w) : r \geq q(p)\}\). Since \(u^o(p)\) is the smallest of the weights in \(\{r \in R(w) : r \geq q(p)\}\), we get \(r_s(p) > r_s(p) - w_t \geq u^o(p)\) and hence, WUQ\(^o\) is also violated.

Here is an example that shows how WUQ-X does not imply WUQ\(^o\).

**Example 8.** Consider three parties, a vote vector \(v = (3, 2, 1)\), and a weight vector \(w = (3, 2, 1)\), and (so \(\omega = 6\)). See that \(q(1) = u^o(1) = 3, q(2) = u^o(2) = 2\) and \(q(3) = u^o(3) = 1\) for the parties. Now, take a seat assignment \(s = (3, 2, 1)\). This does not satisfy WUQ\(^o\) as we get that \(r_s(3) = 3 > u^o(3) = 1\) but note that the removal of the seat that party 3 was assigned, would suffice in helping it go below \(q(3) = 1\).

Next, we ask whether upper quota axioms can be satisfied. A natural candidate is Adams\(_w\), as it is known to satisfy UQ for unit weights (Balinski and Young, 1982). Indeed, Adams\(_w\) even satisfies the stronger notion of WUQ-X, in stark contrast to WLQ-X, which is not satisfiable in general.

**Theorem 16.** Adams\(_w\) satisfies WUQ-X.

**Proof.** Assume there exists a party \(p_x\) that receives more representation than \(q(p_x)\), otherwise, WUQ-X is satisfied by definition. Let \(t\) be the round of Adams\(_w\) such that party \(p_x\) was assigned seat \(t\) and after which \(g_{p_x}(t) > q(p_x)\), and thus, we have that:

\[
g_{p_x}(t) - w_t < q(p_x) = \omega \cdot \frac{v_{p_x}}{n}
\]

We now show that party \(p_x\) does not receive anymore seats. Consider some round \(t^* > t\) where seat \(t^*\) is to be assigned. We know that \(\sum_{p \in [m]} q(p) = \omega\) holds alongside the following:

\[
g_{p_x}(t^*) > g_{p_x}(t) > \omega \cdot \frac{v_{p_x}}{n}.
\]

As at least one seat, namely seat \(t^*\), has not been assigned yet, there must be a party \(p_y\) such that \(g_{p_y}(t^*) < q(p_y) = \omega \cdot \frac{v_{p_y}}{n}\). And it follows that:

\[
\frac{v_{p_x}}{g_{p_x}(t^*)} < \frac{v_{p_x}}{\omega \cdot v_{p_x}/n} = \frac{n}{\omega} = \frac{v_{p_x}}{\omega \cdot v_{p_y}/n} < \frac{v_{p_x}}{g_{p_y}(t^*)}.
\]

Hence, party \(p_y\) has a strictly better ratio than party \(p_x\) so Adams\(_w\) does assign seat \(t^*\) to the latter party. So, we know that removing seat \(t\) would be enough for party \(p_x\) to fall below their weighted quota \(q(p_x)\), and thus, up to this point, we have proven that Adams\(_w\) satisfies WUQ-X.

10
To show that Adamss satisfies WUQ-X, we need the following additional argument: for all the seats \( j < t \) that \( \text{Adamss} \) assigned to party \( p_x \) prior to it being assigned seat \( t \), we have that \( w_j \geq w_t \) and hence, \( g_{p_x}(t) - w_j \leq g_{p_x}(t) - w_t < q(p_x) \). So removing any one of these seats will suffice to ensure that party \( p_x \) does not exceed \( q(p_x) \).

This result represents another “up to any” improvement on prior work (Chakraborty et al., 2021). We now expand our focus to the following envy-freeness axioms (envy-freeness is a well-known fairness notion in the fair division literature).

**Definition 11** (Weighted envy-freeness up to any seat, WEFX). A seat assignment \( s \) provides WEFX if for any two parties \( p_x, p_y \), it holds for every seat \( t \in s(p_y) \) that:

\[
\frac{r_s(p_x)}{v_{p_x}} \geq \frac{(r_s(p_y) - w_t)}{v_{p_y}}.
\]

We can also define the following weakening of WEFX.

**Definition 12** (Weighted envy-freeness up to one seat, WEF1). A seat assignment \( s \) provides WEF1 if for any two parties \( p_x, p_y \), there exists some seat \( t \in s(p_y) \) such that:

\[
\frac{r_s(p_x)}{v_{p_x}} \geq \frac{r_s(p_y) - w_t}{v_{p_y}}.
\]

In our setting, both WEFX and WEF1 ensure that no party prefers the representation afforded to another party. Conceptually, this is similar to the upper-quota notion that states that no party is represented more than it truly deserves. To provide a formal connection between envy-freeness and upper quota, we prove that WUQ-X follows from WEFX.

**Proposition 17.** WEFX implies WUQ-X.

*Proof.* To see that WEFX implies WUQ-X, observe that if WUQ-X is violated, then there exists a party \( p_x \) such that \( (r_s(p_x) - w_t) \geq q(p_x) = \omega \cdot v_{p_x}/n \) for some \( t \in s(p_x) \). On the other hand, for WEFX to hold, for every party \( p_y \in \{p_x\} \) and every \( t \in s(p_x) \) it must be the case that \( r_s(p_y)/v_{p_y} \geq (r_s(p_x) - w_t)/v_{p_x} \). These inequalities imply that \( r_s(p) \geq \omega \cdot v_{p}/n = q(p) \) for every party \( p \in \{p_x\} \), which is not possible if party \( p_x \) exceeded its quota \( q(p_x) \).

On the other hand, it is straightforward to see that WEF1 does not imply WUQ-X.

**Example 9.** Consider two parties, a vote vector \( v = (1, 1) \), and a weight vector \( w = (11, 2, 1) \), and (so \( \omega = 14 \)). See that \( q(1) = q(2) = 7 \). Now, take a seat assignment \( s = (1, 2, 1) \). This does not satisfy WUQ-X as we get that \( r_s(1) - w_3 = 12 - 1 > 7 \) but note that the removal of seat 1 from party 1 would remove any envy from party 2. \( \triangle \)

We also note that the weakening of WEF1 known as WWEF1 (Chakraborty et al., 2021) does not imply WUQ-1. First, we define WWEF1.

**Definition 13** (Weak weighted envy-freeness up to one seat, WWEF1). A seat assignment \( s \) provides WWEF1 if for any two parties \( p_x, p_y \), there exists some seat \( t \in s(p_y) \) such that at least one of inequalities 4 and 5, given below, hold:

\[
\frac{r_s(p_x)}{v_{p_x}} \geq \frac{r_s(p_y) - w_t}{v_{p_y}}, \quad (4)
\]

\[
\frac{r_s(p_x) + w_t}{v_{p_x}} \geq \frac{r_s(p_y)}{v_{p_y}}. \quad (5)
\]

And now, here is an example illustrating that WWEF1 does not imply WUQ-1.

**Example 10** (WWEF1 does not imply WUQ-1). Consider 101 parties, \( v = (100, 1, \ldots, 1) \), and \( w = (1, 1, 1, 1) \). Note that \( q(1) = 2 \) and \( q(p) = 0.02 \) for all \( p \in \{2, \ldots, 101\} \).

We will now show that the seat assignment \( s = (1, 1, 1, 1) \) fails WUQ-1 while satisfying WWEF1. Observe that it satisfies WWEF1 since party 1 exceeded its quota and the second condition of WWEF1 is met as for each party \( p \in \{2, \ldots, 101\} \), we can take a weight-1 seat from party 1 such that \( r_s(p) + 1/v_p = 1 > 0.04 = 4/100 = r_s(1)/v_1 \) holds, and we have that \( r_s(p)/v_p = 0 = r_s(p)/v_p \) for each \( p_x, p_y \in \{2, \ldots, 101\} \).

The fact that WEF1 implies WUQ-1 follows from similar reasoning to that showing that WEFX implies WUQ-X. Now, we show that the Adams method satisfies WEFX.
Apportionment with Weighted Seats

**Theorem 18.** The Adams method satisfies WEFX.

**Proof.** Suppose there are two parties \( p_x, p_y \) with \( r_x(p_x)/v_{p_x} < r_y(p_y)/v_{p_y} \), i.e., party \( p_y \) envies party \( p_x \). Now, consider the last seat \( t \) that was assigned to party \( p_x \) by Adams, in round \( h \). Since this seat was assigned to party \( p_x \), we have that \( v_{p_x}/g_{p_x}(h) > v_{p_y}/g_{p_y}(h) \). And as seat \( t \) was the last seat assigned to party \( p_x \), we get:

\[
\frac{r_x(p_y)}{v_{p_y}} > \frac{g_{p_x}(h)}{v_{p_y}} > \frac{g_{p_y}(h)}{v_{p_x}} = \frac{r_x(p_x) - w_t}{v_{p_x}}.
\]

So, removing seat \( t \) from party \( p_x \) leads to party \( p_y \) no longer envying party \( p_x \), and since all seats assigned to party \( p_x \) prior to seat \( t \) have weight at least as large as seat \( t \), removing any of these seats is sufficient to remove party \( p_y \)'s envy.

This result improves on that of Chakraborty et al. (2021) showing that the standard Adams method can be used to achieve WEF1. Indeed, finding a rule that satisfies WEFX in the setting of Chakraborty et al. (2021) would imply the existence of EFX allocations in the standard fair division setting, which is considered one of the major open questions in fair division (Amanatidis et al., 2022).

Since Adams does not satisfy WLQ-1, can envy-freeness and lower quota be satisfied at the same time? This is not possible, as WEF1 and WLQ-1 are incompatible.

**Proposition 19.** WEF1 and WLQ-1 are incompatible.

**Proof.** Consider three parties with votes \( v = (8, 2, 1) \) and weights \( w = (8, 7, 6) \). Then we have the quotas \( q(1) \approx 15.2 \) for party 1, \( q(2) \approx 3.8 \) for party 2, and \( q(3) \approx 1.9 \) for party 3. To satisfy WLQ-1, two of the seats must be assigned to party 1. However, this leads to one of parties 2 and 3 having no seats: the party without a seat will envy party 1 and be a witness to a violation of WEF1.

Can we at least satisfy upper- and lower-quota axioms at the same time? D'Hondt does not satisfy UQ in the unit-weight case so it cannot satisfy WUQ-1. The example that follows shows this well-known fact that D'Hondt violates UQ even in the unit-weight setting by an arbitrary number of seats Balinski and Young (1982). Note that the following example uses the same election instance that we used to show that WWEF1 does not imply WUQ-1 (see Example 10).

**Example 11** (D'Hondt fails WUQ-1). Consider 101 parties, \( v = (100, 1, \ldots, 1) \), and \( w = (1, 1, 1, 1, 1) \). Note that \( q(1) = 2 \) and \( q(p) = 0.02 \) for all \( p \in \{2, \ldots, 101\} \). D'Hondt assigns all seats to party 1 which violates WUQ-1. Δ

The Greedy method however, is a contender due to its connection to LRM, which satisfies UQ (Balinski and Young, 1982). As it satisfies WLQ-1 it cannot satisfy WEF1, but it does satisfy WUQ-X.

**Theorem 20.** The Greedy method satisfies WUQ-X.

**Proof.** Take a seat assignment \( s \) constructed by the Greedy method. Assume there is a party \( p_x \) that received more representation than \( q(p_x) \), i.e., \( r_x(p_x) = g_{p_x}(k) > q(p_x) \), otherwise, WUQ-X is satisfied. Let \( t \) be the round after which \( g_{p_x}(t) > q(p_x) \) holds, and so \( g_{p_x}(t) - w_t < q(p_x) \) also holds. We argue that party \( p_x \) does not get assigned any seat \( t^* < t \). Observe that in every round \( t' \in [k] \), there always exists a party \( p_y \) such that \( q(p_y) - g_{p_y}(t') \geq 0 \) as we have \( \sum_{p \in [n]} q(p) = \omega \). It then follows, for every round \( t' > t \), that \( q(p_y) - g_{p_y}(t') \geq q(p_x) - g_{p_x}(t^*) \) and hence, party \( p_x \) cannot be assigned seat \( t^* \). Thus, the Greedy method satisfies WUQ-1. However, for all seats \( j < t \) assigned to party \( p_x \) thus far, since \( w_j \geq w_t \), it follows that \( g_{p_x}(t) - w_j < q(p_x) \), and this means that removing any seat assigned to party \( p \) suffices for this party to not be above its weighted quota and thus, WUQ-X is also satisfied.

To close, we note that although our weakest axioms can be achieved by our WSAMs using an arbitrary order of assigning seat weights, our use of the non-increasing weight order proved vital when looking to satisfy stronger axioms (previously WLQ-X-r, and now WEFX and WUQ-X).

## 5 House Monotonicity

Why focus on D'Hondt or Adams when the Greedy method satisfies both WLQ-X-r and WUQ-X? The answer lies with *house monotonicity*, which asks that an increase in the number of available seats should not make any party worse off. Historically, its failure has been the cause of much political animosity (Szpiro, 2010).
Apportionment with Weighted Seats

In standard apportionment, LRM satisfies LQ and UQ but fails house monotonicity, whereas divisor methods satisfy it (Balinski and Young, 1982; Pukelsheim, 2014). Chakraborty et al. (2021) study house monotonicity in their setting, and we shall do similarly here. As our WSAMs differ from the rules studied by Chakraborty et al. (2021), their results on house monotonicity do not apply to our methods. We first consider a strong generalisation of the axiom.

**Definition 14 (Full House Monotonicity, full-HM).** We say a WSAM $F$ satisfies full-HM if for every election instance $(v, w)$ and every $w^* \in \mathbb{N}_{\geq 1}$ such that $w^* = (w^*_i)_{w \in W}$ is a non-increasing weight vector where $W = \{w \in w \cup \{w^*\}$, it holds for $s \in F(v, w)$ and $s^* \in F(v, w^*)$ that $r_s(p) \geq r_{s^*}(p)$ for every party $p \in [m]$.

Full-HM can always be satisfied, for example by assigning all seats to the largest party. Of course, this is arguably not a desirable WSAM and, in particular, violates all quota axioms we have considered so far.

As with much of the work up to this point, we see that dealing with weights provides additional difficulty in importing properties to our setting. For unit-weight elections, all divisor methods satisfy house monotonicity (Balinski and Young, 1982; Pukelsheim, 2014) but unfortunately, our adapted versions of Adams and D’Hondt fail to do the same for full-HM.

**Proposition 21.** Adams$_w$ and D’Hondt$_w$ fail full-HM.

**Proof.** Let us first consider Adams$_w$ and consider an instance with three parties having votes $v = (5, 5, 2)$ and a weight vector $w = (8, 8, 3, 2)$. The seat assignment returned by Adams$_w$ is $s = (1, 2, 3, 3)$, assuming that ties are broken according to the ordering of $v$. Thus, we have $r_s(3) = 5$ for party 3. Suppose a weight-4 seat is added to $w$ so as to obtain the weight vector $w^* = (8, 8, 4, 3, 2)$. Then Adams$_w$ returns $s^* = (1, 2, 3, 1, 2)$. So party 3 go from $r_s(3) = 5$ to $r_{s^*}(3) = 4$.

Consider an instance with three parties having votes $v = (21, 10, 10)$ and a weight vector $w = (2, 2)$. The seat assignment returned by D’Hondt$_w$ is $s = (1, 1)$ giving both seats to party 1 who obtain 4 in representation. Suppose a weight-3 seat is added to $w$ so as to obtain $w^* = (3, 2, 2)$. Then D’Hondt$_w$ returns $s^* = (1, 2, 3)$ that assigns a seat to each party. So party 1 go from $r_s(1) = 4$ to $r_{s^*}(1) = 3$.

We leave open the question whether there is a WSAM that satisfies full-HM along with some of our proportionality axioms. Instead, we try to achieve positive results by restricting the weight associated with an election’s additional seat. This consideration leads us to the following, weaker axiom where, as opposed to full-HM, the extra $k + 1$-th seat must have a weight no larger than any of the original $k$ seats.

**Definition 15 (Minimal House Monotonicity, min-HM).** We say a WSAM $F$ satisfies min-HM if for every election instance $(v, w)$ and every $w^* \in \mathbb{N}_{\geq 1}$ such that $w^* \leq w$ and $w^* = (w^*_i)_{w \in W}$ is a non-increasing weight vector where $W = \{w \in w \cup \{w^*\}$, it holds for $s \in F(v, w)$ and $s^* \in F(v, w^*)$ that $r_s(p) \geq r_{s^*}(p)$ for every party $p \in [m]$.

This axiom provides another difference between our work and that of Chakraborty et al. (2021) as we note that min-HM can only be defined with objective values for items (seats in our case) and thus would not make much sense in the setting of Chakraborty et al. (2021).

Now, back to our analysis of the WSAMs. Positively, divisor methods clearly satisfy min-HM as all seats prior to an extra seat are assigned in the same way.

**Proposition 22.** All divisor methods satisfy min-HM.

While much weaker than full-HM, min-HM is enough to further distinguish between our WSAMs as the Greedy method, given LRM’s failure of min-HM with unit weights, also fails min-HM.

**Proposition 23.** The Greedy method fails min-HM.

**Proof.** Consider three parties, the vote vector $v = (5, 4, 1)$ and a weight vector $w = (4, 3, 2)$. The seat assignment returned by the Greedy method is $s = (1, 2, 3)$. Now for the weight vector $w^* = (4, 3, 2, 1)$, the method assigns the first two seats to parties 1 and 2, respectively. In round 3, note that $\omega \cdot v_p/n - g_p(3) = 1$ for parties $p \in [3]$. Suppose that party 1 is assigned the third seat via tiebreaking. For the next round, parties 2 and 3 remain equally entitled to the last seat. Suppose that tiebreaking leads to this seat being assigned to party 2. The method then returns the seat assignment $s^* = (1, 2, 3)$ with party 3 receiving less representation than in the original election instance.

Alongside looking at other weakenings of full-HM (such as adding a seat that is of at least as much weight as any current seat), we acknowledge the following as the most pressing future work stemming from this section: investigating the existence of a natural WSAM that satisfies the full-HM. Next, we instead study a real-world application of weighted apportionment.
Apportionment with Weighted Seats

<table>
<thead>
<tr>
<th></th>
<th>WEFX</th>
<th>WLQ-(X \cdot r)</th>
<th>WUQ-(X)</th>
<th>WUQ-1</th>
<th>min-HM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adams_ω</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>D’Hondt_ω</td>
<td>✗</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>Greedy Method</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
</tbody>
</table>

Table 1: Whether a WSAM satisfies (✓) or not (✗) an axiom (satisfied by at least one of our WSAMs).

6 Bundestag Case Study

We present here a case study based on the allocation of chair positions to parties in Bundestag committees, first mentioned in Section 1. Our objective is to compare the results produced by our weighted apportionment methods with the historical results, full details of which are publicly available (Feldkamp, 2024; Wikipedia, 2024). We focus on the election of a chairperson for each committee and in doing so, we interpret the size of a committee as a proxy for its importance. We acknowledge that this can only be an approximation of the true value of a committee, but we believe it is accurate enough to give a first impression of the performance of different WSAMs. Moreover, we wish to emphasise that our case-study does not concern the distribution of seats inside the committees (this is settled using standard apportionment methods for each committee independently). Instead, our rules are used to decide which party can choose the head of each committee (of which there is only one per committee). While noting the presence of exceptional cases, it does not seem implausible to us that a party would consider leading two smaller committees nearly as good as leading one bigger committee.

The existing data covers all 20 legislative periods in Germany between 1949 and 2021. For each of these periods, between 4 and 7 parties entered parliament, between 19 and 28 committees were formed, and each committee had between 9 and 49 members.² To construct an election instance for a given legislative period, we take the members of parliament to be the voters, we take the chair positions for the committees of that period to be the seats to be filled, and we use the sizes of those committees as the weights of the seats.

For each of the 20 election instances thus created, we are interested in how the historical Bundestag seat assignment fares in terms of representing parties proportionally and how that assignment compares to the assignments returned by Adams_ω, D’Hondt_ω, and Greedy. Given a seat assignment \(s\), we first ask which of our nine axioms it satisfies. For testing WLQ^o and WUQ^o, we encoded the computations of the obtainable weighted quotas \(\ell^o(p)\) and \(u^o(p)\) into Integer Linear Programs (ILP) and employed an ILP solver to compute them efficiently. As the binary measure of axiom satisfaction provides only limited insight, we introduce three finer-grained measures. The first is average distance to the weighted quota:

\[
\delta(s) = \frac{1}{n} \sum_{p \in [m]} |r_s(p) - q(p)|.
\]

Second, we have average distance below the weighted lower quota:

\[
\delta^- (s) = \frac{1}{\lvert P^- \rvert} \cdot \sum_{p \in P^-} \ell^o(p) - r_s(p),
\]

where \(P^- = \{ p \in [m] \mid r_s(p) < \ell^o(p) \} \) denotes the set of parties whose representation is below the weighted lower quota. Lastly, average distance above the weighted upper quota:

\[
\delta^+ (s) = \frac{1}{\lvert P^+ \rvert} \cdot \sum_{p \in P^+} r_s(p) - u^o(p),
\]

where \(P^+ = \{ p \in [m] \mid r_s(p) > u^o(p) \} \) denotes the set of parties whose representation is above the weighted upper quota.

This leads to twelve measures used to determine the proportionality of a given seat assignment, with our results summarised in Table 2 (the script used to generate the results is available online at github.com/julianchingy/weighted-apportionment.git).

Note that the historical seat assignments perform reasonably well in terms of our measures of quality, but both D’Hondt_ω and Greedy do markedly better. This is borne out by the rate at which the axioms are satisfied, and the results

²In case any relevant data points (such as the size of a committee) changed over the course of a legislative period, we always used the start of that period as our point of reference.
Table 2: Summary of results for the 20 Bundestag committee election instances. For each of the four seat assignments, the table shows: (i) for each axiom, the percentage of election instances for which the axiom is satisfied, and (ii) for each distance measure, the median and maximum distances across the 20 election instances.

of the distance measures. Notably, not only do the Bundestag seat assignments yield worse median and maximum distances than all the WSAMs, but D’Hondt\(_{\omega}\) and Greedy significantly outperform the Bundestag assignments in all three metrics (with significance level \(\alpha = 0.05\)). Of the latter group, Greedy produces lower distance results across the board, even compared to D’Hondt\(_{\omega}\). However, these differences between Greedy and D’Hondt\(_{\omega}\) are not statistically significant (for significance level \(\alpha = 0.05\)). Interestingly, Adams\(_{\omega}\) returned the overall poorest results, particularly with regards to the distance measures, with it only standing out (positively) in testing the envy-freeness axioms. Despite the small sample size, the results suggest that D’Hondt\(_{\omega}\) and Greedy are worth further experimental investigation.

7 Conclusion and Future Work

We introduced a model for apportionment with weighted seats, and generalised apportionment methods and central axioms from the apportionment literature to this model. Direct generalisations of the axioms, we found, yield (mostly) negative results, but mild relaxations are amenable to positive results. The positive outlook is further justified, in particular for the D’Hondt\(_{\omega}\) and Greedy methods, by an experimental case study on Bundestag committee assignments.

Besides the open questions regarding house monotonicity (Section 5), there are two natural directions for future work. The first is a study of other prominent apportionment properties and rules, e.g., population monotonicity and the Sainte-Laguë method. The second is an extension of the weighted-seat notion to more general settings, e.g., multi-winner voting. One step in this direction would be to forego full supply. However, we find that, once making this step, even the weakest of our axioms fail to be satisfied.

Example 12 (WLQ-1 without full supply). Consider four parties, \(v = (3, 3, 3, 1)\) and \(w = (15, 15, 1, \ldots, 1)\) with \(k = 72\). Suppose each party \(p \in [3]\) can receive two seats while party 4 can receive 66 seats. So some party \(p \in [3]\) with \(q(p) = 30\) must be assigned two weight-1 seats and so WLQ-1 cannot be provided. \(\triangle\)

Example 13 (WUQ-1 without full supply). Consider three parties, \(v = (1, 1, 1)\) and \(w = (3, 3, 3, 3)\). Suppose each party \(p \in [2]\) can receive one seat while party 3 can receive three seats. So, party 3 with \(q(p) = 3\) must be assigned three seats of weight 3 and this violates WUQ-1. \(\triangle\)

These examples illustrate that extra care needs to be taken when generalising our work on apportionment with weighted seats to more complex settings.
References


Apportionment with Weighted Seats
