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PROPORTIONALITY FOR CONSTRAINED PUBLIC DECISIONS

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ABSTRACT

We study situations where a group of voters need to take a collective decision over a number of public issues, with the goal of getting a result that reflects the voters' opinions in a proportional manner. Our focus is on interconnected public decisions, where the decision on one or more issues has repercussions on the acceptance or rejection of other public issues in the agenda. We show that the adaptations of classical justified-representation axioms to this enriched setting are always satisfiable only for restricted classes of public agendas. However, the use of suitably adapted well-known decision rules on a class of quite expressive constraints, yields proportionality guarantees that match these justified-representation properties in an approximate sense. We also identify another path to achieving proportionality via an adaptation of the notion of priceability.

1 Introduction

In many situations of collective decision-making, a group of voters is presented with a set of issues for which they are expected to make a binary choice: typically, deciding to either *accept* or *reject* each issue. This setting has recently been studied under the name of *public decisions* (Skowron and Górecki, 2022) and it is of particular interest due to the real-world scenarios captured by it. Notable examples include: instances of *multiple referenda* where the public vote directly on the resolution of political issues; *group activity planning*, where a group of individuals are to choose, as a collective, the activities that the entire group shall partake in; and *committee elections*, where a set of candidates are in the running for multiple positions on a committee and a group of decision-makers must select the committee members (Lang and Xia, 2016).

Given the collective nature of the problem, one of the natural desiderata is that the outcome represents a *fair* compromise for the participating voters. Among the numerous possible interpretations of fairness is the one captured by the notion of *proportional representation*. Proportionality features prominently in many collective choice settings such as that of *apportionment* (Balinski, 2005) and the aforementioned committee elections (Lackner and Skowron, 2023) while being introduced to richer social-choice models such as that of *participatory budgeting (PB)* (Rey and Maly, 2023). Indeed, even when zooming in on the public decisions task, the goal of producing collective outcomes that proportionally reflect the opinions of the voter population has been drawing increasing attention in recent years (Freeman et al., 2020; Masařík et al., 2023; Skowron and Górecki, 2022). However, a component that has so far not received much attention in this growing literature on proportionality is the presence of *constraints* that restrict the possible outcomes that can be returned. In this paper, we focus on answering the question of what one may do when outcomes that would satisfy classical proportionality axioms—and thus be considered fair outcomes—are no longer feasible due to the presence of constraints.

When examining real-world examples of the public-decision model, there are many scenarios where enriching the model with constraints fits naturally: in the case of participatory budgeting, the implementation of one project may be conditional on the acceptance (or rejection) of another; diversity constraints applied to the committee election problem that determine the number of individuals with certain characteristics that may be accepted/rejected; or when selecting the features of some product, only certain feature combinations represent affordable options.

In tackling our task, we build on existing notions of proportionality that have been posed for less rich models and tailor them for the challenges that comes with the existence of constraints. Naturally, this leads us to also consider constrained versions of collective decision rules proposed in the literature and to investigate the extent to which they meet the requirements of our novel constraint-aware notions of proportionality.

Related work. We begin by noting that our constrained public-decision model closely resembles that of *judgment aggregation* and it also naturally fits into the area of collective decisions in *combinatorial domains* (see (Endriss, 2016) and (Lang and Xia, 2016) for general introductions to these two topics, respectively).

Most relevant to our paper is the recent work conducted on fairness in the context of public decisions without constraints (Conitzer et al., 2017; Freeman et al., 2020; Masařík et al., 2023; Skowron and Górecki, 2022). Conitzer et al. (2017) focused on individually proportional outcomes, thus, our work more closely aligns with that of Freeman et al. (2020) and Skowron and Górecki (2022) who adapt the notion of *justified representation* (Aziz et al., 2017; Fernández et al., 2017; Peters and Skowron, 2020) from the literature of *multiwinner voting (MWV)* (Lackner and Skowron, 2023). Moreover, proportionality has also been studied in models of *sequential decision-making* that are relevant to our own as they can be seen as generalisations of the public-decision model without constraints (Bulteau et al., 2021; Chandak et al., 2024; Lackner, 2020). Amongst these sequential decision-making papers, those of Bulteau et al. (2021) and Chandak et al. (2024) relate to our work the most as they also implement justified-representation notions. More recently, Masařík et al. (2023) studied proportionality for a general social-choice model that allows for the modelling of both the unconstrained and constrained versions of the public-decision model. By focusing on the latter, we explore properties that are specifically made for this setting, which in turn allows us to define, and subsequently conduct an analysis of, constrained public-decision rules that are not touched upon by Masařík et al. (2023). Thus, our results complement their work by showing further possibilities, and also limitations, for proportionality within this constrained public-decision model. We also highlight work by Mavrov et al. (2023) who adapted justified representation for the MWV model with arbitrary constraints instead of our focus on the public-decision model. This leads us towards differing approaches in adapting justified representation for constraints and also, analysing quite different rules.

In related fields, previous work studied proportionality in various models that differ from the constrained public-decision model but features collective choices on interconnected propositions: the *belief merging* setting (Haret et al., 2020), interdependent binary issues via *conditional ballots* (Brill et al., 2023), and approval-based shortlisting with constraints (presented in a model of judgment aggregation) (Chingoma et al., 2022).

Contribution. We study the extent to which proportionality can be ensured constrained public-decision setting. First, we introduce the notion of feasible group deviations as a building block that allows the translation of existing proportionality axioms—that are based on varying public-decision interpretations of justified representation—for this setting with constraints.

For each of our axioms, we show that although it is challenging to satisfy these properties in general constrained instances, when one hones in on a restricted—yet highly expressive—class of constraints, we can achieve proportionality guarantees that represent approximations of the desirable justified-representation axioms. In doing so, we also define novel adaptations of recently studied decision rules to our public-decision setting with constraints, namely the method of equal shares (*MES*) and the *MeCorA* rule. Finally, we adapt the priceability notion from the MWV literature, which provides another promising route to introduce proportionality into public decisions under constraints.

Paper outline. We begin by detailing the constrained public-decision model in Section 2. We continue with Section 3 where we discuss two known ways in which justified representation is formalised for public decisions, and also present the notion of deviating groups. Then each of sections 4 and 5 deal with a particular public-decision interpretation of justified representation. Before concluding in Section 7, we deal wit our constrained version of the priceability axiom in Section 6. Note that all omitted proofs can be found as part of the supplementary material.

2 The Model

A finite set of n voters $N = \{1, \ldots, n\}$ has to take a collective decision on a finite set of m binary issues $\mathcal{I} = \{a_1, \ldots, a_m\}$. It is typical in the public decisions setting to consider there only being two available decisions per issue but we instead adopt the following, more general setup. Each issue $a_t \in \mathcal{I}$ is associated with a finite set of *alternatives* called a *domain* $D_t = \{d_t^1, d_t^2, \ldots\} \subseteq X$ where $|D_t| \geqslant 2$ holds for all $t \in [m]$. The design decision of going beyond binary issues is motivated by the wider real-life applicability of this model when more than two alternatives are possible for each issue.

Each voter $i \in N$ submits a *ballot* $\boldsymbol{b}_i = (\boldsymbol{b}_i^1, \ldots, \boldsymbol{b}_i^m) \in D_1 \times \ldots \times D_m$ where $\boldsymbol{b}_i^t = d_t^c$ indicates that voter i chooses the decision d_t^c for the issue issue a_t.

A *profile* $\boldsymbol{B} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n) \in (D_1 \times \ldots \times D_m)^n$ is a vector of the n voters' ballots. An *outcome* $\boldsymbol{w} = (w_1, \ldots, w_m) \in D_1 \times \ldots \times D_m$ is then a vector providing a decision for every issue at stake.

We focus on situations where some *constraints* limit the set of possible collective outcomes: we denote by $\mathcal{C} \subseteq D_1 \times \ldots \times D_m$ the set of *feasible* outcomes. We write $(\boldsymbol{B}, \mathcal{C})$ to denote an *election instance*. By a slight abuse of notation we also refer to $\mathcal{C}$ as the *constraint*, and thus, we refer to elections instances where $\mathcal{C} = D_1 \times \ldots \times D_m$ as *unconstrained* election instances. [1]

Note that voter ballots need not be consistent with the constraints, i.e., for an election instance $(\boldsymbol{B}, \mathcal{C})$, we do not require that $\boldsymbol{b}_i \in \mathcal{C}$ for all voters $i \in N$.[2]

Remark 1. *While not common in work done in the related judgment aggregation model, our assumption that voters ballots need not correspond to feasible outcomes is common in other settings of social choice. In multiwinner voting, voters can approve more candidates than the committee target size while in participatory budgeting, the sum of the costs of a voter's approved projects may exceed the instance's budget. For our setting, we argue that this approach helps capture real-world, constrained decision-making scenarios where either the constraint is* uncertain *when voters submit their ballots, or possibly, the voting process becomes more burdensome for voters as they attempt to create ballots with respect to a (possibly difficult to understand) constraint. For example, consider a group of friends deciding on the travel destinations of their shared holiday across the world, visiting one country in each continent. On a booking platform, there are a certain number of locations that can be selected per continent such as: Amsterdam, Paris and Vienna in Europe; Mexico City and Toronto in North America; Cairo, Nairobi and Cape Town in Africa; and so on. Each friend has a preferred combination of cities and their collective itinerary is subject to factors such as their travel budget or the available flight connections between cities. However, as flight costs and connections may change significantly on a day-to-day basis, it may be unclear which combination of cities are affordable. Therefore, it is not reasonable to impose the requirement, by default that is, that voter ballots are constraint-consistent.*

If needed, we explicitly state when we pivot from this assumption and require that voter ballots be constraint-consistent. At times, we shall restrict ourselves to election instances where $D_t = \{0, 1\}$ holds for every issue a_t. We refer to such cases as *binary* election instances. When necessary, we explicitly state whether any result hinges on the restriction to binary instances. Given an outcome $\boldsymbol{w}$ for a binary instance, the vector $\bar{\boldsymbol{w}} = (\bar{w}_1, \ldots, \bar{w}_m)$ is such that $\bar{w}_t = 1 - w_t$ for all issues $a_t \in \mathcal{I}$.

Now, consider an outcome $\boldsymbol{w}$, a set of issues $S \subseteq \mathcal{I}$ and some vector $\boldsymbol{v} = (1, \ldots, v_m) \in D_1 \times \ldots \times D_m$ (that can be interpreted as either an outcome or voter's ballot). We write $\boldsymbol{w}[S \leftarrow \boldsymbol{v}] = (w'_1, \ldots, w'_m)$ where $w'_t = w_t$ for all issues $a_t \in \mathcal{I} \setminus S$ and $w'_t = v_t$ for all issues $a_j \in S$. In other words, $\boldsymbol{w}[S \leftarrow \boldsymbol{v}]$ is the resultant vector of updating outcome $\boldsymbol{w}$'s decisions on the issues in S by fixing them to those of vector $\boldsymbol{v}$. For a given issue $a_t \in \mathcal{I}$ and a decision $d \in D_t$, we use $N(a_t, d) = \{i \in N \mid \boldsymbol{b}_i^t = d\}$ to denote the set of voters that agree with decision d on issue a_t. Given two vectors $\boldsymbol{v}, \boldsymbol{v}' \in D_1 \times \ldots \times D_m$, we denote the *agreement* between them by $\text{Agr}(\boldsymbol{v}, \boldsymbol{v}') = \{a_t \in \mathcal{I} \mid v_t = v'_t\}$. Then, the *satisfaction* that a voter i obtains from an outcome $\boldsymbol{w}$ corresponds to $u_i(\boldsymbol{w}) = |\text{Agr}(\boldsymbol{b}_i, \boldsymbol{w})|$, i.e., the number of decisions on which the voter i is in agreement with outcome $\boldsymbol{w}$.

3 Proportionality via Justified Representation

This section starts with the observation that classical notions of proportionality fall short when considering interconnected decisions (in the upcoming Example 1), and then follows with our proposed generalisations of such axioms that deal with constraints.

Ideally, when looking to make a proportional collective choice, we would like to meet the following criteria: a group of similarly-minded voters that is an α fraction of the population should have their opinions reflected in an α fraction of the m issues. We wish to define an axiom for our model that captures this idea within our richer framework.

In the setting of multiwinner voting, this is formally captured with the justified representation axioms with one of the most widely studied being *extended justified representation (EJR)* (Aziz et al., 2017). Now, when being studied in the setting of public decisions, there are two different adaptations that have been studied and we shall look at both. One approach intuitively states that 'a group of voters that agree on a set of issues T and represent an α fraction of the voter population, should control a $\alpha \cdot |T|$ number of the total issues in $\mathcal{I}$' (Chandak et al., 2024; Masařík et al., 2023; Skowron and Górecki, 2022). We refer to it as *agreement-EJR*.

[1] Note that while we work formally with the constraint being an enumeration of all feasible outcomes, in practice, it is often possible to represent the set of feasible outcomes in more concise forms—via the use of formulas of *propositional logic*, for example—to help with parsing said constraint and/or speed up computation by exploiting the constraint's representation structure.

[2] This assumption takes our model closer to the particular model of judgment aggregation where the constraints on the output may differ from the constraints imposed on the the voters' input judgments (Endriss, 2018; Chingoma et al., 2022).

This approach differs from the following that is a more faithful translation of the EJR from multiwinner voting: 'a group of voters that agree on, and represent, an α fraction of the issues, and voter population, respectively, should control $\alpha \cdot m$ of the issues in $\mathcal{I}$' (Chandak et al., 2024; Freeman et al., 2020). The requirements on the voter groups that is present in the latter approach are captured by the notion of *cohesiveness* and so we refer to this version of EJR as *cohesiveness-EJR*. Observe that cohesiveness-EJR is stronger than, and implies, agreement-EJR.

Meeting the ideal outlined by both of these notions is not easy in our setting as the constraint $\mathcal{C}$ could rule out a seemingly fair outcome from the onset.

Example 1. Suppose there are two issues $\mathcal{I} = \{a_1, a_2\}$ with constraint $\mathcal{C} = \{(1,0),(0,1)\}$. Then suppose there are two voters $N = \{1,2\}$ with ballots $\boldsymbol{b}_1 = (1,0)$, and $\boldsymbol{b}_2 = (0,1)$ (note that voters 1 and 2 are both, on their own, cohesive groups). Here, both aforementioned EJR interpretations require each voter to obtain at least 1 in satisfaction, i.e., deciding half of the two issues at hand. However, there exists no feasible outcome that provides agreement-EJR or cohesiveness-EJR as one voter $i \in \{1,2\}$ will have satisfaction $u_i(\boldsymbol{w}) = 0$ for any outcome $\boldsymbol{w} \in \mathcal{C}$. △

Example 1 makes clear an issue that we must take into account when defining proportionality properties when there are constraints. That is, a voter group that is an α fraction of the population may lay claim to deciding an α fraction of the issues, but in doing so, they may be resolving, or influencing the decision on, a larger portion of the issues than they are entitled to.

In doing so, we look for meaningful ways to identify, given an outcome $\boldsymbol{w}$, those voter groups that are underrepresented and can justifiably complain at the selection of outcome $\boldsymbol{w}$.

The latter is formalised by the following definition which we use to identify the voter group whose displeasure is justified. Specifically, these are groups that can propose an alternative, feasible outcome $\boldsymbol{w}^*$ that yields greater satisfaction for each group member.

Definition 1 (($S, \boldsymbol{w}$)-deviation)**.** *Given election instance $(\boldsymbol{B}, \mathcal{C})$ and outcome $\boldsymbol{w} \in \mathcal{C}$, a set of voters $N' \subseteq N$ has an* $(S, \boldsymbol{w})$-deviation *if $\emptyset \neq S \subseteq \mathcal{I}$ is a set of issues such that all of the following hold:*

- $S \subseteq \mathrm{Agr}(\boldsymbol{b}_i, \boldsymbol{b}_j)$ *for all $i, j \in N'$ (the voters agree on the decisions on all issues in S).*
- $S \subseteq \mathcal{I} \setminus \mathrm{Agr}(\boldsymbol{b}_i, \boldsymbol{w})$ *for all $i \in N'$*

  (the voters disagree with outcome $\boldsymbol{w}$'s decisions on all issues in S).
- $\boldsymbol{w}[S \leftarrow \boldsymbol{b}_i] \in \mathcal{C}$ *for all $i \in N'$ (fixing outcome $\boldsymbol{w}$'s decisions on issues in S, so as to agree with the voters in N', induces a feasible outcome).*

Intuitively, given an outcome $\boldsymbol{w}$, a voter group having an $(S, \boldsymbol{w})$-deviation indicates the presence of another feasible outcome $\boldsymbol{w}^* \neq \boldsymbol{w}$ where every group member would be better off. Thus, our goal in providing a fair outcome reduces to finding an outcome where every group of voters that has an $(S, \boldsymbol{w})$-deviation is sufficiently represented. We shall use this $(S, \boldsymbol{w})$-deviation notion to convert proportionality axioms from unconstrained settings to axioms that deal with constraints. But first, we look at the following computational question associated with $(S, \boldsymbol{w})$-deviations: given an election instance $(\boldsymbol{B}, \mathcal{C})$ and an outcome $\boldsymbol{w} \in \mathcal{C}$, the problem is to find all groups of voters with an $(S, \boldsymbol{w})$-deviation.

Proposition 1. *Given an election instance $(\boldsymbol{B}, \mathcal{C})$ and an outcome $\boldsymbol{w} \in \mathcal{C}$, there exists an algorithm that finds all groups of voters N' such that there exists an $S \subseteq \mathcal{I}$ with N' having an $(S, \boldsymbol{w})$-deviation, that runs in $O(|\mathcal{C}|^2 mn)$ time.*

Proof. Take $(\boldsymbol{B}, \mathcal{C})$ and outcome $\boldsymbol{w} \in \mathcal{C}$. Consider the following algorithm that operates in $|\mathcal{C}|$ rounds, assessing an outcome $\boldsymbol{w} \in \mathcal{C}$ in each round (with each outcome assessed once throughout): at each round for an outcome $\boldsymbol{w} \in \mathcal{C}$, iterate through all other outcomes $\boldsymbol{w}^* \neq \boldsymbol{w} \in \mathcal{C}$; fix S to be the issues that $\boldsymbol{w}$ and $\boldsymbol{w}^*$ disagree on; in at most mn steps, it can be checked if there is a set of voters that agree with $\boldsymbol{w}^*$ on all issues in S which verifies the existence of a voter group N' with an $(S, \boldsymbol{w})$-deviation; keep track of all such groups N'; if all outcomes have been assessed, terminate, otherwise, move to the next outcome. This algorithm takes $O(|\mathcal{C}|^2 mn)$ time to complete in the worst case, which is polynomial in the input size given our assumptions. □

We offer the following remark in regards to the nature of Proposition 1.

Remark 2. *Proposition 1 can be seen as positive whenever the constraint $\mathcal{C}$ under consideration is 'not too large'. Such an assumption is reasonable for many real-life examples. Consider the quite general, collective task of selecting the features of some product. Our running example of the logo design is an instance of this. Other applicable scenarios include choosing the technical features of a shared computer or the items to be placed in an organisation's common area. In many cases, factors such as a limited budget (or limited space in the case of the common area) may result in very few feature combinations being feasible for said product. These are natural scenarios where we may encounter* *a ‘small’ constraint (according to our definition) with respect to the number of issues at hand and the size of their domains.*

Our goal is to answer the following question: how much representation can we guarantee from some outcome $\boldsymbol{w}$, to a group of voters that has an $(S, \boldsymbol{w})$-deviation and that qualifies as underrepresented?

4 Justified Representation with Cohesiveness

We now propose the following adaptations of cohesiveness-EJR to public decisions with constraints. To adapt cohesiveness-EJR, we adapt cohesiveness from multiwinner voting in a similar manner as done by Freeman et al. (2020). We say that a voter group is T-*agreeing* for some set of issues $T \subseteq \mathcal{I}$ if $T \subseteq \text{Agr}(\boldsymbol{b}_i, \boldsymbol{b}_j)$ holds for all voters $i, j \in N'$ and then we define cohesiveness as the following:

Definition 2 (T-cohesiveness)**.** *For a set of issues $T \subseteq \mathcal{I}$, we say that a set of voters $N' \subseteq N$ is T-*cohesive *if N' is T-agreeing and it holds that $|N'| \geqslant |T| \cdot {}^{n}\!/_{m}$.*

Using T-cohesiveness, we can define EJR for public decisions with constraints (Freeman et al., 2020).

Definition 3 (cohEJR$_\mathcal{C}$)**.** *Given an election $(\boldsymbol{B}, \mathcal{C})$, an outcome $\boldsymbol{w}$ provides* cohEJR$_\mathcal{C}$ *if for every T-cohesive group of voters $N' \subseteq N$ for some $T \subseteq \mathcal{I}$ with an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$, there exists a voter $i \in N'$ such that $u_i(\boldsymbol{w}) \geqslant |T|$.*

Intuitively, cohEJR$_\mathcal{C}$ deems an outcome to be unfair if there exists a T-cohesive voter group with (i) none of its group members having at least $|T|$ in satisfaction, and (ii) ‘flipping’ outcome $\boldsymbol{w}$’s decisions on some of the issues in T leads to some other feasible outcome.

We have the following result that can be interpreted as positive when the size of $\mathcal{C}$ is ‘not too large’.

Proposition 2. *Given an election instance $(\boldsymbol{B}, \mathcal{C})$ and an outcome $\boldsymbol{w} \in \mathcal{C}$, there exists an algorithm that decides in $O((\max_{t\in[m]} |D_t|)^m |\mathcal{C}|^3 mn)$ time whether outcome $\boldsymbol{w}$ provides cohEJR$_\mathcal{C}$.*

Proof. From Proposition 1 we know that, given an outcome $\boldsymbol{w}$, we can find all groups with some $(S, \boldsymbol{w})$-deviation for some $S \subseteq \mathcal{I}$ in $O(|\mathcal{C}|^2 mn)$ time. There can be at most $(\max_{t\in[m]} |D_t|)^m(|\mathcal{C}| - 1)$ such groups (recall that $\max_{t\in[m]} |D_t|$ is the maximal size of any issue’s domain). Then, for each group N' with an $(S, \boldsymbol{w})$-deviation, we can check their size in polynomial time and thus verify whether they are a T-cohesive with $S \subseteq T$, and if so, we can check if there exists any voter $i \in N'$ with $u_i(\boldsymbol{w}) > |T|$. □

Now, Chandak et al. (2024) have already shown that, in general, cohesiveness-EJR is not always satisfiable in their sequential decisions model. This negative result carries over to the unconstrained public-decision setting. Although we shall, in the sections to follow, analyse the extent to which we can achieve positive results with cohesiveness-EJR in our constrained setting, this negative result motivates the study of the following weaker axiom—which is an adaptation of the multiwinner JR axiom—that can always be satisfied in the public-decision setting without constraints (Bulteau et al., 2021; Chandak et al., 2024; Freeman et al., 2020).

Definition 4 (cohJR$_\mathcal{C}$)**.** *Given an election instance $(\boldsymbol{B}, \mathcal{C})$, an outcome $\boldsymbol{w}$ provides* cohJR$_\mathcal{C}$ *if for every T-cohesive group of voters $N' \subseteq N$ for some $T \subseteq \mathcal{I}$ with an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$ where $|S| = |T| = 1$, there exists a voter $i \in N'$ such that $u_i(\boldsymbol{w}) \geqslant 1$.*

Unfortunately, when considering arbitrary constraints, even cohJR$_\mathcal{C}$ cannot always be achieved. Note that this even holds for binary election instances.

Proposition 3. *There exists an election instance where no outcome provides cohJR$_\mathcal{C}$.*

Proof. Consider the binary election instance with issues $\mathcal{I} = \{a_1, a_2\}$ and a constraint $\mathcal{C} = \{(0,1), (0,0)\}$. Suppose that $N = \{1, 2\}$, where $\boldsymbol{b}_1 = (1,1)$ and $\boldsymbol{b}_2 = (1,0)$. Note that for both outcomes $\boldsymbol{w} \in \mathcal{C}$, one voter will have satisfaction of 0 while being a T-cohesive group with an $(S, \boldsymbol{w})$-deviation for $|S| = |T| = 1$. As each voter is half of the population, they may ‘flip’ issue a_2 to deviate towards the alternative feasible outcome, which provides them greater satisfaction than the current one. □

Let us now restrict the constraints that we consider. To do so, we introduce notation for the *fixed decisions* for a set of outcomes $C \subseteq \mathcal{C}$, which are the issues in $\mathcal{I}$ whose decisions are equivalent across all the outcomes in C. For a set of outcomes $C \subseteq \mathcal{C}$, we represent this as:

$$\mathcal{I}_{\text{fix}}(C) = \{a_t \in \mathcal{I} \mid \text{there exists some } d \in D_t \text{ such that } w_t = d \text{ for all } \boldsymbol{w} \in C\}.$$

Definition 5 (No Fixed Decisions (NFD) property)**.** *We say a constraint $\mathcal{C}$ has the* NFD *property if $\mathcal{I}_{fix}(\mathcal{C}) = \emptyset$ holds for $\mathcal{C}$.*

Remark 3. *At first glance, this NFD property seems more than a reasonable requirement but rather a property that should be assumed to hold by default. We argue however, that by doing so, we will neglect election instances where decisions that are fixed from the get-go may contribute to the satisfaction of voters and, specifically for our goal, these fixed decisions may aid in giving the voters their fair, proportional representation. It is this reason, why we did not restrict ourselves to election instances where the NFD property holds.*

Now, we show that with the NFD property, the cohEJR$_\mathcal{C}$ axiom can always be provided, albeit only for 'small' election instances. We begin with cases where the number of feasible outcomes is limited to two.

Proposition 4. *For election instances $(\boldsymbol{B}, \mathcal{C})$ with $|\mathcal{C}| = 2$ where $\mathcal{C}$ has the NFD property, cohEJR$_\mathcal{C}$ can always be satisfied.*

Proof. Take some feasible outcome $\boldsymbol{w} \in \mathcal{C}$. Observe that when $|\mathcal{C}| = 2$, if property NFD holds, then the two feasible outcomes differ on the decisions of all issues. Thus, it is only possible for T-cohesive groups with an $(S, \boldsymbol{w})$-deviation for $|S| \leqslant |T| = m$ to have an allowable deviation from $\boldsymbol{w}$ to the only other feasible outcome. This means only the entire voter population have the potential to deviate. And if such deviation to $\boldsymbol{w}'$ exists, then outcome $\boldsymbol{w}'$ sufficiently represents the entire voter population. □

Now we ask the following: can we guarantee cohEJR$_\mathcal{C}$ when $m \leqslant 3$? We answer in the positive when we restrict ourselves to binary election instances.

Proposition 5. *For binary election instances $(\boldsymbol{B}, \mathcal{C})$ with $m \leqslant 3$ where the constraint $\mathcal{C}$ has the NFD property, cohEJR$_\mathcal{C}$ can always be provided.*

Proof. The case for $m = 1$ is trivially satisfied so we present the proof as two separate cases where the number of issues is either $m = 2$ or $m = 3$.

Case $\boldsymbol{m = 2}$: Observe that for two issues (i.e., $m = 2$) there are 7 possible constraints $\mathcal{C}$ satisfying the NFD property. Take one such $\mathcal{C}$ and a feasible outcome $\boldsymbol{w} = (d_x, d_y) \in \mathcal{C}$ where $d_x, d_y \in \{0, 1\}$. Let us consider now groups of voters with an $(S, \boldsymbol{w})$-deviation over some set of issues $S \subseteq T$ who are witness to a violation of cohEJR$_\mathcal{C}$. As $m = 2$, the agreement among voters and the deviation may concern at most two issues, i.e., $|S|, |T| \in \{1, 2\}$.

First, consider $|T| = 1$. Since $|S| \leqslant |T|$ and $S \neq \emptyset$, we have $|S| = 1$ for any T-cohesive group (which is thus of size $|N'| \geqslant n/2$) wishing to perform a $(S, \boldsymbol{w})$-deviation from $\boldsymbol{w}$ to some other feasible outcome $\boldsymbol{w}' \in \mathcal{C}$. If there is a voter $i \in N'$ such that $u_i(\boldsymbol{w}) \geqslant 1$, group N' would be sufficiently satisfied—therefore, cohEJR$_\mathcal{C}$ is ensured and we are done. Otherwise, we have that all $i \in N'$ are unanimous and that $u_i(\boldsymbol{w}) = 0$; hence, $\boldsymbol{b}_i = (1 - d_x, 1 - d_y)$ for all $i \in N'$. There are two possible outcomes (deviations) that differ from $\boldsymbol{w}$ in only one coordinate. If neither outcome is in $\mathcal{C}$, then no feasible deviation is possible for N' and we are done. Otherwise, assume without loss of generality that $\boldsymbol{w}' = (1 - d_x, d_y) \in \mathcal{C}$. Now, if there is a voter $i \in N \setminus N'$ such that $u_i(\boldsymbol{w}) \geqslant 1$, then we are done (as the group $N \setminus N'$ would be sufficiently satisfied if it were T-cohesive for $|T| = 1$). Else, it means that all voters $j \in N \setminus N'$ are unanimous on ballot $\boldsymbol{b}_j = (d_x, 1 - d_y)$. But then, since $\mathcal{C}$ satisfies property NFD, there exists some outcome $\boldsymbol{w}'' \in \mathcal{C}$ such that $\boldsymbol{w}''_2 = 1 - d_y$. Then, $u_i(\boldsymbol{w}'') \geqslant 1$ for all $i \in N$ and no deviation is possible.

Finally, consider $|T| = m = 2$. In order for a group N' that is T-cohesive to have a $(S, \boldsymbol{w})$-deviation for $|S| \leqslant |T|$, it must be the case that $N' = N$, and $u_i(\boldsymbol{w}) = 0$ for all $i \in N$. By property NFD, there must be some outcome $\boldsymbol{w}' \neq \boldsymbol{w} \in \mathcal{C}$, and thus $u_i(\boldsymbol{w}') \geqslant 1$ for all $i \in N$.

Case $\boldsymbol{m = 3}$: Let $(\boldsymbol{B}, \mathcal{C})$ be an election instance satisfying the conditions in the statement. We now reason on the existence of possible T-cohesive groups that are a witness to the violation of cohEJR$_\mathcal{C}$, for each possible size $1 \leqslant |T| \leqslant 3$ of the set T.

For $|T| = 1$, suppose by contradiction that for all $\boldsymbol{w} \in \mathcal{C}$, there is some voter group N' such that $|N'| \geqslant n/3$ and each voter in N' has satisfaction of 0. Thus, for all voters $i \in N'$ we have $\boldsymbol{b}_i = \bar{\boldsymbol{w}}$. Moreover, for a T-cohesive group with an $(S, \boldsymbol{w})$-deviation for $|S| = |T| = 1$ to be possible, there has to exist a $\boldsymbol{w}' \in \mathcal{C}$ whose decisions differ from $\boldsymbol{w}$ in exactly one issue, i.e., $\text{Agr}(\boldsymbol{w}, \boldsymbol{w}') = 2$. To fit all these disjoint T-cohesive groups for $|T| = 1$, one for each outcome in $\mathcal{C}$, it must be that $n \geqslant |\mathcal{C}| \cdot n/3$, hence $|\mathcal{C}| \leqslant 3$ must hold. If $|\mathcal{C}| = 1$, the NFD property cannot be met. If $|\mathcal{C}| = 2$, then the two feasible outcomes cannot differ in the decision of only one issue while also satisfying the NFD property. For $|\mathcal{C}| = 3$, to get a T-cohesive voter group with an $(S, \boldsymbol{w})$-deviation for $|S| = |T| = 1$ at every $\boldsymbol{w} \in \mathcal{C}$, the three feasible outcomes must differ by at most one decision, contradicting the NFD property.

For $|T| = 2$, we only consider $(S, \boldsymbol{w})$-deviations from a T-cohesive group N' with $|S| \in \{1, 2\}$. Consider the case of $|S| = 1$. W.l.o.g., assume $\boldsymbol{w} = (0, 0, 0)$ and assume that there exists a T-cohesive group N' (where $|N'| \geqslant n \cdot {}^2\!/_3$) with every voter having satisfaction < 2, with an $(S, \boldsymbol{w})$-deviation towards outcome, e.g., $\boldsymbol{w}' = (1, 0, 0)$. The case for $(0, 1, 0)$ is similar. Now one of the N' voters has satisfaction of 2. If ${}^n\!/_3$ voters now have an allowable deviation (satisfaction of 0 with the current outcome), by NFD one of the outcomes $\{010, 111, 011, 110\}$ must be in $\mathcal{C}$. Observe that any of them provides satisfaction of at least 2 to all cohesive groups of $|T| = 2$, and at least satisfaction 1 to every cohesive group of $|T| = 1$. Now we look at the case for $|S| = 2$. W.l.o.g., consider the outcome $\boldsymbol{w} = (0, 0, 0)$ and assume that there exists a T-cohesive group N' (where $|N'| \geqslant n \cdot {}^2\!/_3$) with an $(S, \boldsymbol{w})$-deviation towards outcome, e.g., $\boldsymbol{w}' = (1, 1, 0)$. Thus, there is some voter i in N' with satisfaction $u_i(\boldsymbol{w}') \geqslant 2$. At this point, the only possible further $(S, \boldsymbol{w})$-deviation could arise for $|S| = 1$ in case there are ${}^n\!/_3$ voters in $N \setminus N'$ each have a satisfaction of 0 for $\boldsymbol{w}'$, i.e., each has the ballot $(0, 0, 1)$ and either one of the outcomes in $\{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}$ is in $\mathcal{C}$. Now take instead that $u_i(\boldsymbol{w}') = 2$ and consider two cases where either voter i agrees or disagrees with the voters in $N \setminus N'$ on the decision of issue a_3. First, assume that voter $i \in N'$ *agrees* with the voters in $N \setminus N'$ on issue a_3 (so voter i had the ballot $\boldsymbol{b}_i = (1, 1, 1)$). Then if either $(0, 1, 1) \in \mathcal{C}$ or $(1, 1, 1) \in \mathcal{C}$ holds, we have that cohEJR$_\mathcal{C}$ is provided. And if $(0, 0, 1) \in \mathcal{C}$ holds, then voters in $N \setminus N'$ are entirely satisfied and the voters in N' may only have an $(S, \boldsymbol{w})$-deviation for $|S| \leqslant |T| = 2$ if either $(0, 1, 1) \in \mathcal{C}$ or $(1, 1, 1) \in \mathcal{C}$ holds (as they only 'flip' issues they disagree with), which means that cohEJR$_\mathcal{C}$ is provided. In the second case, assume that voter $i \in N'$ *disagrees* with the voters in $N \setminus N'$ on issue a_3 and so, voter i had the ballot $\boldsymbol{b}_i = (1, 1, 0)$. This means that $u_i(\boldsymbol{w}') = 3$ holds, hence, any outcome that the voters in $N \setminus N'$ propose given they have an $(S, \boldsymbol{w})$-deviation for $|S| = 1$, would be one that provides cohEJR$_\mathcal{C}$.

Finally, a T-cohesive group for $|T| = 3$ implies a unanimous profile; if there exists an allowable $(S, \boldsymbol{w})$-deviation for $|S| \leqslant |T| = 3$, then the outcome in $\mathcal{C}$ maximising the sum of agreement with the profile provides cohEJR$_\mathcal{C}$. ☐

We leave it open whether the above result holds if we do not restrict our view to binary election instances. Unfortunately, the good news ends there as we provide a example showing that cohJR$_\mathcal{C}$ cannot be guaranteed when do not have $m \leqslant 3$ (even for binary election instances).

Proposition 6. *There exists an election instance $(\boldsymbol{B}, \mathcal{C})$ where $m > 3$ and the constraint $\mathcal{C}$ has the NFD property but no cohJR$_\mathcal{C}$ outcome exists.*

Proof. Suppose there is a binary election instance with a constraint $\mathcal{C} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4\}$ for $m = 8$ such that $\boldsymbol{w}_1 = (0, 0, 0, \ldots, 0)$, $\boldsymbol{w}_2 = (0, 0, 1, \ldots, 1)$, $\boldsymbol{w}_3 = (1, 1, 1, \ldots, 1)$, $\boldsymbol{w}_4 = (1, 1, 0, \ldots, 0)$. Consider now a profile of four voters where $\boldsymbol{b}_i = \boldsymbol{w}_i$. Given that $m = 8$, note that for every outcome $\boldsymbol{w} \in \mathcal{C}$, there exists some voter that deserves 2 in satisfaction by being T-cohesive for $|T| = 2$ with an $(S, \boldsymbol{w})$-deviation but with zero in satisfaction. And by cohJR$_\mathcal{C}$, such a voter would be entitled to at least 1 in satisfaction, so there is no outcome in $\mathcal{C}$ that provides cohJR$_\mathcal{C}$. ☐

We now turn our attention towards a weakening of cohEJR$_\mathcal{C}$ that takes inspiration from EJR-1 studied in the context of participatory budgeting.

Definition 6 (cohEJR$_\mathcal{C}$-1)**.** *Given an election $(\boldsymbol{B}, \mathcal{C})$, an outcome $\boldsymbol{w}$ provides* cohEJR$_\mathcal{C}$-1 *if for every T-cohesive group of voters $N' \subseteq N$ for some $T \subseteq \mathcal{I}$ with an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$, there exists a voter $i \in N'$ such that $u_i(\boldsymbol{w}) \geqslant |T| - 1$.*

As cohEJR$_\mathcal{C}$ implies cohEJR$_\mathcal{C}$-1, the results of Propositions 4 and 5 immediately apply to cohEJR$_\mathcal{C}$-1.

Corollary 7. *For binary election instances $(\boldsymbol{B}, \mathcal{C})$ with $|\mathcal{C}| = 2$ where the constraint $\mathcal{C}$ has the NFD property, cohEJR$_\mathcal{C}$-1 can always be provided.*

Corollary 8. *For binary election instances $(\boldsymbol{B}, \mathcal{C})$ with $m \leqslant 3$ where the constraint $\mathcal{C}$ has the NFD property, cohEJR$_\mathcal{C}$-1 can always be provided.*

Note that for the computational result for cohEJR$_\mathcal{C}$ in Proposition 2, a simple alteration of the proof given for Proposition 2 (replacing the value $|T|$ with $|T| - 1$ in the final satisfaction check) yields a corresponding computational result for cohEJR$_\mathcal{C}$-1.

Proposition 9. *Given an election instance $(\boldsymbol{B}, \mathcal{C})$ and an outcome $\boldsymbol{w} \in \mathcal{C}$, there exists an algorithm that decides in $O((\max_{t \in [m]} |D_t|)^m |\mathcal{C}|^3 mn)$ time whether outcome $\boldsymbol{w}$ provides cohEJR$_\mathcal{C}$-1.*

For the result of stating that cohEJR$_\mathcal{C}$ can be provided when $m = 2$ given that NFD holds (see Proposition 5), we can show something stronger for cohEJR$_\mathcal{C}$-1 by dropping the assumption that the NFD property holds.

Proposition 10. *For election instances $(\boldsymbol{B}, \mathcal{C})$ with $m = 2$, cohEJR$_\mathcal{C}$-1 can always be provided.*

Proof. Consider an election over two issues, where a T-cohesive group of voters has an $(S, \boldsymbol{w})$-deviation for some outcome $\boldsymbol{w}$, as per Definition 6. Observe that, when $m = 2$, $(S, \boldsymbol{w})$-deviation are only possible for $|S| \in \{1, 2\}$. Take a T-cohesive group N' for $|T| = 1$ with an $(S, \boldsymbol{w})$-deviation from $\boldsymbol{w}$ to some other feasible outcome $\boldsymbol{w}' \in \mathcal{C}$. Even if $u_i(\boldsymbol{w}) = 0$ for every voter $i \in N'$, we have $u_i(\boldsymbol{w}) \geqslant |T| - 1 = 1 - 1 = 0$, and thus cohEJR$_{\mathcal{C}}$-1 is satisfied. Take now a T-cohesive group N' for $|T| = 2$: for them to deviate, it must be the case that $N' = N$, and $u_i(\boldsymbol{w}) = 0$ for all $i \in N$. If they have an $(S, \boldsymbol{w})$-deviation for $|S| = |T| = 2$, the outcome $\boldsymbol{w'}$ they wish to deviate to must increase the satisfaction of each voter by at least 1, which thus satisfies $u_i(\boldsymbol{w}) \geqslant |T| - 1 = 2 - 1 = 1$, and thus cohEJR$_{\mathcal{C}}$-1. □

Can we show that an outcome providing cohEJR$_{\mathcal{C}}$-1 always exists when there are more than three issues, unlike for cohEJR$_{\mathcal{C}}$? Unfortunately, this is not the case, even assuming property NFD, as the same counterexample used to prove Proposition 6 yields the following (so also for binary election instances).

Proposition 11. *There exists an election instance $(\boldsymbol{B}, \mathcal{C})$ where $m > 3$ and the constraint $\mathcal{C}$ has the NFD property but there exists no outcome that provides cohEJR$_{\mathcal{C}}$-1.*

We demonstrate that the challenge of satisfying cohEJR$_{\mathcal{C}}$-1 lies in the constraints. To do so, we show that in the setting without constraints, it is always possible to find an outcome that provides cohEJR$_{\mathcal{C}}$-1. To do so, we define the constrained version of MES that has been studied for the public-decision setting without constraints. Our adaptation allows for the prices associated with fixing the outcome's decisions on issues to vary. This contrasts with the unconstrained MES that fixes the prices of every issue's decision to n from the onset. And this pricing is determined by a particular pricing type λ.

Definition 7 (MES$_{\mathcal{C}}$)**.** *The rule runs for at most m rounds. Each voter has a budget of m. In every round, for every undecided issue a_t in a partial outcome $\boldsymbol{w}^*$, we identify those issue-decision pairs (a_t, d) where fixing some decision $d \in D_t$ on issue a_t allows for a feasible outcome to be returned in future rounds. If no such issue-decision pair exists, then the rule stops. Otherwise, for every such pair (a_t, d), we calculate the minimum value for $\rho_{(a_t,d)}$ such that if each voter in $N(a_t, d)$ were to pay either $\rho_{(a_t,d)}$ or the remainder of their budget, then these voters could afford to pay the price $\lambda(a_t, d)$ (determined by the pricing type λ). If there exists no such value for $\rho_{(a_t,d)}$, then we say that the issue-decision pair (a_t, d) is* not affordable *in round, and if in a round, there are no affordable issue-decision pairs, the rule stops. Otherwise, we update $\boldsymbol{w}^*$ by setting decision d on issue a_t for the pair (a_t, d) with a minimal value $\rho_{(a_t,d)}$ (breaking ties arbitrarily, if necessary) and have each voter in $N(a_t, d)$ either paying $\rho_{(a_t,d)}$, or the rest of their budget. Note that MES$_{\mathcal{C}}$ may terminate with not all issues being decided and we assume that all undecided issues are decided arbitrarily.*

A natural candidate for a pricing type is the standard pricing of unconstrained MES where the price for every issue-decision pair (a_t, d) is set to $\lambda(a_t, d) = n$. And with this pricing, that we refer to as *unit* pricing λ_{unit}, we can show that MES$_{\mathcal{C}}$ satisfies cohEJR$_{\mathcal{C}}$-1 for unconstrained, binary elections.

Proposition 12. *For binary election instances, when $\mathcal{C} = \{0, 1\}^m$, MES$_{\mathcal{C}}$ with unit pricing λ_{unit} satisfies cohEJR$_{\mathcal{C}}$-1.*

Proof. Take an outcome $\boldsymbol{w}$ returned by MES$_{\mathcal{C}}$ with unit pricing λ_{unit} and consider a T-cohesive group of voters N'. Let us assume that for every voter $i \in N'$, it holds that $u_i(\boldsymbol{w}) < |T| - 1$ and then set $\ell = |T| - 1$. So to conclude the run of MES$_{\mathcal{C}}$, each voter in N' paid for at most $\ell - 1 = |T| - 2$ issues.

Now, assume that the voters in N' paid at most $m/(\ell+1)$ for any decision on an issue. We know that each voter has at least the following funds remaining at that moment:

$$m - (\ell - 1)\frac{m}{\ell + 1} = \frac{2m}{\ell + 1} = \frac{2m}{|T|} \geqslant \frac{2n}{|N'|}.$$

The last step follows from the group N' being . So now we know that the voters in N' hold at least $2n$ in funds when some at the end of MES$_{\mathcal{C}}$'s run. Thus, we know that at least two issues have not been funded and for at least one of these two issues, at least half of N' agree on the decision of this issue (as the election instance is binary) and they hold enough funds to pay for it, hence, we have a contradiction to MES$_{\mathcal{C}}$ terminating.

Now, assume that some voter i in N' paid more than $m/(\ell+1)$ for a decision on an issue. Since we know that at the end of MES$_{\mathcal{C}}$'s execution, each voter in N' paid for at most $\ell - 1 = |T| - 2$ issues, then at the round r that voter i paid more than $m/(\ell+1)$ for an issue's decision, the voters in N' collectively held at least $2n$ in funds. Since at least two issues in were not funded, there exists some issue that could have been paid for in round r, where voters each pay $m/(\ell+1)$, contradicting the fact that voter i paid more than $m/(\ell+1)$ in round r. So, we have that this group of voters N' cannot exist and that MES$_{\mathcal{C}}$ satisfies cohEJR$_{\mathcal{C}}$-1. □

This result provides us with an axiom 'close to' EJR that we know is always satisfiable when the issues have size-two domains and there are no constraints.

5 Justified Representation with Agreement

Given the mostly negative results regarding the cohesiveness-EJR notion, we move on to justified representation based on agreement. We justify this move as the notion based on agreement is weaker and yields more positive results in the unconstrained setting. Thus, by assessing it here, we are able to establish a baseline of what can be achieved in terms of EJR-like proportionality guarantees in our constrained model. First, we formalise agreement-based EJR with the following axiom.

Definition 8 (agrEJR$_\mathcal{C}$)**.** *Given an election* $(\boldsymbol{B}, \mathcal{C})$*, an outcome* $\boldsymbol{w}$ *provides* agrEJR$_\mathcal{C}$ *if for every* T*-agreeing group of voters* $N' \subseteq N$ *for some* $T \subseteq \mathcal{I}$ *with an* $(S, \boldsymbol{w})$*-deviation for some* $S \subseteq T$ *with* $|S| \leqslant |T| \cdot {}^{|N'|}/_{n}$*, there exists a voter* $i \in N'$ *such that* $u_i(\boldsymbol{w}) \geqslant {}^{|N'|}/_{n} \cdot |T|$.

Now, in more unfortunate news, we find that agrEJR$_\mathcal{C}$ is not always satisfiable in general. In fact, the counterexample of Proposition 6 suffices to show this as each voter requires at least 1 in satisfaction for to agrEJR$_\mathcal{C}$ to be satisfied.

Proposition 13. *There exists an election instance where no outcome provides agrEJR*$_\mathcal{C}$ *(even when the NFD property holds for* $\mathcal{C}$*).*

We now focus on a particular class of constraints as we import agreement-EJR into our setting. Specifically, we consider a class that allows us to talk about how restrictive, and thus how costly, the fixing of a particular issue-decision pair is.

Akin to work by Rey et al. (2023, 2020), we consider constraints $\mathcal{C}$ that can be equivalently expressed as a set of implications *Imp*$_\mathcal{C}$, where each implication in *Imp*$_\mathcal{C}$ is a propositional formula with the following form: $\ell_{(a_x,d_x)} \to \ell_{(a_y,d_y)}$. This class of constraints allows us, for instance, to express simple dependencies and conflicts such as 'selecting x means that we must select y' and 'selecting x means that y cannot be selected', respectively. These constraints correspond to *propositional logic formulas* in *2CNF*.

Example 2. Take a set of issues $\mathcal{I} = \{a, b, c, d, e\}$ for a binary election instance. Here is an example of an implication set:

- *Imp*$_\mathcal{C}$ $= \{(a, 1) \to (b, 1), (c, 1) \to (e, 0), (d, 1) \to (e, 0)\}$. Here, accepting a means that b must also be accepted while accepting either c or d requires the rejection of e.

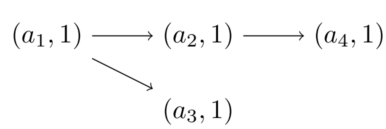
△

Given a (possibly partial) outcome $\boldsymbol{w} \in \mathcal{C}$ and the set *Imp*$_\mathcal{C}$, we construct a directed *outcome implication graph* $G_{\boldsymbol{w}} = \langle H, E \rangle$ where $H = \bigcup_{a_t \in \mathcal{I}} \{(a_t, d) \mid d \in D_t\}$ as follows:

1. Add the edge $((a_x, d_x), (a_y, d_y))$ to E if $\ell_{(a_x,d_x)} \to \ell_{(a_y,d_y)} \in$ *Imp*$_\mathcal{C}$ and $w_y \neq d_y$;
2. Add the edge $((a_y, d_y^*), (a_x, d_x^*))$ for all $d_y^* \neq d_y \in D_y, d_x^* \neq d_x \in D_x$ to E if $\ell_{(a_x,d_x)} \to \ell_{(a_y,d_y)} \in$ *Imp*$_\mathcal{C}$ and $w_x = d_x$.

Given such a graph $G_{\boldsymbol{w}}$ for an outcome $\boldsymbol{w}$, we use $G_{\boldsymbol{w}}(a_x, d_x)$ to denote the set of all vertices that belong to some path in $G_{\boldsymbol{w}}$ having vertex (a_x, d_x) as the source (note that $G_{\boldsymbol{w}}(a_x, d_x)$ excludes (a_x, d_x)).

Example 3. Consider a binary election instance and take a set of issues $\mathcal{I} = \{a_1, a_2, a_3, a_4\}$ and the implication set *Imp*$_\mathcal{C}$ $= \{(a_1, 1) \to (a_2, 1), (a_1, 1) \to (a_3, 1), (a_2, 1) \to (a_4, 1)\}$ of some constraint $\mathcal{C}$. Consider the outcome implication graph for $\boldsymbol{w}_1 = (0, 0, 0, 0)$ (vertices with no adjacent edges are omitted for readability):



Then, we have $G_{\boldsymbol{w}}(a_1, 1) = \{(a_2, 1), (a_3, 1), (a_4, 1)\}$ and therefore $|G_{\boldsymbol{w}}(a_1, 1)| = 3$. △

Thus, for an issue-decision pair (a_x, d_x), we can count the number of affected issues in setting a decision d_x for the issue a_x. This leads us to the following class of constraints.

Definition 9 (k-restrictive constraints)**.** *Take some constraint $\mathcal{C}$ expressible as a set of implications $Imp_{\mathcal{C}}$. For some positive integer $k \geqslant 2$, we say that $\mathcal{C}$ is* k-restrictive *if for every outcome $\boldsymbol{w} \in \mathcal{C}$, it holds that:*

$$\max\left\{ |G_{\boldsymbol{w}}(a_x, d_x)| \;\middle|\; (a_x, d_x) \in \bigcup_{a_t \in \mathcal{I}} \{(a_t, d) \mid d \in D_t\} \right\} = k - 1$$

where $G_{\boldsymbol{w}}$ is the outcome implication graph constructed for outcome $\mathbf{w}$ and the implication set $Imp_{\mathcal{C}}$.

Intuitively, with a k-restrictive constraint, if one were to fix/change an outcome $\boldsymbol{w}$'s decision for one issue, this would require fixing/changing $\boldsymbol{w}$'s decisions on at most $k-1$ other issues. So intuitively, when dealing with k-restrictive constraints, we can quantify (at least loosely speaking) how 'difficult' it is to satisfy a constraint via the use of this value k. Thus, we can use this value k to account for the constraint's difficulty when designing proportionality axioms.

Before assessing how k-restrictive constraints affect our goal of providing proportionality, we touch on the computational complexity of checking, for some constraint $\mathcal{C}$, whether there exists a set of implications *Imp*$_{\mathcal{C}}$ that is equivalent to $\mathcal{C}$. For the case of binary elections, this problem been studied under the name of *Inverse Satisfiablility* and it has been shown that for formulas in 2CNF, the problem is in P (Kavvadias and Sideri, 1998). So in the remainder of the paper, when we refer to a k-restrictive constraint $\mathcal{C}$, we thus assume that $\mathcal{C}$ is expressible using an implication set *Imp*$_{\mathcal{C}}$.

We now import the agreement-EJR notion and an approximate variant into our framework with constraints.

Definition 10 (α-agrEJR$_{\mathcal{C}}$-β)**.** *Given an election $(\boldsymbol{B}, \mathcal{C})$, some $\alpha \in (0,1]$ and some positive integer β, an outcome $\boldsymbol{w}$ provides* α-agrEJR$_{\mathcal{C}}$-β *if for every T-agreeing group of voters $N' \subseteq N$ for some $T \subseteq \mathcal{I}$ with an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$ with $|S| \leqslant |T| \cdot {}^{|N'|}/_{n}$, there exists a voter $i \in N'$ such that $u_i(\boldsymbol{w}) \geqslant \alpha \cdot {}^{|N'|}/_{n} \cdot |T| - \beta$.*

With this axiom, we formalise agreement-EJR to our constrained public-decision model with the presence of the multiplicative and additive factors allowing us to measure how well rules satisfy this notion even if they fall short providing the ideal representation.[3] Note that for the sake of readability, when we have either $\alpha = 1$ or $\beta = 0$, we omit them from the notation when referring to α-agrEJR$_{\mathcal{C}}$-β.

Example 4. Suppose there are four issues $\mathcal{I} = \{a_1, a_2, a_3, a_4\}$ and consider a constraint $\mathcal{C} = \{(1,1,0,0), (1,1,1,0)\}$. Then suppose there are two voters with ballots $\boldsymbol{b}_1 = (1,1,1,1)$ and $\boldsymbol{b}_2 = (0,0,0,0)$ so each voter deserves at least 2 in satisfaction according to agreement-EJR. See that outcome $\boldsymbol{w} = (1,1,0,0)$ provides agrEJR$_{\mathcal{C}}$ while the outcome $\boldsymbol{w}' = (1,1,1,0)$ only provides ${}^{1}/_{2}$-agrEJR$_{\mathcal{C}}$ as voter 2 only obtains 1 in satisfaction whilst having a sufficiently small $(S, \boldsymbol{w})$-deviation for the issue a_3 (deviating to outcome $\boldsymbol{w}$). △

We now analyse MES$_{\mathcal{C}}$ with respect to this axiom for the class of k-restrictive constraints. We say that for MES$_{\mathcal{C}}$, the price for an issue-decision pair (a_x, d) given a partial outcome $\boldsymbol{w}^*$ is $\lambda(a_x, d) = n \cdot (|G_{\boldsymbol{w}^*}(a_x, d)| + 1)$ and we refer to this as a *fixed* pricing λ_{fix}. Then we can show the following for binary election instances.

Theorem 14. *For binary election instances $(\boldsymbol{B}, \mathcal{C})$ where $\mathcal{C}$ is k-restrictive for some k, MES$_{\mathcal{C}}$ with fixed pricing λ_{fix} satisfies ${}^{1}/_{k}$-agrEJR$_{\mathcal{C}}$-1.*

Proof. For a binary election instance $(\boldsymbol{B}, \mathcal{C})$ where $\mathcal{C}$ is k-restrictive, take an outcome $\boldsymbol{w}$ returned by MES$_{\mathcal{C}}$ with fixed pricing λ_{fix}. Consider a T-agreeing voter group N'. Let us assume that for every $i \in N'$, it holds that $u_i(\boldsymbol{w}) < {}^{|N'|}/_{nk} \cdot |T| - 1$ and then set $\ell = {}^{|N'|}/_{nk} \cdot |T| - 1$. So to conclude MES$_{\mathcal{C}}$, each voter $i \in N'$ paid for at most $\ell - 1 = {}^{|N'|}/_{nk} \cdot |T| - 2$ issues. Note that for a k-restrictive constraint $\mathcal{C}$, the maximum price MES$_{\mathcal{C}}$ with fixed pricing λ_{fix} sets for any issue-decision pair is nk (as at most k issues are fixed for a MES$_{\mathcal{C}}$ purchase). Now, assume that the voters in N' paid at most ${}^{m}/_{(\ell+1)}$ for any decision on an issue. We know that each voter has at least the following funds remaining at that moment:

$$m - (\ell - 1)\frac{m}{\ell + 1} = \frac{2m}{\ell + 1} = \frac{2m}{{}^{|N'|}/_{kn} \cdot |T|} = \frac{2mnk}{|N'||T|} \geqslant \frac{2nk}{|N'|}.$$

We now have that voter group N' holds at least $2nk$ in funds at the rule's end. Thus, we know that at least k issues have not been funded and for at least one of these k issues, at least half of N' agree on the decision for it (as the election is a binary instance) while having enough funds to pay for it. Hence, we have a contradiction to MES$_{\mathcal{C}}$ terminating.

Now, assume that some voter $i \in N'$ paid more than ${}^{m}/_{(\ell+1)}$ for fixing an issue's decision. Since we know that at the end of MES$_{\mathcal{C}}$'s run, each voter in N' paid for at most $\ell - 1$ issues, then at the round r that voter i paid more than

[3] Observe that we include the axiom's size requirement on the set S such that a group has an $(S, \boldsymbol{w})$-deviation in order to prohibit considering cases such as a single voter only having an $(S, \boldsymbol{w})$-deviation for $S = \mathcal{I}$ while not intuitively being entitled to that much representation.

$m/(\ell+1)$, the voters group N' collectively held at least $2nk$ in funds. Since at least k issues in were not funded, there exists some issue that could have been paid for in round r, where voters each pay $m/(\ell+1)$. This contradicts the fact that voter i paid more than $m/(\ell+1)$ in round r. So, we have that this group of voters N' cannot exist which concludes the proof. ☐

Towards an even more positive result, and one where we are not limited to binary election instances, we now provide an adaptation of the *MeCorA* rule (Skowron and Górecki, 2022) . In the unconstrained public-decision model, MeCorA is presented by Skowron and Górecki (2022) as an auction-style variant of MES that allows voter groups to change the decision of an issue all while increasing the price for any further change to this issue's decision. In our constrained model, groups are allowed to pay for changes to the decisions on sets of issues, as long as these changes represent a feasible deviation.

Definition 11 (MeCorA$_\mathcal{C}$)**.** *Take some constant $\epsilon > 0$. Start by setting $\lambda_t = 0$ as the current price of every issue $a_t \in \mathcal{I}$, endow each voter $i \in N$ with a personal budget of m and take some arbitrary, feasible outcome $\boldsymbol{w} \in \mathcal{C}$ as the current outcome. A groups of voters can 'update' the current outcome $\boldsymbol{w}$'s decisions on some issues $S \subseteq \mathcal{I}$ if the group:*

(i) *can propose, for each issue $a_t \in S$, a new price $\lambda_t^* \geqslant \lambda_t + \epsilon$,*

(ii) *can afford the sum of new prices for issues in S, and*

(iii) *has an $(S, \boldsymbol{w})$-deviation.*

The rule then works as follows. Given a current outcome $\boldsymbol{w}$, it computes, for every non-empty $S \subseteq \mathcal{I}$, the smallest possible value $\rho_{(t,S)}$ for each issue $a_t \in S$ such that for some N', if voters in N' each pay $\rho_S = \sum_{a_t \in S} \rho_{(t,S)}$ (or their remaining budget), then N' is able to 'update' the decisions on every $a_t \in S$ as per conditions $(i) - (iii)$. If there exists no such voter group for issues S then it sets $\rho_S = \infty$.

If $\rho_S = \infty$ for every $S \subseteq \mathcal{I}$, the process terminates and returns the current outcome $\boldsymbol{w}$. Otherwise, it selects the set S with the lowest value ρ_S (any ties are broken arbitrarily) and does the following:

1. *updates the current outcome $\boldsymbol{w}$'s decisions on issues in S to the decisions agreed upon by the voters with the associated $(S, \boldsymbol{w})$-deviation,*
2. *updates the current price of every issue $a_t \in S$ to λ_t^*,*
3. *returns all previously spent funds to all voters who paid for the now-changed decisions on issues in S,*
4. *and finally, for each voter in N', deduct $\sum_{a_t \in S} \rho_{(t,S)}$ from their personal budget (or the rest of their budget).*

Next, we show the representation guarantees can be achieved on instances with k-restrictive constraints via the use of modified version of MeCorA$_\mathcal{C}$. Moreover, we can drop the restriction to binary election instances that was key for the result of Theorem 14. In this MeCorA$_\mathcal{C}$ variant, we first partition the voter population into groups where members of each group agree on some set of issues. Then, for each group, its members may only pay to change some decisions as a collective and only on those issues that they agree on. Contrarily to MeCorA$_\mathcal{C}$, voter groups cannot pay to change some decisions if this leads to the group's members gaining 'too much' satisfaction from the altered outcome (i.e., a voter group exceeding their proportional share of their agreed-upon issues, up to some additive factor q that parameterises the rule).

Definition 12 (Greedy MeCorA$_\mathcal{C}$-q)**.** *The set of the voters N is partitioned into p disjoints sets $N(T_1), \ldots, N(T_p)$ such that:*

(i) *for every $x \in \{1, \ldots, p\}$, a voter group $N(T_x) \subseteq N$ is T_x-agreeing for some $T_x \subseteq \mathcal{I}$, and*

(ii) *for all $x \in \{1, \ldots, p-1\}$, it holds that $|N(T_x)| \cdot |T_x| \geqslant |N(T_{x+1})| \cdot |T_{x+1}|$*

As with MeCorA$_\mathcal{C}$, voter groups shall pay to change the decisions of some issues during the rule's execution. However, given the initial partition, during the run of Greedy MeCorA$_\mathcal{C}$-q, the voters in $N(T_x)$ may only change decisions for the issues in T_x.

Moreover, if a voter group $N(T_x)$ for some $x \in \{1, \ldots, p\}$ wishes to change some decisions at any moment during the process, this change does not lead to any voter in $N(T_x)$ having satisfaction greater than $|N(T_x)|/n \cdot |T_x| - q$ with the updated outcome. Besides these two differences, the rule works exactly as MeCorA$_\mathcal{C}$.

Now, we can show the following for Greedy MeCorA$_\mathcal{C}$-q working on a k-restrictive constraint. For this result, we require the additional assumption that voter ballots represent feasible outcomes in $\mathcal{C}$.

Theorem 15. *For election instances $(\boldsymbol{B}, \mathcal{C})$ where voters' ballots are consistent with the constraint $\mathcal{C}$ and $\mathcal{C}$ is k-restrictive for some $k \geqslant 2$, Greedy MeCorA$_\mathcal{C}$-$(k-1)$ satisfies agrEJR$_\mathcal{C}$-$(k-1)$.*

Proof. Take an outcome $\boldsymbol{w}$ returned by Greedy MeCorA$_\mathcal{C}$-$(k-1)$. Assume that $\boldsymbol{w}$ does not provide agrEJR$_\mathcal{C}$-$(k-1)$. Thus, there is a T-agreeing group N' such that $u_i(\boldsymbol{w}) < {|N'|}/{n} \cdot |T| - k + 1 = \ell$ holds for every $i \in N'$. Now, consider the partition of voters $N(T_1), \ldots, N(T_p)$ constructed by Greedy MeCorA$_\mathcal{C}$-$(k-1)$ to begin its run. Assume first that there is some $x \in \{1, \ldots, p\}$ such that $N' = N(T_x)$, i.e., voters N' appear in their entirety in said partition. We then have $T = T_x$. Moreover, voters in N' each contribute to at most ℓ decisions at any moment of the run of Greedy MeCorA$_\mathcal{C}$-$(k-1)$, as this is the limit the rule imposes on their total satisfaction. We now consider two cases. Assume that the voters in N' contributed at most ${m}/{(\ell+k-1)}$ to change some decisions during the rule's execution. It follows that each voter has at least the following funds remaining: $m - (\ell - 1) \cdot {m}/{(\ell+k-1)} \geqslant {nmk}/{|N'||T|}$.

In this case, the voters in N' would have at least ${nmk}/{|T|}$ in collective funds, so it follows that each distinct $(S, \boldsymbol{w})$-deviation available to N' must cost at least ${nmk}/{|T|}$. As N' is T-agreeing, it must be that N' has at least a ${(|T|-\ell+1)}/{k}$ many $(S, \boldsymbol{w})$-deviations due to $\mathcal{C}$ being k-restrictive and as the voters' ballots are consistent with $\mathcal{C}$.

Now, consider the case where some voter in N' contributed more than ${m}/{(\ell+k-1)}$ to change some decisions. The first time that this occurred, the change of decisions did not lead to any voter in N' obtaining a satisfaction greater than $\ell = {|N'|}/{n} \cdot |T| - k + 1$ (otherwise the rule would not allow these voters to pay for the changes). Thus, each voter in N' must have contributed to at most $\ell - 1$ issues before this moment. From the reasoning above, it must hold that in this moment, each voter held at least ${nmk}/{|N'||T|}$ in funds with there being at least ${(|T|-\ell+1)}/{k}$ feasible deviations available to N' and each such deviation costing at least ${nmk}/{|T|}$. So in both cases, for the $(S, \boldsymbol{w})$-deviations that are present in T that voters in N' wish to make, outcome $\boldsymbol{w}$'s decisions must have been paid for by voters within the remaining voter population $N \setminus N'$. And so, these decisions must have cost the voters in $N \setminus N'$ at least:

$$\frac{nmk}{|T|} \cdot \left(\frac{|T| - \ell + 1}{k}\right) = \frac{nm}{|T|} \cdot \left(|T| - \frac{|N'|}{n} \cdot |T| + k\right)$$
$$> \frac{nm}{|T|} \cdot \left(\frac{n|T| - |N'||T|}{n}\right) = m(n - |N'|).$$

However, voters $N \setminus N'$ have at most $m(n - |N'|)$ in budget. Thus, the rule cannot have terminated with the voter group N' existing.

Now, assume that the group N' did not appear in their entirety within the partition $N(T_1), \ldots, N(T_p)$ made by Greedy MeCorA$_\mathcal{C}$-$(k-1)$. This means that some voter $i \in N'$ is part of another voter group $N(T_x)$ that is T_x-agreeing such that ${|N(T_x)|}/{n} \cdot |T_x| \geqslant {N'}/{n} \cdot |T|$. Now, recall that for each voter group $N(T)$ in the partition, the voters in $N(T)$ have the same satisfaction to end the rule's execution (as they only pay to flip decisions as a collective). Thus, from the arguments above, it holds for this voter $i \in N' \cap N(T_x)$ that $u_i(\boldsymbol{w}) \geqslant {|N(T_x)|}/{n} \cdot |T_x| - k + 1 \geqslant {|N'|}/{n} \cdot |T| - k + 1$, which contradicts the assumption that every voter in N' has satisfaction less than ${|N'|}/{n} \cdot |T| - k + 1$. ☐

We now offer another way towards producing proportional outcomes when using k-restrictive constraints. It is a constrained adaptation of the *Local Search Proportional Approval Voting (LS PAV)* rule from the MWV literature that is a polynomial-time computable rule that is known to satisfy EJR (Aziz et al., 2017). In the MWV setting, the rule begins with an arbitrary committee of some fixed size k and in iterations, searches for any swaps between committee members and non-selected candidates that brings about an increase of the *PAV score* by at least ${n}/{k^2}$. In our model, the PAV score of some feasible outcome $\boldsymbol{w} \in \mathcal{C}$ is defined to be PAV$(\boldsymbol{w}) = \sum_{i \in N} \sum_{t=1}^{u_i(\boldsymbol{w})} {1}/{t}$. We can then lift Local Search PAV to our setting with constraints.

Definition 13 (Local Search PAV$_\mathcal{C}$, LS PAV$_\mathcal{C}$)**.** *Beginning with an arbitrary outcome $\boldsymbol{w} \in \mathcal{C}$ as the current winning outcome, the rule looks for all possible deviations. If there exists an $(S,\boldsymbol{w})$-deviation for some voter group to some outcome $\boldsymbol{w}' \in \mathcal{C}$ such that $PAV(\boldsymbol{w}') - PAV(\boldsymbol{w}) \geqslant {n}/{m^2}$, i.e., the new outcome $\boldsymbol{w}'$ yields a PAV score that is at least ${n}/{m^2}$ higher than that of $\boldsymbol{w}$, then the rule sets $\boldsymbol{w}'$ as the current winning outcome. The rule terminates once there exists no deviation that improves on the PAV score of the current winning outcome by at least ${n}/{m^2}$.*

As there is a maximum obtainable PAV score, LS PAV$_\mathcal{C}$ is guaranteed to terminate. The question is how long this rule takes to return an outcome when we have to take k-restrictive constraints into account.

Proposition 16. *For elections instances where $\mathcal{C}$ is k-restrictive (where k is a fixed constant), LS PAV$_\mathcal{C}$ terminates in polynomial time.*

Proof. We show that given an outcome $\boldsymbol{w}$, finding all possible deviations can be done in polynomial time for a k-restrictive constraint $\mathcal{C}$. This can be done by exploiting the presence of the implication set $Imp_{\mathcal{C}}$. Note that the size of the implication set $Imp_{\mathcal{C}}$ is polynomial in the number of issues. So we can construct the outcome implication graph of $Imp_{\mathcal{C}}$ and the outcome $\boldsymbol{w}$ in polynomial time. Then for each issue $a_t \in \mathcal{I}$, we can find the set $G_{\boldsymbol{w}}(a_t, d)$ for some $d \neq w_t \in D_t$ in polynomial time and the issue-decision pairs represent the required additional decisions to be fixed in order to make a deviation from outcome $\boldsymbol{w}$ by changing the $\boldsymbol{w}$'s decision on issue a_t to d. Doing this for each issue a_t allows us to find a deviation that can improve the PAV score, if such a deviation exists. With similar reasoning used in other settings (Aziz et al., 2017; Chandak et al., 2024), we end by noting that since there is a maximum possible PAV score for an outcome, and each improving deviation increases the PAV score by at least n/m^2, the number of improving deviations that LS-PAV$_{\mathcal{C}}$ makes is polynomial in the number of issues m. ☐

Off the back of this positive computational result, we present the degree to which LS PAV$_{\mathcal{C}}$ provides proportional outcomes with regards to the α-agrEJR$_{\mathcal{C}}$-β axiom.

Theorem 17. *For election instances $(\boldsymbol{B}, \mathcal{C})$ where the voters' ballots are consistent with the constraint $\mathcal{C}$ and $\mathcal{C}$ is k-restrictive for some $k \geqslant 2$, LS-PAV$_{\mathcal{C}}$ satisfies $2/(k+1)$-agrEJR$_{\mathcal{C}}$-$(k-1)$.*

Proof. For an election instance $(\boldsymbol{B}, \mathcal{C})$ where $\mathcal{C}$ is k-restrictive for $k \geqslant 2$, take an outcome $\boldsymbol{w}$ returned by LS-PAV$_{\mathcal{C}}$ and consider a group of voters N' that agree on some set of issues T. Let us assume that for every voter $i \in N'$, it holds that $u_i(\boldsymbol{w}) < 2/k+1 \cdot |N'|/n \cdot |T| - k + 1$ and then set $\ell = 2/k+1 \cdot |N'|/n \cdot |T| - k + 1$. We use r_i to denote the number of outcome $\boldsymbol{w}$'s decisions that a voter $i \in N$ agrees with.

For each voter $i \in N \setminus N'$, we calculate the maximal reduction in PAV score that may occur from a possible deviations by LS-PAV$_{\mathcal{C}}$ when $\mathcal{C}$ is k-restrictive. This happens when for each of at most r_i/k deviations, we decrease their satisfaction by k and remove $\sum_{t=0}^{k-1} 1/(r_i - t)$ in PAV score. So for these voters in $N \setminus N'$, we deduct at most the following:

$$\sum_{N \setminus N'} \frac{r_i}{k} \cdot \left(\sum_{t=0}^{k-1} \frac{1}{r_i - t} \right) \leqslant \sum_{N \setminus N'} \frac{r_i}{k} \cdot \left(\frac{\sum_{t=1}^{k} t}{r_i} \right) = \frac{k+1}{2} \cdot (n - |N'|).$$

Now, so there are $|T| - (\ell - 1) = |T| - \ell + 1$ issues that all voters in N' agree on but they disagree with outcome $\boldsymbol{w}$'s decisions on these issues. Since we assume the constraint is k-restrictive, then for each of these $|T| - \ell + 1$ issues, they fix at most $k - 1$ other issues and thus, there are at least $(|T| - \ell + 1)/k$ feasible deviations that can be made by LS-PAV$_{\mathcal{C}}$ amongst these issues. For the voters in N', we now consider the minimal increase in PAV score that may occur from these possible deviations by LS-PAV$_{\mathcal{C}}$. For each such deviation, we increase their satisfaction by at least k and thus, for a voter $i \in N'$, we increase the PAV score by $\sum_{t=1}^{k} 1/(r_i + t)$. Since for each voter $i \in N'$ we have $r_i \leqslant \ell - 1$, and as there are at least $(|T| - \ell + 1)/k$ feasible deviations in T, it follows that we add at least the following to the PAV score:

$$\frac{|T| - \ell + 1}{k} \cdot \left(\sum_{i \in N'} \sum_{t=1}^{k} \frac{1}{r_i + t} \right) \geqslant \frac{|T| - \ell + 1}{k} \cdot \left(\sum_{i \in N'} \sum_{t=1}^{k} \frac{1}{\ell + t - 1} \right)$$

Taking into account that $k \geqslant 2$ and $\ell = 2|N'||T|/(n(k+1)) - k + 1$, then with further simplification, we find that at least the following is added to the PAV score:

$$> \frac{n(k+1)}{2} - |N'| + \frac{n(k+1)}{|T|} \geqslant \frac{k+1}{2} \cdot (n - |N'|) + \frac{n(k+1)}{|T|}.$$

So the total addition to the PAV score due to satisfying voters in N' is strictly greater than the PAV score removed for the added dissatisfaction of voters in $N \setminus N'$ (which is at most $(k+1)(n - |N'|)/2$). And specifically, this change in score is at least $n(k+1)/|T| > n/|T|$ and thus, at least one of the $(|T| - \ell + 1)/k$ many deviations must increase the PAV score by more than:

$$\frac{k}{|T| - \ell + 1} \cdot \frac{n}{|T|} \geqslant \frac{1}{|T|} \cdot \frac{n}{|T|} \geqslant \frac{n}{|T|^2} \geqslant \frac{n}{m^2}.$$

Thus, LS-PAV$_{\mathcal{C}}$ would not terminate but would instead make this deviation in order to increase the total PAV score. Thus, contradicting that such a group N' cannot exist. ☐

With this result, we have a rule that when focused on k-restrictive constraints, is both polynomial-time computable and provides substantial proportional representation guarantees (assuming voter ballots are constraint consistent).

6 Proportionality via Priceability

With this section, we offer an alternative to the justified-representation-like interpretation of proportional representation, and this is through the notion of priceability (Lackner and Skowron, 2023). Recent work has shown the promise of this market-based approach for a general social choice model (Masařík et al., 2023) and the sequential choice model (Chandak et al., 2024). We look to employ it for constrained public decisions (albeit looking at a weaker priceability axiom than the axiom that Masařík et al. (2023) studied).

Definition 14 (Priceability)**.** *Each voter has a personal budget of m and they have to collectively fund the decisions on some issues, with each decision coming with some price. A* price system $\boldsymbol{ps} = (\{p_i\}_{i\in N}, \{\pi_{(a_t,d)}\}_{(a_t,d)\in H})$ *where* $H = \bigcup_{a_t\in\mathcal{I}}\{(a_t, d) \mid d \in D_t\}$ *is a pair consisting of (i) a collection of* payment functions $p_i : \mathcal{I} \times \{0, 1\} \to [0, b]$, *one for each voter $i \in N$, and (ii) a collection of* prices $\pi_{(a_t,d)} \in \mathbb{R}_{\geqslant 0}$, *one for each decision pair (a_t, d) for $a_t \in \mathcal{I}$ and $d \in D_t$. We consider priceability with respect to outcomes $\boldsymbol{w} \in \mathcal{C}$ where decisions are made on all issues. We say that an outcome $\boldsymbol{w} = (w_1, \dots, w_m)$ is* priceable *if there exists a price system* $\boldsymbol{ps}$ *such that:*

- (P1) : *For all $a_t \in \mathcal{I}$ and $d \in D_t$, it holds that if $d \neq \boldsymbol{b}_i^t$ we have $p_i(a_t, d) = 0$, for every $i \in N$.*
- (P2) : $\sum_{(a_t,d)\in H} p_i(a_t, d) \leqslant m$ *for every $i \in N$ where it holds that $H = \bigcup_{a_t\in\mathcal{I}}\{(a_t, d) \mid d \in D_t\}$.*
- (P3) : $\sum_{i\in V} p_i(a_t, d) = \pi_{(a_t,w_t)}$ *for every $a_t \in \mathcal{I}$.*
- (P4) : $\sum_{i\in V} p_i(a_t, d) = 0$ *for every $a_t \in \mathcal{I}$ and every $d \neq w_t \in D_t$.*
- (P5) : *There exists no group of voters N' with an $(S, \boldsymbol{w})$-deviation for some $S \subseteq \mathcal{I}$, such that for each $a_t \in S$:*

$$\sum_{i\in N'}\left(m - \sum_{(a'_t,d')\in H} p_i(a'_t, d')\right) > \pi_{(a_t,w_t)}$$

where $H = \bigcup_{a_t\in\mathcal{I}}\{(a_t, d) \mid d \in D_t\}$.

Condition (P1) states that each voter only pays for decisions that she agrees with; (P2) states that a voter does not spend more than her budget m; (P3) states that for every decision in the outcome, the sum of payments for this decision is equal to its price; (P4) states that no payments are made for any decision not in the outcome; and, finally, (P5) states that for every set of issues S, there is no group of voters N' agreeing on all decisions for issues in S, that collectively hold more in unspent budget to 'update' outcome $\boldsymbol{w}$'s decision on every issue $a_t \in S$ to a decision that they all agree with (where 'updating' these issues leads to a feasible outcome). We illustrate priceability in our setting with the following example of a binary election instance.

Example 5. Take four issues $\mathcal{I} = \{a_1, a_2, a_3, a_4\}$ and a constraint $\mathcal{C} = \{(1, 1, 1, 1), (1, 1, 0, 0)\}$. Suppose there are two voters with ballots $\boldsymbol{b}_1 = (1, 1, 1, 1)$ and $\boldsymbol{b}_2 = (0, 0, 0, 0)$. Note that outcome $\boldsymbol{w} = (1, 1, 1, 1)$ is not priceable as any price system where voter 1 does not exceed her budget would have voter 2 having enough in leftover budget to cause a violation of condition (P5) (with her entire budget being leftover, she can afford more than price of the $(S, \boldsymbol{w})$-deviation to outcome $\boldsymbol{w}$). On the other hand, $\boldsymbol{w}' = (1, 1, 0, 0)$ is priceable where we set the price of this outcome's decisions to 1. △

The following result gives some general representation guarantees whenever we have priceable outcomes.

Proposition 18. *Consider a priceable outcome $\boldsymbol{w}$ with price system $\boldsymbol{ps} = (\{p_i\}_{i\in N}, \{\pi_{(a_t,d)}\}_{(a_t,d)\in H})$ where $H = \bigcup_{a_t\in\mathcal{I}}\{(a_t, d) \mid d \in D_t\}$. Then, for every T-cohesive group of voters $N' \subseteq N$ for some $T \subseteq \mathcal{I}$ with an $(S, \boldsymbol{w})$-deviation for some $S \subseteq T$, it holds that:*

$$\sum_{i\in N'} u_i(\boldsymbol{w}) \geqslant \frac{n}{q} \cdot |T| - |S|$$

where $q = \max\{\pi_{(a_t,w_t)}\}_{a_t\in S}$.

Proof. Take a priceable outcome $\boldsymbol{w}$ and consider a T-cohesive group of voters N'. Suppose that $\sum_{i\in N'} u_i(\boldsymbol{w}) < {}^{n}/_{q} \cdot |T| - |S|$ where $q = \max\{\pi_{(a_t,w_t)}\}_{a_t\in S}$. As a group, the voters N' have a budget of $m|N'|$. Now, the voters in N' collectively contributed to at most ${}^{n}/_{q} \cdot |T| - |S| - 1$ decisions in outcome $\boldsymbol{w}$, and for each decision, the price was at most q (as q is the the price system's maximal price). So, we have that voter group N' has at least the following in leftover budget:

$$m|N'| - q \cdot \left(\frac{n}{q} \cdot |T| - |S| - 1\right) \geqslant m \cdot \frac{n|T|}{m} - n|T| + q|S| + q = q \cdot (|S| + 1).$$

Note we made use of the fact that N' is T-cohesive. Thus, we know that N' has strictly more than $q|S|$ in funds and for each issue in $a_t \in S$, holds more than in funds than $q \geqslant \pi_{(a_t,w_t)}$. This presents a violation of condition P5 of priceability. Hence, voter group N' cannot exist. □

However, we now must ascertain whether priceable outcomes always exist, regardless of the nature of the constraint. We see that this is possible thanks to the rule we have already defined, namely MeCorA$_{\mathcal{C}}$.

The next result shows that MeCorA$_{\mathcal{C}}$ captures the notion of priceability.

Proposition 19. *MeCorA$_{\mathcal{C}}$ always returns priceable outcomes.*

Proof. Let $\boldsymbol{w} = (w_1, \ldots, w_m)$ be the outcome returned by MeCorA$_{\mathcal{C}}$. We define the following price system **ps**: For each issue $a_t \in \mathcal{I}$, fix the prices $\pi_{(a_t,w_t)} = \pi_{(a_t,d)} = \lambda_t$ for all $d \neq w_t \in D_t$ where λ_t is issue a_t's last MeCorA$_{\mathcal{C}}$ price (before being set to ∞) prior to the rule's termination. Fix the payment functions p_i for each voter to the money they spent to end the execution of MeCorA$_{\mathcal{C}}$. Observe that the priceability conditions (P1)-(P4) clearly hold: since we have that, to end MeCorA$_{\mathcal{C}}$'s run, voters do not pay for decisions that (i) they do not agree with (condition (P1)) and (ii) are not made by outcome $\boldsymbol{w}$ (condition (P4)); MeCorA$_{\mathcal{C}}$ limits each voter a budget of m (condition (P2)) (P2); and the sum of payments for decisions made by outcome $\boldsymbol{w}$ will equal exactly $\pi_{(a_t,w_t)} = \lambda_t$ (condition (P3)). Now, for condition (P5), note that if such a group of voters N' existed for some set of issues S, then MeCorA$_{\mathcal{C}}$ would not have terminated as this group of voters could have changed the decisions of these issues in S while increasing each issues' prices. □

This is a positive result that, combined with that of Proposition 18, gives us a rule that always returns us priceable outcomes for any election instance.

7 Conclusion

We considered two different interpretations of justified representation from multiwinner voting and adapted them to a public-decision model with constraints. In analysing the feasibility of the axioms, we devised restricted classes of constraints (the NFD property and simple implications). While we could show mostly negative results for the satisfaction of cohesiveness-EJR under constraints, we were able to adapt successfully three known rules (MES, Local Search PAV and MeCorA)

to yield positive proportional guarantees that meet, in an approximate sense, the requirements of agreement-EJR. Additionally, we defined a suitable notion of priceability and showed that our adaptation of MeCorA always returns priceable outcomes.

Our work opens up a variety of paths for future research. First, assessing a class of constraints that are more expressive than the simple implications seems a natural starting point in extending our work. Then, on a more technical level, it would be interesting to check if the representation guarantees that are offered by MES$_{\mathcal{C}}$, LS-PAV$_{\mathcal{C}}$ and Greedy MeCorA$_{\mathcal{C}}$-$(k-1)$ still hold for a wider range of election instances. Regarding our adaptation of priceability, the question is open as to whether there are more constrained public-decision that always produce complete priceable outcomes. Given that we opted to represent the constraints as an enumeration of all feasible outcomes, it is natural to ask what occurs to results such as Proposition 1 when we consider the constraint takes a particular form of representation, e.g., $\mathcal{C}$ is represented as a Boolean formula of propositional logic. We also note some lingering computational questions such as the computational complexity of (i) computing outcomes for rules such as MES$_{\mathcal{C}}$ and Greedy MeCorA$_{\mathcal{C}}$-$(k-1)$ for general constraints, and (ii) of checking whether a given feasible outcome is priceable. Finally, the list of proportionality notions to be tested on the constraints test-bed is not exhausted, with the proportionality degree (Lackner and Skowron, 2023) most notably still to be considered.

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